Chapter 5

Quantum Noise and Squeezed States in Josephson Junctions

5.1 Introduction

5.1.1 Motivation

During the past decade, squeezed states of light [11] have attracted much attention in the atomic-molecular-optical physics community. In these states, the quantum fluctuations of one of the canonical variables (e.g., the electric field) can be squeezed below its vacuum-state value at the expense of increasing the quantum fluctuations of its conjugate variable. Examples of canonical conjugate variables are the position $x$ and momentum $p$ of a particle confined in one dimension, the electric field intensity and phase for light, as well as the charge number difference $n$ and phase difference $\phi$ for a superconducting Josephson junction. The use of squeezed states for the exploitation of the limits of the uncertainty principle provides nearly noiseless optical measurements [79], unique opportunities for the study of QED (e.g., enhance the lifetime of atomic excited states), higher signal-to-noise ratio for gravitational wave detectors [69], and the promise of improved optical information transmission [11]. These and many other applications of squeezed states are described in [69], [11], [57], and references therein.

Is it possible to control quantum fluctuations in condensed matter systems, in analogy to these optical situations? We have explored this question in several condensed matter examples [18, 19, 20]. Indeed, quantum fluctuations can produce important measurable effects [15] in these systems even when temperatures are not very low. Moreover, other non-quantum-optics analogs of squeezed states are now being vigorously investigated by several groups [16].

The modulation of the quantum fluctuations in the photon field is made possible by the
very high precision achieved by optical measurements. Although most condensed matter experiments do not produce such high-accuracy results, two exceptions to this rule are the quantum Hall effect and the Josephson effect. For instance, the most precise measurement of energy levels has been achieved not in the traditional context of atomic-optical physics, but with superconducting tunneling Josephson junctions [23], to an astounding accuracy of three parts in $10^{19}$. Furthermore, Josephson junctions are currently being used in an increasingly large number of applications involving high-precision measurements [80]. Indeed, the goal of several single-electron transport devices [81] is to operate in the limiting case where fluctuations in junction charge number $N$ are totally suppressed for all times (i.e., $\Delta N = 0$). For instance, the quantum pump proposed by Qian Niu [81, 82] acts like a "controlled conveyor belt" or "turnstile" of electrons, where electrons are transported by a time-dependent electric field. This remarkable device might provide an accurate standard of current. In Niu's quantum pump, $\Delta N$ is always zero since the time-dependent electric field produces a constant current of electrons. Thus the phase is totally undetermined, and both conjugate variables can be treated as classical quantities. Here, we are interested in studying quantum noise and JJJs are ideal candidates for the possible manipulation of quantum fluctuations.

Here we explore the quantum fluctuation properties of a Josephson junction, and therefore point out a possible way of manipulating the accuracy limitations imposed by quantum noise. In light of the recent advances in micro-fabrication and the general interest in the possible control and minimization of quantum noise, we ask the following question: how do quantum fluctuations affect Cooper pair tunneling in a Josephson junction? To answer this, we need to first compute its quantum fluctuations and understand how they behave in different physical situations involving Josephson junctions. This is indeed the problem we focus on here. We would like to develop a theory of the quantum mechanical squeezed states in Josephson junctions, and identify and list their properties. In particular, we diagonalize several Hamiltonians corresponding to different configurations containing Josephson junctions, find their eigenstates, and calculate the corresponding fluctuations. We also construct the time-evolution operators for the various cases considered here. From them, and with different initial states, we calculate the time evolution of the variances of the conjugate variables of the system. These provide a measure of the quantum fluctuations of the charge and the phase difference of the Josephson junction.
5.1.2 Quantum Noise

There are various kinds of quantum noise. If a pair of quantum variables do not commute with each other, the product of the variances of these two quantities has a lower limit which is determined by the uncertainty principle. This kind of quantum noise, originating from the non-vanishing commutator of the conjugate variables, is an intrinsic property of an isolated system. However, no physical system is truly isolated; no matter how a system is chosen, it always has an environment to interact with. Furthermore, all the environmental degrees of freedom are ultimately governed by quantum mechanics and often involve non-commuting variables. Therefore, each degree of freedom has its own quantum noise due to the uncertainty principle, and this noise is also embodied in the system variables due to the interaction between the system and the environment; this kind of fluctuation can be considered as external to the system.

Another kind of quantum noise is shot noise. It originates from the discreteness of the carriers in the transport phenomena. For example, the tunneling events of quasi-particles in a Josephson junction are mostly uncorrelated and random. Such randomness naturally introduces a noise into the tunneling current, which is shot noise.

In this chapter (and the whole thesis) we only deal with the first kind of quantum noise—the intrinsic one originating from the non-vanishing commutator of the conjugate variables. The shot noise can be disregarded because of the coherence in the supercurrent of a Josephson junction. We also neglect the coupling of the junction to its environment. In addition, we focus our attention on the zero temperature case, so that no quasiparticle is excited, and no thermal noise is involved. We expect that our results will still persist at sufficiently low (but finite) temperatures.

5.1.3 Summary of This Chapter

In this chapter, we study the intrinsic quantum fluctuations of the charge and phase of a Josephson junction in various circumstances. Section 5.1 explains the motivation of this study. In Section 5.2 we consider a Josephson junction in a variety of situations, i.e., coupled to one or several of the following circuit elements: a capacitor, an inductor (in a superconducting ring), and an applied current-source. The Hamiltonian for each case is constructed. We use the small-phase approximation (described in Section 5.2) because we work in the strong coupling limit and treat the metastable states as nearly localized. In Section 5.3 we find the second quantized forms of the Hamiltonians. We then proceed to solve for the
ground and excited states near the potential minima of the various configurations that were
described in Section 5.4. The corresponding ground states are squeezed vacuum or coher­
ent states, while the excited states are a class of squeezed number states. In Section 5.5
we calculate the quantum fluctuations of the phase $\phi$ and Cooper pair number $n$ over the
junction for all the cases considered. We also construct the approximate time evolution
operators for the configurations described in Section 5.6. In Section 5.7 we present another
approach to study this problem: the rotating wave approximation. This is basically also a
first-order perturbation in energy. Section 5.8 presents a discussion and also lists several
open problems in the field. Finally, Section 5.9 summarizes the conclusions for the chapter.

Appendices A, B, F, and G are related to this chapter. Appendices A and B summarize
some properties of squeezed and coherent states. Appendix F presents a brief review of
the quantized LC circuit model, and Appendix G presents derivations of results used in the
main body of the chapter.

5.2 Hamiltonians for System Configurations Containing a
Josephson Junction

5.2.1 General Cases

A Josephson junction [83, 84, 85] is a weak link between two superconductors. A super­
current can tunnel through it without any dissipation. The supercurrent $I$ and the phase
difference $\phi$ between the two superconductors satisfy the following equations

$$I = I_0 \sin \phi,$$
$$\frac{d\phi}{dt} = \frac{2e}{h} V. \quad (5.1)$$

Here $I_0$ is a critical current, below which only the supercurrent exists at $T = 0$; and $V$ is
the voltage drop over the junction when quasiparticle tunneling exists.

Josephson junctions are generally damped systems, especially if quasiparticles are in­
volved. Furthermore, in the Josephson equations the phase difference $\phi$ is treated as a
classical quantity instead of a quantum mechanical operator. However, in the limit of zero
temperature and small current, the system is dissipationless. Therefore, it is possible in
this limit to write an effective Hamiltonian for an ideal junction.

Here we treat the phase difference $\phi$ as an operator, and we also include in our effective
Hamiltonian its conjugate variable, the charge number $n$, in the form of charging energy. We
want to study the quantum noise originating from the non-vanishing commutator between
the phase and the charge number.

We first consider an isolated junction (i.e., decoupled from its environment) near its ground state, at temperature $T = 0$. A capacitance is always present in Josephson junctions, especially in small-area ones (see, e.g., [86, 87], and references therein). A standard model [85, 89] for a simple Josephson junction is an ideal junction in parallel with a capacitor $C$. Its Hamiltonian is,

$$H_1 = \frac{Q^2}{2C} - E_J \cos \phi,$$

where the operators $Q$ and $\phi$ satisfy the commutation relation

$$[\phi, Q] = 2ie. \quad (5.3)$$

Rewriting $Q$ as $Q = e^*n = 2en$ and $E_C = 4e^2/C$, $H_1$ takes the form

$$H_1 = \frac{4e^2 n^2}{2C} - E_J \cos \phi$$

$$= \frac{E_C}{2} n^2 - E_J \cos \phi,$$

where

$$[\phi, n] = i. \quad (5.5)$$

Here we only consider the coherent Cooper pair tunneling and neglect quasi-particle tunneling and shunting Ohmic resistance, which actually originates from the quasiparticle tunneling.

