Supplementary Information: Experimental quantum forgery of quantum optical money

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In this supplementary material we provide more details on our theoretical approach. We recall definitions and some properties of a few known optimal axially-symmetric quantum cloners, which we have implemented experimentally in this work. These include the universal cloner (UC), the phase-covariant cloner (PCC), and the mirror-phase-covariant cloner (MPCC). We also present the expansion coefficients $\hat{K}_{l,m}$. Moreover, we present an additional figure of the measured fidelities.

Axially-symmetric quantum cloning

The figure of merit for the quantum cloning machines is the fidelity of their clones. The special case, where the qubits are uniformly distributed around the poles of the Bloch sphere corresponds to the axially-symmetric cloning described in Ref. [1]. This class of cloning machines includes both phase-covariant cloners (PCC) and mirror-phase-covariant cloners (MPCC) as special cases. The former is a cloning process optimized for a given qubit distribution, where there is a higher chance of cloning a qubit corresponding to one of the poles than the other pole (see Fig. S1). The latter optimal cloning process is optimized for mirror-symmetric distribution on a Bloch sphere (see Fig. S1). Note that the optimal universal cloner (UC) is a special case of MPCC.

An arbitrary optimal $1 \to 2$ cloning of qubits given by an axially-symmetric distribution can be expressed as a unitary transformation[1]

$$|\circlearrowleft\rangle_a|\circlearrowleft\circlearrowright\rangle_c \;\to\; \Lambda_+|\circlearrowleft\circlearrowleft\rangle_{a,b}|\circlearrowright\rangle_c + \bar{\Lambda}_+|\psi\rangle_{a,b}|\circlearrowleft\rangle_c, \quad \text{(S1)}$$
$$|\circlearrowright\rangle_a|\circlearrowleft\circlearrowright\rangle_c \;\to\; \Lambda_-|\circlearrowright\circlearrowright\rangle_{a,b}|\circlearrowleft\rangle_c + \bar{\Lambda}_-|\psi\rangle_{a,b}|\circlearrowright\rangle_c, \quad \text{(S2)}$$

where $\bar{\Lambda}_\pm = \sqrt{1-\Lambda_\pm^2}$, $|\psi\rangle = (|\circlearrowleft\circlearrowright\rangle + |\circlearrowright\circlearrowleft\rangle)/\sqrt{2}$, and a, b, c stand for the two copy modes and the ancillary mode, respectively. For the PCC, $\Lambda_\pm \in \{0,1\}$ and $\Lambda_+ = 1-\Lambda_-$. For example, if $p(\circlearrowleft) \gg p(\circlearrowright)$, then $\Lambda_+ = 1$ and $\Lambda_- = 0$. In particular, the cloning transformation for the mirror-phase-covariant cloner[2] can be expressed as

$$|\circlearrowleft\rangle_a|\circlearrowleft\circlearrowright\rangle_c \;\to\; \Lambda|\circlearrowleft\circlearrowleft\rangle_{a,b}|\circlearrowright\rangle_c + \bar{\Lambda}|\psi\rangle_{a,b}|\circlearrowleft\rangle_c, \quad \text{(S3)}$$
$$|\circlearrowright\rangle_a|\circlearrowleft\circlearrowright\rangle_c \;\to\; \Lambda|\circlearrowright\circlearrowright\rangle_{a,b}|\circlearrowleft\rangle_c + \bar{\Lambda}|\psi\rangle_{a,b}|\circlearrowright\rangle_c. \quad \text{(S4)}$$

Quantum banknote 1 is cloned in the optimal way if $\Lambda = 0.88$. The optimal universal cloning (UC) is a special case of the MPCC and it corresponds to $\Lambda = \sqrt{2/3}$.

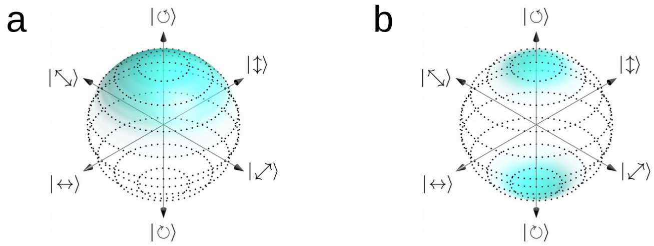

FIG. S1: Assorted axially-symmetric qubit distributions over the Bloch sphere, where the shade corresponds to the probability of choosing a qubit at the specific point on the sphere. The first example (a) is an axially-symmetric qubit distribution, which is optimally cloned by PCC. The next example (b), corresponds to an axially-symmetric qubit distribution with additional mirror symmetry. This distribution is optimally cloned by MPCC.