When a Josephson junction is current-biased, the system is no longer closed. However, it can still be described by an effective Hamiltonian, with an additional linear interaction term:

$$H_2 = \frac{E_C}{2} n^2 - E_J \cos \phi + \frac{I \Phi_0}{2\pi} \phi.$$

Here $I$ is the biasing current and $\Phi_0 = h/2e$ is the superconducting flux quantum in the MKS units.

When a Josephson junction is in a superconducting loop with an external flux $\Phi_e = \Phi_0 \phi_e / 2\pi$, the Hamiltonian has the form

$$H_3 = \frac{E_C}{2} n^2 - E_J \cos \phi + \frac{\Phi_0^2}{8\pi^2 L} (\phi - \phi_e)^2,$$

where $L$ is the inductance of the loop. In the special case when the external flux vanishes, the Hamiltonian takes the simplified form

$$H_4 = \frac{E_C}{2} n^2 - E_J \cos \phi + \frac{\Phi_0^2}{8\pi^2 L} \phi^2.$$

These different configurations are summarized in Table 5.1.
Table 5.1: The four different configurations of a Josephson Junction in an LC Circuit considered in Chapters 4 and 5. Here JJ refers to Josephson Junction, SC refers to Super- Conducting, $\Phi$ is the magnetic flux through a superconducting ring, while $\phi$ is the phase difference between the two superconducting leads of a Josephson junction. These two quantities are related by $\Phi = \Phi_0(\phi/2\pi)$, where $\Phi_0$ is the flux quantum $\Phi_0 = h/2e$. It can be proved from the Josephson equations that the superconducting phase difference $\phi$ here is equivalent to the phase on the capacitor, which is defined in Appendix F. Therefore, it is not a coincidence that they have the same relation to the magnetic flux. In $H_2$, $E_S = hI/2e$, where $I$ is the biasing current of the junction.

5.2.2 Form of the Hamiltonians near a Potential Minimum

Motivation for the Small Phase Approximation

In this section, we investigate the intrinsic quantum fluctuation properties of the various Hamiltonians described in the previous section. For this purpose, we focus on the localized or nearly-localized states of a Josephson junction, and neglect its tunneling features. Below we consider each individual physical situation.

In the case of an isolated single junction, the problem is similar to that of an electron in a periodic potential, with $E_J$ corresponding to the strength of the potential and $E_C$ corresponding to the inverse mass of the electron. The eigenstates of such a periodic system are Bloch states. However, when an electron is in a deep potential well, it will be tightly bound to that site most of the time. Therefore, even though an electron in a periodic potential is always in an extended state, for deep enough potential wells we can consider its localized regime in the tight-binding limit. Similarly, for a Josephson junction with $E_J \gg E_C$, we can expand the $\cos \phi$ term at a potential minimum. In doing so, we substitute the periodic potential with a parabolic-type potential in the phase-representation. We thus put our emphasis on the local properties of the phase $\phi$ instead of its transport properties. This kind of approximation has been used in a variety of problems, including spin-density
waves [90] and Josephson junction arrays [91]. Needless to say, phase variations during a tunneling event do affect the local properties of a junction and give rise to larger fluctuations. However, in the limit when \( T = 0 \) and \( E_J \gg E_C \), transport effects are small, so that we can neglect them to a first order approximation.

When there is a biasing current, the system is in the so-called “washboard” potential. When the biasing current is large enough, any localized state is unstable; i.e., the phase of the system will inevitably tunnel from one local potential minimum to another. However, when the biasing current is small \( I \Phi_0 \ll \sqrt{E_J E_C} \), so that the lifetime of a metastable state is long compared to the characteristic time inside the potential well, we can study the properties of the metastable state at the potential minima as if it were a localized state.

When there is a superconducting ring associated with the Josephson junction, the situation is somewhat different. Now the potential is a parabolic one modulated by a sinusoidal term. There is a definite potential minimum, and the quantum fluctuation properties around this global minimum are of interest.

In summary, to study the intrinsic quantum fluctuation properties of a Josephson junction, it is physically justified to expand the interaction potential term around a local (or a global) minimum. The analytical solution of the full problem, without approximations, is quite difficult and would require a much more numerical approach, which is beyond the scope of this work.

**Single Junction: Isolated or Current-Biased**

For a single isolated junction, there is an infinite number of potential minima, which are all equivalent to each other. We expand the potential energy around \( \phi = 0 \):

\[
H_1 = \frac{E_C}{2} n^2 - E_J \left( 1 - \frac{1}{2} \phi^2 + \frac{1}{24} \phi^4 \right). \tag{5.10}
\]

When the junction is current-biased, the sinusoidal potential is tilted and becomes the well-known “washboard” potential.

\[
V_2 = -E_J \cos \phi + \frac{I \Phi_0}{2\pi} \phi. \tag{5.11}
\]

Again, there is an infinite number of equivalent local potential minima, \( \phi_{m2} \); and any one of them satisfies

\[
\left. \frac{\partial V_2}{\partial \phi} \right|_{\phi_{m2}} = \left. \left( E_J \sin \phi + \frac{I \Phi_0}{2\pi} \right) \right|_{\phi_{m2}} = 0, \tag{5.12}
\]

\[
\left. \frac{\partial^2 V_2}{\partial \phi^2} \right|_{\phi_{m2}} = \left. E_J \cos \phi \right|_{\phi_{m2}} > 0. \tag{5.13}
\]
In other words,

\[
\sin \phi_{m2} = -\frac{I\Phi_0}{2\pi E_J},
\]

(5.14)

\[
\cos \phi_{m2} = \sqrt{1 - \left(\frac{I\Phi_0}{2\pi E_J}\right)^2}.
\]

(5.15)

Now we can expand the potential energy \(V_2\) around \(\phi_{m2}\) to investigate properties of the localized states. Writing \(\phi = \phi_{m2} + \phi_{\text{local}}\), the potential is then expanded as

\[
V_2 = -E_J \cos (\phi_{m2} + \phi_{\text{local}}) + \frac{I\Phi_0}{2\pi} (\phi_{m2} + \phi_{\text{local}})
\]

\[
= -E_J (\cos \phi_{m2} \cos \phi_{\text{local}} - \sin \phi_{m2} \sin \phi_{\text{local}}) + \frac{I\Phi_0}{2\pi} (\phi_{m2} + \phi_{\text{local}}).
\]

(5.16)

Since we are concerned with quantum states in the vicinity of a local potential minimum, the variable \(\phi_{\text{local}}\) is a small quantity. Therefore we use the following expansions

\[
\cos \phi_{\text{local}} = 1 - \frac{1}{2} \phi_{\text{local}}^2 + \frac{1}{24} \phi_{\text{local}}^4,
\]

(5.17)

\[
\sin \phi_{\text{local}} = \phi_{\text{local}} - \frac{1}{6} \phi_{\text{local}}^3.
\]

(5.18)

Here we have taken higher-order corrections into account so that the nonlinear feature of the sinusoidal functions can be described better. The potential now takes the form

\[
V_2 = -E_J \left\{ \left(1 - \frac{1}{2} \phi_{\text{local}}^2 + \frac{1}{24} \phi_{\text{local}}^4\right) \cos \phi_{m2} - \left(\phi_{\text{local}} - \frac{1}{6} \phi_{\text{local}}^3\right) \sin \phi_{m2} \right\}
\]

\[
+ \frac{I\Phi_0}{2\pi} (\phi_{m2} + \phi_{\text{local}})
\]

\[
= \left(-E_J \cos \phi_{m2} + \frac{I\Phi_0}{2\pi} \phi_{m2}\right) + \frac{1}{2} (E_J \cos \phi_{m2}) \phi_{\text{local}}^2 - \frac{1}{6} (E_J \sin \phi_{m2}) \phi_{\text{local}}^3
\]

\[
- \frac{1}{24} (E_J \cos \phi_{m2}) \phi_{\text{local}}^4
\]

\[
= \left(-E_J \cos \phi_{m2} + \frac{I\Phi_0}{2\pi} \phi_{m2}\right) + \frac{E_J}{2} \sqrt{1 - \left(\frac{I\Phi_0}{2\pi E_J}\right)^2} \phi_{\text{local}}^2 + \frac{I\Phi_0}{12\pi} \phi_{\text{local}}^3
\]

\[
- \frac{E_J}{24} \sqrt{1 - \left(\frac{I\Phi_0}{2\pi E_J}\right)^2} \phi_{\text{local}}^4.
\]

(5.19)

The constant term (which is a function of \(\phi_{m2}\)) gives the value of the local potential minimum, while the terms related to \(\phi_{\text{local}}\) determine the quantum fluctuation properties of the metastable states at the minimum. It can be seen that these later terms are only related to \(E_J\) and \(I\), while the actual value of \(\phi_{m2}\) is irrelevant.