In the case of an axially-symmetric distribution, where all equatorial states appear with the same probability, there is a single parameter[1]

$$\Gamma = \frac{\sqrt{2}\gamma_-(\gamma_+ - 1)}{\gamma_+^2 - \gamma_-^2}, \quad \text{(S5)}$$

that can be used to select the optimal cloning transformation, which in the case of quantum banknotes 1 and 2 depends only on the probabilities $p(\circlearrowleft)$ and $p(\circlearrowright)$, because $\gamma_\pm = p(\circlearrowleft) \pm p(\circlearrowright)$. If $|\Gamma| > 1$, the optimal cloner is the PCC. Alternatively, if $|\Gamma| = 0$, the optimal cloner is the MPCC. Here, we analyse the case where $p(\circlearrowleft) = p(\circlearrowright)$ or $p(\circlearrowright) = 0$.

Hybrid quantum-classical cloning

Our linear-optical implementation of the quantum cloning machine works probabilistically. Thus, to im-

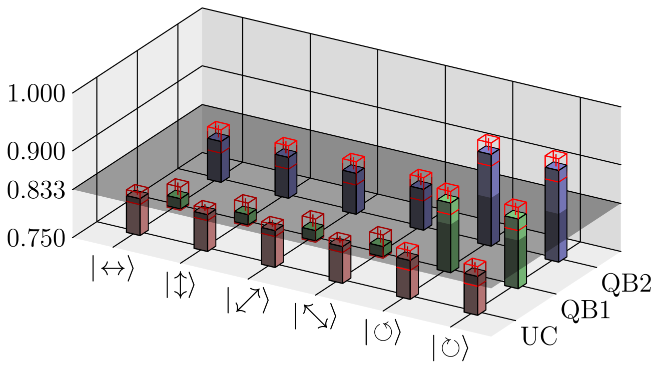

FIG. S2: Single-copy fidelities $\bar{F}_i$ for the states with different linear and circular polarizations measured during the encoding of our two quantum banknotes 1 and 2 (QB1 for $\epsilon = 0$ and QB2 for $\epsilon = 0.4$), and the optimal universal cloner (UC for $\epsilon = 0$). The grey surface shows the theoretical fidelity of the universal cloner. The red frames show the error bars of the measured fidelities. This figure can be compared with Figs. 5 and 6 in the main article.

prove the cloning efficiency, we apply, interchangeably, the optimal quantum and deterministic classical copying processes. Specifically, we classically (or quantumly) copied a fraction ϵ (or $1-\epsilon$) of the input photons. Thus, ϵ can be treated as a hybridization parameter. The measured fidelities are presented in Fig S2. The copied quantum banknotes are depicted in Figs. S3 and S4.

Expansion coefficients $\hat{K}_{l,m}$

The expansion coefficients $\hat{K}_{l,m}$, given in Eq. (15), of the operator $\hat{R}$, given in Eq. (14), can be written in the form of block matrices as follows:

$$\hat{K}_{0,0} = \frac{\sqrt{\pi}}{12}\left(\begin{array}{c|c} 3\hat{A}+\hat{B} & 2\hat{C}^{\mathrm{T}} \\ \hline 2\hat{C} & 3\hat{A}-\hat{B} \end{array}\right), \tag{S6}$$

$$\hat{K}_{1,0} = \frac{1}{2}\sqrt{\frac{\pi}{12}}\left(\begin{array}{c|c} \hat{A}+\hat{B} & \hat{0} \\ \hline \hat{0} & -\hat{A}+\hat{B} \end{array}\right), \tag{S7}$$

$$\hat{K}_{1,1} = -\sqrt{\frac{\pi}{24}}\left(\begin{array}{c|c} \hat{C} & \hat{A} \\ \hline \hat{0} & \hat{C} \end{array}\right), \tag{S8}$$

$$\hat{K}_{2,0} = -\frac{1}{6}\sqrt{\frac{\pi}{5}}\left(\begin{array}{c|c} -\hat{B} & \hat{C}^{\mathrm{T}} \\ \hline \hat{C} & \hat{B} \end{array}\right), \tag{S9}$$

$$\hat{K}_{2,1} = -\frac{1}{2}\sqrt{\frac{\pi}{30}}\left(\begin{array}{c|c} \hat{C} & \hat{B} \\ \hline \hat{0} & -\hat{C} \end{array}\right), \tag{S10}$$