Notice that the fourth-order term in the Hamiltonian has a negative sign, which at first sight might appear to be a non-stabilizing term. Recall that in the standard Landau
argument for phase transitions, a free energy is expanded in terms of powers of the order parameter, and only the first powers are kept (assuming a small value of the order parameter close to $T_c$). There, the $\phi^4$ term provides stability when the $\phi^2$ term is not stable, i.e., when the coefficient of $\phi^2$ is either zero or negative. In our case, we have a dynamic model instead of a statistical model. The coefficients in our Hamiltonians are all microscopic constants that do not depend on temperature (to first order approximation), and the coefficient of $\phi^2$ (i.e., $E_J$) does not change sign. These are significantly different from the standard Landau phenomenology for phase transitions. Indeed, in the small-$\phi$ approximation, the $\phi^2$ term of the JJ energy expansion in $\phi$ provides stability to the $\phi = 0$ solution.

The essential idea of our approach is as follows. The dominant $\phi^2$ term provides a confining harmonic oscillator potential, which produces time-independent fluctuations in the canonical conjugate coordinates. It is the (smaller) $\phi^4$ term that induces time-dependent modulations on the fluctuations of the conjugate variables. Higher-order terms (e.g., $\phi^6$, and $\phi^8$) also modulate in time the time-independent harmonic fluctuations, but since these higher-order terms are much smaller than the (already small) $\phi^4$ term, we will ignore them here. Notice that even for the large-$\phi$ case when $\phi = 1$, the $\phi^4$ term is 12 times smaller than the $\phi^2$ case (see Eq. (5.10)). Thus, the small-$\phi$ limit we use here is robust for a relatively wide range of values of $\phi$.

**Josephson Junction in a Superconducting Ring**

When a Josephson junction is in a superconducting ring, the situation is different. Now the potential is a parabola modulated by a cosine function in $\phi$-space. Therefore, there generally exists a global potential minimum, where the state is localized. Furthermore, if the system falls into one of the local potential minima, it tends to eventually relax into the global one. Otherwise, these local metastable potential minima should have similar quantum fluctuation properties as in the case of the washboard potential. Now let us take a closer look at the relevant Hamiltonians.

If there is an external flux $\Phi_e = \Phi_0 \phi_e/2\pi$ through the superconducting ring, the Hamiltonian is

$$H_3 = \frac{E_C}{2} n^2 + V_3,$$

$$V_3 = -E_J \cos \phi + \frac{\Phi_0^2}{8\pi^2 L} (\phi - \phi_e)^2.$$

The potential minima are at $\phi_{m3}$ where $\partial V / \partial \phi|_{\phi_{m3}} = 0$ and $\partial^2 V / \partial \phi^2|_{\phi_{m3}} > 0$. In other words, $\phi_{m3}$ satisfies $E_J \sin \phi_{m3} + \Phi_0^2 (\phi_{m3} - \phi_e)/4\pi^2 L = 0$ and $E_J \cos \phi_{m3} + \Phi_0^2/4\pi^2 L > 0$. 

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In the neighborhood of \( \phi_{m3} \), the phase can be expressed as \( \phi = \phi_{m3} + \phi_{\text{local}} \). The potential then has the form

\[
V_3 = -E_J \cos(\phi_{m3} + \phi_{\text{local}}) + \frac{\Phi_0^2}{8\pi^2 L} (\phi_{m3} + \phi_{\text{local}} - \phi_e)^2
\]

\[
= -E_J \left( \cos \phi_{m3} \cos \phi_{\text{local}} - \sin \phi_{m3} \sin \phi_{\text{local}} \right) + \frac{\Phi_0^2}{8\pi^2 L} \left\{ (\phi_{m3} - \phi_e)^2 + 2(\phi_{m3} - \phi_e)\phi_{\text{local}} + \phi_{\text{local}}^2 \right\}.
\]  (5.22)

Now we again expand \( \cos \phi_{\text{local}} \) and \( \sin \phi_{\text{local}} \) as in Eq. (5.14) and (5.15); the potential \( V_3 \) can then be simplified to

\[
V_3 = \left\{ -E_J \cos \phi_{m3} + \frac{\Phi_0^2}{8\pi^2 L} (\phi_{m3} - \phi_e)^2 \right\} + \frac{1}{2} \left( E_J \cos \phi_{m3} + \frac{\Phi_0^2}{4\pi^2 L} \right) \phi_{\text{local}}^2
\]

\[
- \frac{1}{6} (E_J \sin \phi_{m3}) \phi_{\text{local}}^3 - \frac{1}{24} (E_J \cos \phi_{m3}) \phi_{\text{local}}^4,
\]  (5.23)

while

\[
E_J \sin \phi_{m3} = -(\phi_{m3} - \phi_e)\Phi_0^2/4\pi^2 L,
\]  (5.24)

\[
E_J \cos \phi_{m3} > -\Phi_0^2/4\pi^2 L.
\]  (5.25)

Therefore, the Hamiltonian for a Josephson junction in a superconducting ring threaded with an external flux \( \Phi_e \) takes the simplified form

\[
H_3 = \frac{E_C}{2} n^2 + \left\{ -E_J \cos \phi_{m3} + \frac{\Phi_0^2}{8\pi^2 L} (\phi_{m3} - \phi_e)^2 \right\} + \frac{1}{2} \left( E_J \cos \phi_{m3} + \frac{\Phi_0^2}{4\pi^2 L} \right) \phi_{\text{local}}^2
\]

\[
- \frac{1}{6} (E_J \sin \phi_{m3}) \phi_{\text{local}}^3 - \frac{1}{24} (E_J \cos \phi_{m3}) \phi_{\text{local}}^4.
\]  (5.26)

When the external flux through the ring is zero, the form of the Hamiltonian will remain the same, but the equations satisfied by \( \phi_{m4} \) are different:

\[
E_J \sin \phi_{m4} = -\frac{\Phi_0^2}{4\pi^2 L} \phi_{m4},
\]  (5.27)

\[
E_J \cos \phi_{m4} > -\frac{\Phi_0^2}{4\pi^2 L}.
\]  (5.28)

Furthermore, it can be seen from the above equations that \( \phi_{m4} = 0 \) is the global potential energy minimum. The expansion around this point is even simpler than around a metastable point:

\[
H_4 = \frac{E_C}{2} n^2 - E_J + \frac{1}{2} \left( E_J + \frac{\Phi_0^2}{4\pi^2 L} \right) \phi_{\text{local}}^2 - \frac{1}{24} E_J \phi_{\text{local}}^4,
\]  (5.29)

\[
\phi_{m4} = 0.
\]  (5.30)
Notice that the third-order term in $\phi_{\text{local}}$ has vanished. Indeed, from the form of the original potential, it can be seen that the external flux only shifts the position of the minimum of the parabolic potential. If $\Phi_e = 0$, the global minimum of the total potential is at $\phi = 0$. Moreover, the change of $\Phi_e$ does not directly affect the quantum fluctuation properties of the total potential because it only leads to a horizontal shift in the parabolic part of the potential.