$$\hat{K}_{2,2} = \sqrt{\frac{\pi}{30}}\left(\begin{array}{c|c} \hat{0} & \hat{C} \\ \hline \hat{0} & \hat{0} \end{array}\right), \tag{S11}$$

where $\hat{A} = \mathrm{diag}[2,2,2,2]$, $\hat{B} = \mathrm{diag}[2,0,0,-2]$, $C_{ij} = (\delta_{i,2}+\delta_{i,3})\delta_{j,1} + (\delta_{j,2}+\delta_{j,3})\delta_{i,4}$, and $\hat{0}$ is a 4×4 matrix of zeros. Moreover, it follows from Eq. (13) that $\hat{K}_{l,m} = (-1)^m \hat{K}^{\mathrm{T}}_{l,-m}$.

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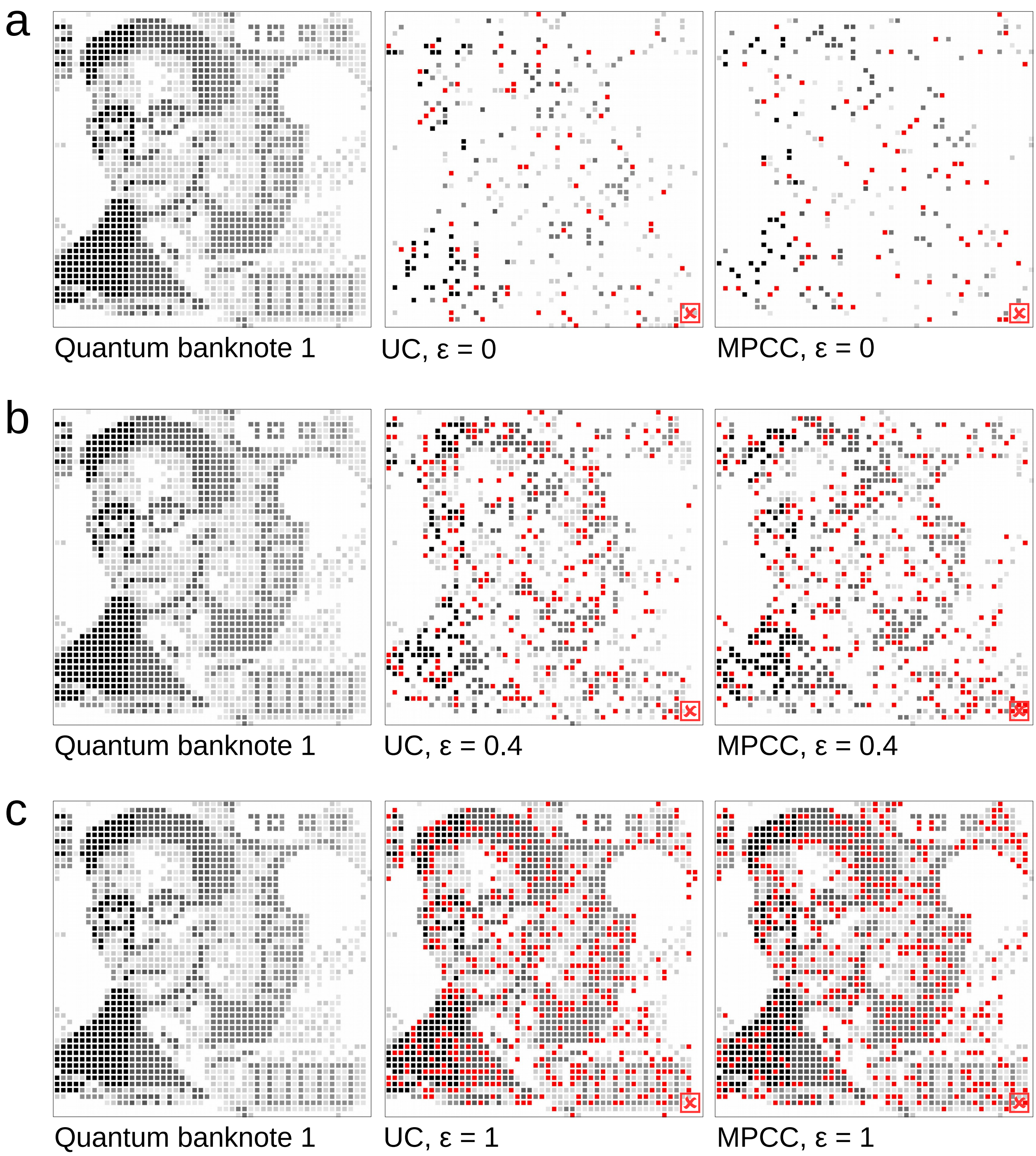

FIG. S3: Experimental quantum banknote 1 is copied probabilistically, and subsequently verified, with the optimal 1-to-2 linear optical cloning machine shown in Fig. 3 . One observes that the copies, which are provided with the best possible cloning machines, are noisy and, thus, the sequences of qubits are damaged (shown in red). The performance of the cloning process depends on the statistics of photon polarisations and on the hybridisation parameter ϵ, as shown in panels (a), (b), and (c).