5.3 Second Quantized form

5.3.1 Single Isolated Junction

Let us first consider a free oscillator,

$$H_1 = \frac{E_C}{2} n^2 + \frac{E_J}{2} \phi^2 - \frac{E_J}{24} \phi^4 - E_J. \quad (5.31)$$

We introduce a pair of creation and annihilation operators

$$a_1 = \frac{1}{\sqrt{2}} \left\{ \left( \frac{E_J}{E_C} \right)^{1/4} \phi + i \left( \frac{E_C}{E_J} \right)^{1/4} n \right\}, \quad (5.32)$$

$$a_1^\dagger = \frac{1}{\sqrt{2}} \left\{ \left( \frac{E_J}{E_C} \right)^{1/4} \phi - i \left( \frac{E_C}{E_J} \right)^{1/4} n \right\}, \quad (5.33)$$

and also the following definitions

$$\hbar \omega_1 = \sqrt{E_J E_C}, \quad (5.34)$$

$$\lambda_1 = \sqrt{\frac{E_J}{E_C}}, \quad (5.35)$$

so that

$$\phi = \frac{1}{\sqrt{2}} \left( \frac{E_C}{E_J} \right)^{1/4} (a_1 + a_1^\dagger) = \frac{1}{\sqrt{2} \lambda_1} (a_1 + a_1^\dagger), \quad (5.36)$$

$$n = \frac{1}{i \sqrt{2}} \left( \frac{E_J}{E_C} \right)^{1/4} (a_1 - a_1^\dagger) = -i \sqrt{\frac{\lambda_1}{2}} (a_1 - a_1^\dagger). \quad (5.37)$$

Notice that the number states for $a_1$ and $a_1^\dagger$ are not the Cooper-pair-number eigenstates. Instead, they are the eigenstates of the operator $a_1^\dagger a_1$. In the $a$-representation, the charge-number operator $n$ plays a role which is similar to the momentum in a simple harmonic oscillator (see Appendix F). The Hamiltonian for an isolated single junction now takes the form

$$H_1 = \hbar \omega_1 \left( a^\dagger a + \frac{1}{2} \right) - \frac{\hbar \omega_1}{96 \lambda_1} (a + a^\dagger)^4$$

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\[
H_1 = \hbar \omega_1 \left( 1 - \frac{1}{8\lambda_1} \right) a^\dagger a - \frac{\hbar \omega_1}{16\lambda_1} (a^2 + a^\dagger_1^2)
- \frac{\hbar \omega_1}{96\lambda_1} (a^4 + 4a^3 + 6a^2 + 4a^\dagger a^3 + a^4) + \text{constant.} \quad (5.38)
\]

From its definition, \( \omega_1 \) is the plasma frequency of the junction and our quantum here [83]. The lowest-order terms of \( H_1 \) are proportional to \( a^2 \) and \( a^\dagger_1^2 \), and there is no linear term.

### 5.3.2 Current-Biased Junction

In the cases of a current-biased junction and a junction within a superconducting ring, the Hamiltonians are more complicated because of the linear shifts in the potentials produced by the current and the flux.

For a current-biased junction, in the vicinity of any local potential minimum, the Hamiltonian is

\[
H_2 = \frac{E_C}{2} n^2 + \frac{E_J}{2} \sqrt{1 - \left( \frac{I\Phi_0}{2\pi E_J} \right)^2} \phi^2_{\text{local}} + \frac{I\Phi_0}{12\pi} \phi^3_{\text{local}} - \frac{E_J}{24} \sqrt{1 - \left( \frac{I\Phi_0}{2\pi E_J} \right)^2} \phi^4_{\text{local}}, \quad (5.39)
\]

where we have already dropped the constant term, which does not affect the quantum noise.

To take into account the shift in the potential, we redefine the Josephson coupling energy as

\[
E_{J2} = E_J \sqrt{1 - \left( \frac{I\Phi_0}{2\pi E_J} \right)^2}, \quad (5.40)
\]

The eigenfrequency of the system and the junction parameter then have to be redefined as

\[
\omega_2 = \frac{1}{\hbar} \sqrt{E_{J2} E_C}, \quad (5.41)
\]
\[
\lambda_2 = \sqrt{\frac{E_{J2}}{E_C}}, \quad (5.42)
\]

and the creation and annihilation operators become

\[
a_2 = \frac{1}{\sqrt{2}} \left\{ \left( \frac{E_{J2}}{E_C} \right)^{1/4} \phi_{\text{local}} + i \left( \frac{E_C}{E_{J2}} \right)^{1/4} n \right\}, \quad (5.43)
\]
\[
a_2^\dagger = \frac{1}{\sqrt{2}} \left\{ \left( \frac{E_{J2}}{E_C} \right)^{1/4} \phi_{\text{local}} - i \left( \frac{E_C}{E_{J2}} \right)^{1/4} n \right\}, \quad (5.44)
\]

so that

\[
\phi_{\text{local}} = \frac{1}{\sqrt{2}} \left( \frac{E_C}{E_{J2}} \right)^{1/4} (a_2 + a_2^\dagger), \quad (5.45)
\]
\[
n = \frac{1}{i\sqrt{2}} \left( \frac{E_{J2}}{E_C} \right)^{1/4} (a_2 - a_2^\dagger), \quad (5.46)
\]
Now the Hamiltonian $H_2$ takes the form

\[
H_2 = \hbar \omega_2 \left( a_2^\dagger a_2 + \frac{1}{2} \right) + \frac{I \Phi_0}{24 \sqrt{2 \pi}} \left( \frac{E_C}{E_{J2}} \right)^{3/4} \left( a_2^\dagger + a_2 \right)^3 - \frac{E_C}{96} \left( a_2^\dagger + a_2 \right)^4
\]

\[
= \hbar \omega_2 \left( a_2^\dagger a_2 + \frac{1}{2} \right) + \frac{I \Phi_0}{8 \sqrt{2 \pi} \lambda_2} \left( a_2^\dagger + a_2 \right) - \frac{\hbar \omega_2}{32 \lambda_2} \left\{ 8a_2^\dagger a_2 + 1 + 2 \left( a_2^2 + a_2^3 \right) \right\}
\]

\[
+ \frac{I \Phi_0}{24 \sqrt{2 \pi} \lambda_2^3} \left( a_2^4 + 3a_2^2 a_2 + 3a_2^4 a_2 + a_2^3 \right)
\]

\[
- \frac{\hbar \omega_2}{96 \lambda_2} \left( a_2^4 + 4a_2^3 a_2 + 6a_2^2 a_2^2 + 4a_2^3 a_2^2 + a_2^4 \right)
\]

\[
= \hbar \omega_2 \left( 1 - \frac{1}{8 \lambda_2} \right) a_2^\dagger a_2 + \frac{I \Phi_0}{8 \sqrt{2 \pi} \lambda_2^{3/2}} \left( a_2^\dagger + a_2 \right) - \frac{\hbar \omega_2}{16 \lambda_2} \left( a_2^2 + a_2^3 \right)
\]

+ constant + higher order terms.

(5.48)

The difference between $H_1$ and $H_2$ lies in the linear term in $H_2$, which represents a driving force (the effect of the biasing current).

### 5.3.3 Josephson Junction in a Superconducting Ring

When the Josephson junction is in a superconducting ring, the situation is somewhat similar to the current-biased junction. Recall that the Hamiltonian (with an external flux) is

\[
H_3 = \frac{E_C}{2} n^2 + \frac{1}{2} \left( E_J \cos \phi_{m3} + \frac{\Phi_0^2}{4 \pi^2 L} \right) \phi_{\text{local}}^2 - \frac{1}{6} (E_J \sin \phi_{m3}) \phi_{\text{local}}^3
\]

\[- \frac{1}{24} (E_J \cos \phi_{m3}) \phi_{\text{local}}^4 + \text{constant},
\]

(5.49)

where $\phi_{m3}$ satisfies

\[
E_J \sin \phi_{m3} = -\frac{\Phi_0^2}{4 \pi^2 L} (\phi_{m3} - \phi_c).
\]

(5.50)

Consequently, the renormalized Josephson coupling energy and the eigenfrequency of the system are

\[
E_{J3} = E_J \cos \phi_{m3} + \frac{\Phi_0^2}{4 \pi^2 L},
\]

(5.51)

\[
\omega_3 = \frac{1}{\hbar} \sqrt{E_C E_{J3}},
\]

(5.52)

\[
\lambda_3 = \sqrt{\frac{E_{J3}}{E_C}}.
\]

(5.53)

The creation and annihilation operators are now defined as

\[
a_3 = \frac{1}{\sqrt{2}} \left\{ \left( \frac{E_{J3}}{E_C} \right)^{1/4} \phi_{\text{local}} + i \left( \frac{E_C}{E_{J3}} \right)^{1/4} n \right\},
\]

(5.54)

\[
a_3^\dagger = \frac{1}{\sqrt{2}} \left\{ \left( \frac{E_{J3}}{E_C} \right)^{1/4} \phi_{\text{local}} - i \left( \frac{E_C}{E_{J3}} \right)^{1/4} n \right\},
\]

(5.55)
so that $\phi_{\text{local}}$ and $n$ can be expressed as

$$\phi_{\text{local}} = \frac{1}{\sqrt{2}} \left( \frac{E_C}{E_{J3}} \right)^{1/4} (a_3 + a_3^\dagger),$$  \hspace{1cm} (5.56)

$$n = \frac{1}{i\sqrt{2}} \left( \frac{E_{J3}}{E_C} \right)^{1/4} (a_3 - a_3^\dagger).$$  \hspace{1cm} (5.57)

The Hamiltonian $H_3$ is now

$$H_3 = \hbar \omega_3 \left( a_3^\dagger a_3 + \frac{1}{2} \right) - \frac{\hbar \omega_3}{12\sqrt{2}\lambda_3} \frac{E_J \sin \phi_{m3}}{E_{J3}} \left( a_3^\dagger + a_3 \right)^3$$

$$- \frac{\hbar \omega_3}{96\lambda_3} E_J \cos \phi_{m3} \left( \frac{E_{J3}}{E_J} \right)^{1/4} \left( a_3^\dagger + a_3 \right)^4$$

$$= \hbar \omega_3 \left( a_3^\dagger a_3 + \frac{1}{2} \right) - \frac{\hbar \omega_3}{4\sqrt{2}\lambda_3} \frac{E_J \sin \phi_{m3}}{E_{J3}} \left( a_3^\dagger + a_3 \right)^3$$

$$- \frac{\hbar \omega_3}{96\lambda_3} E_J \cos \phi_{m3} \left( \frac{E_{J3}}{E_J} \right)^{1/4} \left( a_3^\dagger + a_3 \right)^4$$

$$= \hbar \omega_3 \left( 1 - \frac{E_J \cos \phi_{m3}}{8\lambda_3 E_{J3}} \right) a_3^\dagger a_3 - \frac{\hbar \omega_3 E_J \sin \phi_{m3}}{4\sqrt{2}\lambda_3 E_{J3}} \left( a_3^\dagger + a_3 \right)^3$$

$$- \frac{\hbar \omega_3 E_J \cos \phi_{m3}}{16\lambda_3 E_{J3}} \left( a_3^\dagger + a_3 \right)^2 + \text{constant + higher order terms}. \hspace{1cm} (5.59)$$

If the external flux through the superconducting ring vanishes, we can still use all the definitions and formulae above, by simply substituting $\phi_e$ by 0. However, if we further limit our focus onto the global minimum, the expressions for the Hamiltonian $H_4$ and the relevant quantities become more compact because in this case $\phi_{m4} = 0$. We thus have

$$E_{J4} = E_J + \frac{\Phi_0^2}{4\pi^2 L},$$  \hspace{1cm} (5.60)

$$\omega_4 = \frac{1}{\hbar} \sqrt{E_C E_{J4}},$$  \hspace{1cm} (5.61)

$$\lambda_4 = \sqrt{\frac{E_{J4}}{E_C}},$$  \hspace{1cm} (5.62)

$$\cos \phi_{m4} = 1,$$  \hspace{1cm} (5.63)

$$a_4 = \frac{1}{\sqrt{2}} \left\{ \left( \frac{E_{J4}}{E_C} \right)^{1/4} \phi_{\text{local}} + i \left( \frac{E_C}{E_{J4}} \right)^{1/4} n \right\},$$  \hspace{1cm} (5.64)

$$\phi_{\text{local}} = \frac{1}{\sqrt{2}} \left( \frac{E_C}{E_{J4}} \right)^{1/4} (a_4 + a_4^\dagger),$$  \hspace{1cm} (5.65)

$$n = \frac{1}{i\sqrt{2}} \left( \frac{E_{J4}}{E_C} \right)^{1/4} (a_4 - a_4^\dagger),$$  \hspace{1cm} (5.66)

$$H_4 = \hbar \omega_4 \left( 1 - \frac{E_J}{8\lambda_4 E_{J4}} \right) a_4^\dagger a_4 - \frac{\hbar \omega_4 E_J}{16\lambda_4 E_{J4}} (a_4^\dagger)^2 + a_4^2) + \text{constant + higher order terms}. \hspace{1cm} (5.67)$$

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The parameters for all the above configurations are summarized in Table 5.2.

5.3.4 Approximate Form of the Hamiltonians

From the Schrödinger equation, the state vector of an arbitrary system (with a time-independent Hamiltonian) can be formally written as

$$|\psi(t)\rangle = e^{iHt/\hbar}|\psi(0)\rangle. \quad (5.69)$$

In general, $|\psi(t)\rangle$ can be expanded in a coherent state basis:

$$|\psi(t)\rangle = \int d^2\alpha\ d^2\beta\ |\alpha\rangle\langle\alpha|e^{iHt/\hbar}\beta\rangle\langle\beta|\psi(0)\rangle. \quad (5.70)$$

In the short-time limit, $e^{i\int Ht/\hbar} \approx 1 + iHt/\hbar$. Furthermore, let us restrict ourselves to the cases when $|\psi(0)\rangle$ is the ground state or a low-energy excited state. When this condition is true, $\langle\beta|\psi(0)\rangle$ should have its largest value for small $|\beta|$. Since $t$ is small, $|\psi(t)\rangle$ should not be too different from $|\psi(0)\rangle$. Therefore, the above integral is significant only when $|\alpha|$ is small. With both $\alpha$ and $\beta$ small, i.e., $|\alpha| \ll 1$ and $|\beta| \ll 1$, $\langle\alpha|\left(a^\dagger a^3 + a^3 a\right)|\beta\rangle \approx \max\{|\alpha|^3, |\beta|^3\} \ll \max\{|\alpha|^2, |\beta|^2\} \approx \langle\alpha|\left(a^\dagger a^2 + a^2 a\right)|\beta\rangle$. In addition, the higher order terms also have very small coefficients. Therefore, as a first-order approximation, we can drop all the terms that are to third or higher-orders in $a$ and $a^\dagger$. Such an approximation greatly simplifies the Hamiltonians in the various cases. For instance, the Hamiltonian for the single isolated junction becomes

$$H_1 = \hbar\omega_1 \left(1 - \frac{1}{8\lambda_1}\right) a_{\uparrow}^\dagger a_{\uparrow} - \frac{\hbar\omega}{16\lambda_1} \left(a_{\uparrow}^2 + a_{\uparrow}^{\dagger 2}\right). \quad (5.71)$$

The Hamiltonian of the current-biased junction takes the form

$$H_2 = \hbar\omega_2 \left(1 - \frac{1}{8\lambda_2}\right) a_{\uparrow}^\dagger a_{\downarrow} + \frac{I\Phi_0}{8\sqrt{2}\pi\lambda_2^{3/2}} \left(a_{\uparrow}^\dagger + a_{\downarrow}\right) - \frac{\hbar\omega_2}{16\lambda_2} \left(a_{\uparrow}^{\dagger 2} + a_{\downarrow}^2\right). \quad (5.72)$$

The Hamiltonian of a Josephson junction in a superconducting ring becomes

$$H_3 = \hbar\omega_3 \left(1 - \frac{E_J\cos\phi_{m3}}{8\lambda_3 E_{J,3}}\right) a_{\uparrow}^\dagger a_3 - \frac{\hbar\omega_3 E_J\sin\phi_{m3}}{4\sqrt{2}\lambda_3 E_{J,3}} \left(a_{\uparrow}^\dagger + a_3\right) - \frac{\hbar\omega_3 E_J\cos\phi_{m3}}{16\lambda_3 E_{J,3}} \left(a_{\uparrow}^{\dagger 2} + a_3^2\right). \quad (5.73)$$