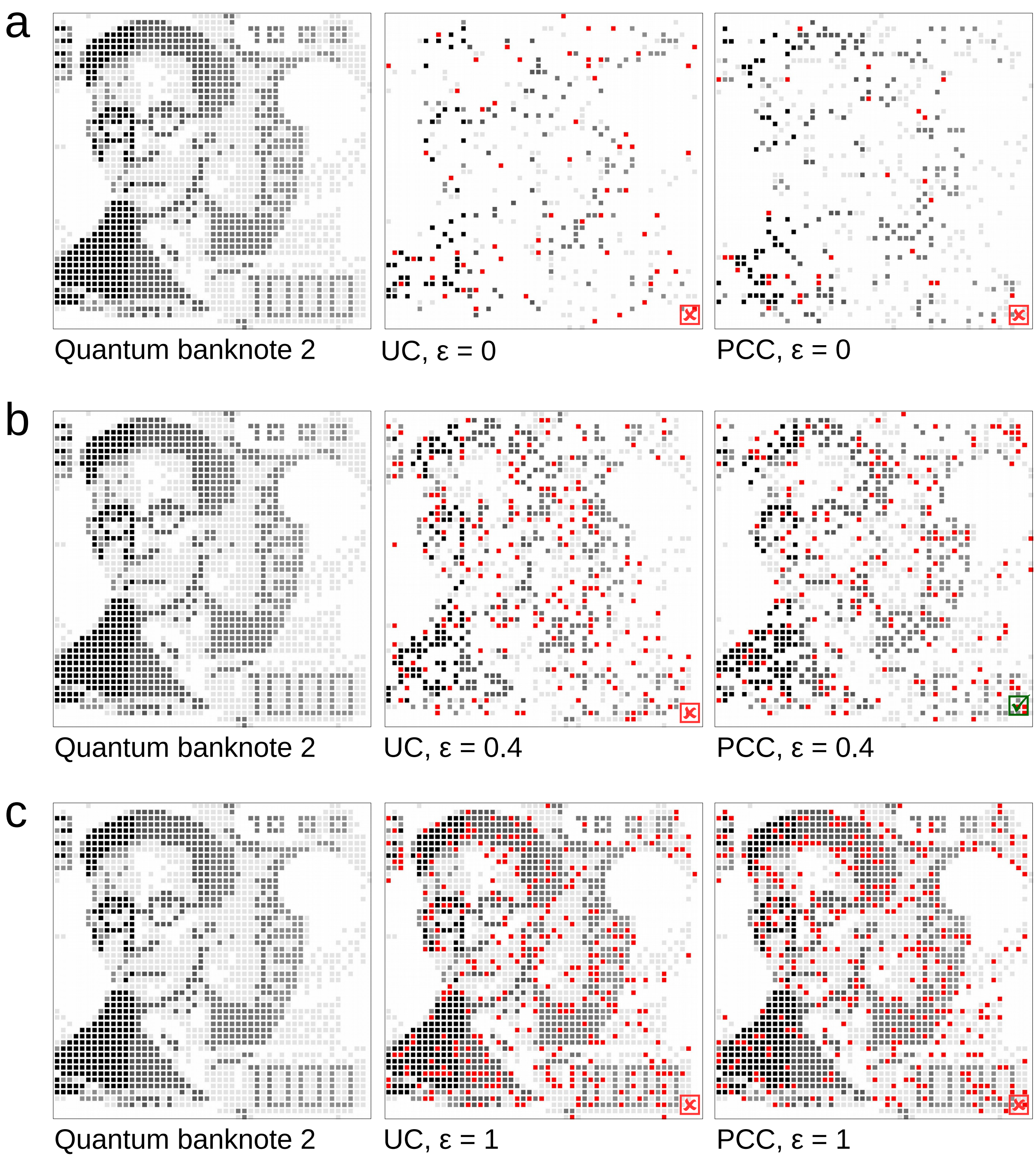

FIG. S4: Same as in Fig. S3 but for quantum banknote 2.