Finally, when there is no external flux through the ring, and we consider only the global potential minimum, the Hamiltonian is

$$H_4 = \hbar\omega_4 \left(1 - \frac{E_J}{8\lambda_4 E_{J,4}}\right) a_{\uparrow}^\dagger a_{\downarrow} - \frac{\hbar\omega_4 E_J}{16\lambda_4 E_{J,4}} \left(a_{\uparrow}^{\dagger 2} + a_{\downarrow}^2\right). \quad (5.74)$$
<table>
<thead>
<tr>
<th></th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effective JJ coupling $E_{J1}$</td>
<td>$E_{J1} = E_J$</td>
<td>$E_{J2} = E_J\sqrt{1 - \left(\frac{I^2 \Phi_0}{2E_J}\right)^2}$</td>
<td>$E_{J3} = E_J \cos \phi_{m3} + \Phi_0^2/4\pi^2 L$</td>
<td>$E_{J4} = E_J + \Phi_0^2/4\pi^2 L$</td>
</tr>
<tr>
<td>Energy quantum $\hbar \omega_i$</td>
<td>$\hbar \omega_1 = \sqrt{E_J E_C}$</td>
<td>$\hbar \omega_2 = \sqrt{E_J E_C}$</td>
<td>$\hbar \omega_3 = \sqrt{E_J E_C}$</td>
<td>$\hbar \omega_4 = \sqrt{E_J E_C}$</td>
</tr>
<tr>
<td>JJ parameter $\lambda_i$</td>
<td>$\lambda_1 = \sqrt{E_J / E_C}$</td>
<td>$\lambda_2 = \sqrt{E_J / E_C}$</td>
<td>$\lambda_3 = \sqrt{E_J / E_C}$</td>
<td>$\lambda_4 = \sqrt{E_J / E_C}$</td>
</tr>
<tr>
<td>Nonlinearity parameter $\delta_i$</td>
<td>$\delta_1 = 1/16 \lambda_1$</td>
<td>$\delta_2 = 1/16 \lambda_2$</td>
<td>$\delta_3 = E_J \cos \phi_{m3} / 16 \lambda_3 E_J$</td>
<td>$\delta_4 = \sqrt{E_J^3 E_C E_{J3}^3 / 16}$</td>
</tr>
<tr>
<td>Renormalized frequency $\omega$</td>
<td>$\omega_1(1 - 2\delta_1)$</td>
<td>$\omega_2(1 - 2\delta_2)$</td>
<td>$\omega_3(1 - 2\delta_3)$</td>
<td>$\omega_4(1 - 2\delta_4)$</td>
</tr>
<tr>
<td>Linear coefficient $\alpha_i$</td>
<td>0</td>
<td>$\alpha_2 = I \Phi_0 / 8 \sqrt{2\pi \lambda_2^3}$</td>
<td>$\alpha_3 = -\hbar \omega_3 E_J \sin \phi_{m3} / 4 \sqrt{2 \lambda_3 E_J}$</td>
<td>0</td>
</tr>
<tr>
<td>Displacement factor $d_i$</td>
<td>0</td>
<td>$-\alpha_2 (1 + 2\delta_2) / \hbar \omega_2$</td>
<td>$-\alpha_3 (1 + 2\delta_3) / \hbar \omega_3$</td>
<td>0</td>
</tr>
<tr>
<td>Quadratic coefficient $\beta_i$</td>
<td>$-\delta_1$</td>
<td>$-\delta_2$</td>
<td>$-\delta_3$</td>
<td>$-\delta_4$</td>
</tr>
<tr>
<td>$</td>
<td>\text{Squeezing factor}</td>
<td>= s_i$</td>
<td>$\delta_1$</td>
<td>$\delta_2$</td>
</tr>
</tbody>
</table>

Table 5.2: The parameters for the different Josephson junction configurations considered in Chapters 4 and 5. In the table, $E_J$ is the Josephson coupling energy for an ideal junction; $E_C$ is the charging energy associated with the relevant junction; $\Phi_0 = h/2e$ is the superconducting magnetic flux quantum; $\phi_{m3}$ is the phase of a local potential minimum for case 3; $L$ is the inductance of the superconducting ring; and finally, $I$ is the biasing current for case 2.
The four simplified Hamiltonians belong to two groups. The first group (denoted by $H_A$) includes $H_1$ and $H_4$, with a free oscillator term and two second-order perturbation terms. The second group (denoted by $H_B$) includes $H_2$ and $H_3$, which have two linear terms in addition to the free oscillator term and the second-order perturbation terms. Below we will find that these linear terms lead to higher-energy ground states for the relevant systems.

Why the four diverse situations we originally considered can be simplified into only two categories? The reason lies in the special form of the Josephson coupling energy. Although each one of these Hamiltonians has its particular character, they all share the charging and Josephson coupling energy terms. When $E_J$ is the dominant energy scale in the system, the Josephson coupling is the governing factor in all of these systems, which leads to similar properties. More specifically, the linear and quadratic energy terms are more of global importance. If we focus on a local potential minimum, the special features should mostly be determined by the sinusoidal function.

5.4 Ground and Excited States of Various Hamiltonians

In this section we focus on the ground states of the various systems we are interested in.

5.4.1 Ground and Excited States of the Hamiltonians $H_1$ and $H_4$

The Hamiltonian $H_1$ for an isolated junction and $H_4$ for a junction in a superconducting ring (at zero flux) at the global potential minimum have the same form $H_A$:

$$H_A = \hbar \omega a^\dagger a + \beta a^2 + \beta^* a^\dagger a^2. \quad (5.75)$$

This Hamiltonian can be easily diagonalized by introducing operators $b$ and $b^\dagger$:

$$b = \mu a + \nu a^\dagger, \quad (5.76)$$
$$b^\dagger = \mu^* a^\dagger + \nu^* a. \quad (5.77)$$

From these expressions we obtain

$$b^\dagger b = (|\mu|^2 + |\nu|^2)a^\dagger a + \mu \nu^* a^2 + \mu^* \nu a^\dagger a^2 + |\nu|^2. \quad (5.78)$$

Therefore, if we require that

$$\beta = \frac{\hbar \omega \mu \nu^*}{|\mu|^2 + |\nu|^2}, \quad (5.79)$$

the Hamiltonian $H_A$ is diagonalized to

$$H_A = \frac{\hbar \omega}{|\mu|^2 + |\nu|^2} b^\dagger b - \frac{\hbar \omega |\nu|^2}{|\mu|^2 + |\nu|^2}. \quad (5.80)$$
To satisfy the boson commutator \([b, b^\dagger] = 1\), \(\mu\) and \(\nu\) must satisfy the equality

\[ |\mu|^2 - |\nu|^2 = 1. \quad (5.81) \]

From this equation and Eq. (5.79), we can solve for the transformation coefficients \(\mu\) and \(\nu\) up to a global phase factor:

\[
|\mu|^2 = \frac{1}{2\sqrt{1 - 4\delta^2}} + \frac{1}{2} \\
\simeq 1 + \delta^2, \quad (5.82)
\]

\[
|\nu|^2 = \frac{1}{2\sqrt{1 - 4\delta^2}} - \frac{1}{2} \\
\simeq \delta^2, \quad (5.83)
\]

\[
|\mu|^2 + |\nu|^2 = \frac{1}{\sqrt{1 - 4\delta^2}} \\
\simeq 1 + 2\delta^2, \quad (5.84)
\]

\[
\phi_\mu - \phi_\nu = \phi_\beta + 2m\pi, \quad (5.85)
\]

\[
\delta = \frac{|\beta|}{\hbar\omega}. \quad (5.86)
\]

Here \(m\) is an arbitrary integer, and \(\delta\) is a small quantity because the perturbation terms are much smaller than the free oscillator term. Now the Hamiltonian takes on the simpler form

\[
H_A = \hbar\omega \sqrt{1 - 4\delta^2} b^\dagger b - \frac{1}{2}\hbar\omega \left(1 - \sqrt{1 - 4\delta^2}\right) \\
\simeq \hbar\omega(1 - 2\delta^2) b^\dagger b - \hbar\omega\delta^2. \quad (5.87)
\]

The ground state of this Hamiltonian satisfies

\[ b|0\>_b = 0. \quad (5.88) \]

A squeezing operator \(S(\xi)\) [11, 58] can be introduced to represent the transformation between \(b\) and \(a\):

\[
S(\xi) = \exp \left(\frac{\xi a^2 - \xi^* a^\dagger a}{2}\right), \quad (5.89)
\]

where \(\xi = se^{i\theta}\) is a complex squeezing factor. Its physical meaning is described in [58] and also in Figs. 5.1 and 5.2. Briefly, the squeezing operator acting on the vacuum or a coherent state periodically reduces or "squeezes" the uncertainty of one of the conjugate coordinates (e.g., \(n\) and \(\phi\), or \(x\) and \(p\)) below its minimum uncertainty or coherent-state value. It can be easily shown that

\[
S^{-1}(\xi) a S(\xi) = a \cosh s - a^\dagger e^{i\theta} \sinh s. \quad (5.90)
\]
Therefore, if we let

\[ \mu = \cosh s, \]  
\[ \nu = -e^{\theta} \sinh s, \]  

then

\[ b = S^{-1}(\xi) a S(\xi). \]  

Together with Eqs. (5.82) and (5.83), we can express the squeezing factor \( \xi \) in terms of \( \delta \) and \( \phi_\beta \):

\[ s = \arccosh \mu, \]  
\[ \theta = -\phi_\beta - \pi. \]  

Recall that the ground state of the system satisfies \( b|0\rangle_b = 0 \), thus

\[ S^{-1}(\xi) a S(\xi)|0\rangle_b = 0, \]  

which leads to

\[ a \{S(\xi)|0\rangle_b\} = 0. \]  

In other words, \( S(\xi)|0\rangle_b \) is the vacuum state for \( a \):

\[ S(\xi)|0\rangle_b = |0\rangle_a, \]  

or \( |0\rangle_b = S^{-1}(\xi)|0\rangle_a \). Namely, the ground state of the Hamiltonian \( H_A \) is

\[ |\text{ground}\rangle_A = S^{-1}(\xi)|0\rangle_a = S(-\xi)|0\rangle_a, \]  

where \( |0\rangle_a \) is the vacuum state in the \( a \)-representation. In other words, the ground state of \( H_A \) is a squeezed vacuum state with a squeezing factor of \( -\xi \). Intuitively, the dominant \( \phi^2 \) term in the original \( H_1 \) and \( H_4 \) provides a confining harmonic oscillator potential (represented by the \( a^\dagger a \) term in \( H_A \)), which produces the minimum fluctuations allowed by the uncertainty principle in the canonical conjugate coordinates. It is the (smaller) \( \phi^4 \) term in the original \( H_1 \) and \( H_4 \) (which gives rise to the \( a^2 \) and \( a^{12} \) terms) that induces modulations on the fluctuations of the conjugate variables, therefore leading to the squeezing effect.

The excited states of the Hamiltonian \( H_A \) are number states in the \( b \)-representation:

\[ |\text{excited}\rangle_A = |n\rangle_b = \frac{b^\dagger^n}{\sqrt{n!}} |0\rangle_b. \]  

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Figure 5.1: Schematic diagram of the uncertainty areas in the phase difference and charge number ($\phi, n$) phase space of: (a) the vacuum state, (b) a "number" state, (c) a coherent state, and (d) a squeezed state. In these diagrams, the phases are measured in units of \((E_C/E_J)^{1/4}\), while the charge numbers in units of \((E_J/E_C)^{1/4}\). In addition, the "number" state here is not an eigenstate of the charge number operator \(n\). It is actually an eigenstate of the harmonic part of the Hamiltonians. Notice that the coherent state has the same uncertainty area as the vacuum state, and that both areas are circular, while the squeezed state has an elliptical uncertainty area. Therefore, in the direction parallel to the \(\theta/2\) line, the squeezed state has a smaller noise than both the vacuum and coherent states.
Figure 5.2: Schematic diagram of the time evolution of both the expectation value $\langle \phi \rangle(t)$ and the fluctuation $\langle (\Delta \phi)^2 \rangle(t)$ of the phase operator $\phi$ for different states: vacuum (a), number (b), coherent (c), as well as squeezed (d) and (e). Here dashed lines represent $\langle \phi \rangle(t)$, while solid lines represent the envelopes $\langle \phi \rangle \pm \sqrt{\langle (\Delta \phi)^2 \rangle}$, which provide the upper and lower bounds for the fluctuating quantity $\phi(t)$.

(a) For the vacuum state $|0\rangle$, $\langle \phi \rangle(t) = 0$ and $\langle (\Delta \phi)^2 \rangle(t) = 2\sqrt{E_C/E_J}$.
(b) For a number state $|n\rangle$, $\langle \phi \rangle(t) = 0$ and $\langle (\Delta \phi)^2 \rangle(t) = 2n+1$.

(c) For a coherent state $|\alpha\rangle$, $\langle \phi \rangle(t) = 2\text{Re}(\alpha e^{-i\omega t}) = 2|\alpha|\cos \omega t$, which means that $\alpha$ is real, and $\langle (\Delta \phi)^2 \rangle(t) = 2$.

(d) For a squeezed state $|\alpha e^{-i\omega t}\rangle$, $\xi(t)$, where the squeezing factor $\xi(t)$ satisfies $\xi(t) = re^{-2i\omega t}$, $\langle \phi \rangle(t) = 2|\alpha|\cos \omega t$, which means that $\alpha$ is real, and its fluctuation is $\langle (\Delta \phi)^2 \rangle(t) = 2(e^{-2r}\cos^2 \omega t + e^{2r}\sin^2 \omega t)$.

(e) A squeezed state, as in (d). Now the expectation value of $\phi$ is $\langle \phi \rangle(t) = 2|\alpha|\sin \omega t$, which means that $\alpha$ is purely imaginary, and the fluctuation $\langle (\Delta \phi)^2 \rangle$ has the same time-dependence as in (d). Notice that the squeezing effect now appears at the times when $\langle \phi \rangle(t)$ reaches its maxima while in (d) the squeezing effect is present at the times when $\langle \phi \rangle(t)$ is close to zero.
Since $b^\dagger$ is related to $a^\dagger$ by $b^\dagger = S^{-1}(\xi)a^\dagger S(\xi)$, the excited states of $H_A$ can be simplified as

$$|\text{excited}\rangle_A = \frac{1}{\sqrt{n!}} \left( S^{-1}(\xi)a^\dagger S(\xi) \right)^n S^{-1}(\xi) |0\rangle_a = \frac{1}{\sqrt{n!}} S^{-1}(\xi) a^\dagger |n\rangle_a = S(-\xi)|n\rangle_a.$$  (5.101)

In other words, the excited states of $H_A$ are not the number states in the $a$-representation, $|n\rangle_a$, nor the number states $|n\rangle_n$ in the charge-number representation, where the Cooper-pair number is a good quantum number. Instead, $|\text{excited}\rangle_A$ is a "squeezed" version of the number states $|n\rangle_a$ (see Section 5.5.3 for more details).

For the isolated-junction Hamiltonian $H_1$, the parameters in $H_A$ should be substituted by

$$\omega = \omega_1 \left( 1 - \frac{1}{8\lambda_1} \right),$$  \hspace{1cm} (5.102)

$$\beta = -\frac{\hbar \omega_1}{16\lambda_1},$$  \hspace{1cm} (5.103)

$$\delta_1 = \frac{|\beta|}{\hbar \omega_1} = \frac{1}{16\lambda_1 - 2}.$$  \hspace{1cm} (5.104)

In the large-ratio limit, $\lambda_1 = \sqrt{E_J/E_C} \gg 1$, between the Josephson coupling energy and the charging energy, the quantity $\delta_1 \cong 1/16\lambda_1 \ll 1$ is very small. Therefore,

$$|\mu|^2 \cong 1 + \delta_1^2 \cong 1 + \left( \frac{1}{16\lambda_1} \right)^2,$$  \hspace{1cm} (5.105)

$$|\nu|^2 \cong \delta_1^2 \cong \left( \frac{1}{16\lambda_1} \right)^2.$$  \hspace{1cm} (5.106)

Notice that the phase of $\beta$ is $\pi$, so that we can set $\phi_\mu = 0$ and $\phi_\nu = \pi$. Furthermore, the squeezing factor is

$$\xi_1 = s_1 \cong \delta_1 = \frac{1}{16\lambda_1} = \frac{1}{16} \sqrt{\frac{E_C}{E_J}},$$  \hspace{1cm} (5.107)

with the phase angle of $\xi_1$ satisfying $\theta_1 = 0$.

For $H_4$, which is the Hamiltonian of a Josephson junction at the global potential minimum in a superconducting loop without external flux, the parameters are

$$\omega = \omega_4 \left( 1 - \frac{E_J}{8\lambda_4 E_{J4}} \right),$$  \hspace{1cm} (5.108)

$$\beta = -\frac{\hbar \omega_4 E_J}{16\lambda_4 E_{J4}}.$$  \hspace{1cm} (5.109)
Thus, now we have
\[ \delta_4 \cong \frac{E_J E_C^{1/2}}{16 E_J^3} = \frac{1}{16} \sqrt{\frac{E_C^3}{E_J^3}}, \]
and the squeezing factor becomes
\[ \xi_4 = \delta_4 \cong \frac{1}{16} \sqrt{\frac{E_C^3}{E_J^3}}, \]
with the phase angle of $\xi_4$ satisfying $\theta_4 = 0$.

5.4.2 Ground and Excited States of the Hamiltonians $H_2$ and $H_3$

Recall that the Hamiltonian $H_2$ describes a current-biased Josephson junction and $H_3$ a Josephson junction in a superconducting ring with external flux. Both $H_2$ and $H_3$ have the similar form $H_B$:
\[ H_B = \hbar \omega a^\dagger a + \alpha a + \alpha^* a^\dagger + \beta a^2 + \beta^* a^\dagger^2. \] (5.112)

Systems represented by $H_2$ are diagonalizable, as in the case of $H_A$. Again, we can introduce a pair of operators $b$ and $b^\dagger$:
\begin{align*}
    b &= \mu a + \nu a^\dagger, \\
    b^\dagger &= \mu^* a^\dagger + \nu^* a.
\end{align*} (5.113) (5.114)

Namely, $b = S^{-1}(-\xi) a S(-\xi)$, where $S(-\xi) = \exp(-\xi^* a^2 / 2 + \xi a^\dagger / 2)$ is a squeezing operator and $-\xi = \exp(i\theta + \pi)$ is the squeezing factor; $\xi$ is related to $\mu$ and $\nu$ by $\mu = \cosh s$ and $\nu = -\exp(i\theta) \sinh s$. If we define
\[ \delta = \frac{|\beta|}{\hbar \omega}, \]
(5.115)
$\mu$ and $\nu$ then satisfy
\begin{align*}
    |\mu|^2 &\cong 1 + \delta^2, \\
    |\nu|^2 &\cong \delta^2, \\
    \phi_\mu - \phi_\nu &= \phi_\beta.
\end{align*} (5.116) (5.117) (5.118)

The Hamiltonian can thus be simplified to
\begin{align*}
    H_B &= \frac{\hbar \omega}{|\mu|^2 + |\nu|^2} b^\dagger b - \frac{\hbar \omega |\nu|^2}{|\mu|^2 + |\nu|^2} + (\alpha^* - \alpha^* \nu^*) b + (\alpha^* \mu - \alpha \nu) b^\dagger \\
    &\cong \hbar \omega (1 - 2\delta^2) b^\dagger b - \hbar \omega b^2 + \gamma b + \gamma^* b^\dagger,
\end{align*} (5.119)
where
\[ \gamma = \alpha^* - \alpha^* \nu^*. \] (5.120)
Now we introduce the operator $c$ which diagonalizes $H_B$

$$c = b + \frac{\gamma^*}{\hbar \omega}(1 + 2\delta^2), \quad (5.121)$$

so that

$$c^\dagger c = b^\dagger b + \frac{\gamma^*}{\hbar \omega}(1 + 2\delta^2)b + \frac{\gamma^*}{\hbar \omega}(1 + 2\delta^2)b^\dagger + \left\{ \frac{|\gamma|}{\hbar \omega}(1 + 2\delta^2) \right\}^2, \quad (5.122)$$

and

$$H_B = \hbar \omega(1 - 2\delta^2)c^\dagger c - \hbar \omega \delta^2 - \frac{|\gamma|^2}{\hbar \omega}(1 + 2\delta^2). \quad (5.123)$$

The ground state of this Hamiltonian satisfies

$$c|0\rangle_c = 0. \quad (5.124)$$

If we define a displacement operator $D$ as

$$D_b(\eta) = \exp \left( \eta b^\dagger - \eta^* b \right), \quad (5.125)$$

which "displaces" $b$ by $\eta$

$$D_b^{-1}(\eta) b D_b(\eta) = b + \eta, \quad (5.126)$$

we can express the operator $c$ as

$$c = D_b^{-1} \left( \frac{\gamma^*}{\hbar \omega^\prime} \right) D_b \left( \frac{\gamma^*}{\hbar \omega^\prime} \right) = D_b^{-1} \left( \frac{\gamma^*}{\hbar \omega^\prime} \right) \hat{S}_a(-\xi) D_b \left( \frac{\gamma^*}{\hbar \omega^\prime} \right), \quad (5.127)$$

where

$$\omega^\prime = \omega(1 - 2\delta^2). \quad (5.128)$$

Thus

$$D_b^{-1}(\gamma^*/\hbar \omega^\prime) b D_b(\gamma^*/\hbar \omega^\prime) |0\rangle_c = 0, \quad (5.129)$$

or

$$b \{ D_b(\gamma^*/\hbar \omega^\prime) |0\rangle_c \} = 0. \quad (5.130)$$

Therefore,

$$D_b(\gamma^*/\hbar \omega^\prime) |0\rangle_c = |0\rangle_b. \quad (5.131)$$

In other words,

$$|0\rangle_c = D_b^{-1} \left( \frac{\gamma^*}{\hbar \omega^\prime} \right) |0\rangle_b. \quad (5.132)$$
According to the discussion in the previous section

$$|0\rangle_b = S_a^{-1}(\xi)|0\rangle_a. \quad (5.133)$$

Therefore, the ground state of $H_B$ can be expressed as

$$|\text{ground}\rangle_B = |0\rangle_c = D_b^{-1} \left( \frac{\gamma^*}{\hbar \omega'} \right) S_a^{-1}(\xi)|0\rangle_a. \quad (5.134)$$

Recall that $D^{-1}(\eta) = D(-\eta)$ and $S^{-1}(\xi) = S(-\xi)$. Thus the ground state of $H_B$ is

$$|0\rangle_c = D_b \left( -\frac{\gamma^*}{\hbar \omega'} \right) S_a(-\xi)|0\rangle_a$$

$$= D_a \left( -\frac{1 + 2\delta^2}{\hbar \omega} \left| a^* |\mu|^2 + |\nu|^2 - 2a^* |\nu| \right| \right) S_a(-\xi)|0\rangle_a$$

$$\approx D_a \left( -\frac{a^* - 2a\delta e^{i(\theta - \delta)}}{\hbar \omega} \right) S_a(-\xi)|0\rangle_a, \quad (5.135)$$

which is a squeezed coherent state in the "a" representation, with $\xi = \delta$.

The excited eigenstates of $H_B$ can be obtained in a similar manner as for $H_A$. They are the number states in the $c$-representation

$$|\text{excited}\rangle_B = |n\rangle_c \quad (5.136)$$

$$= \frac{(c^*)^n}{\sqrt{n!}} |0\rangle_c$$

$$= \frac{1}{\sqrt{n!}} \left[ D_b^{-1} \left( \frac{\gamma^*}{\hbar \omega'} \right) S_a^{-1}(\xi) a^\dagger S_a(\xi) D_b \left( \frac{\gamma^*}{\hbar \omega'} \right) \right]^n D_b^{-1} \left( \frac{\gamma^*}{\hbar \omega'} \right) S_a^{-1}(\xi)|0\rangle_a$$

$$= D_b^{-1} \left( \frac{\gamma^*}{\hbar \omega'} \right) S_a^{-1}(\xi) a^\dagger n |0\rangle_a$$

$$= D_b^{-1} \left( \frac{\gamma^*}{\hbar \omega'} \right) S_a^{-1}(\xi)|n\rangle_a$$

$$= D_b \left( -\frac{\gamma^*}{\hbar \omega'} \right) S_a(-\xi)|n\rangle_a. \quad (5.137)$$

Therefore, the excited states of $H_B$ are displaced and squeezed number states in the $a$-representation.

In the case of $H_2$, which describes a current-biased Josephson junction, the parameters are

$$\omega = \omega_2 \left(1 - \frac{1}{8\lambda_2}\right), \quad (5.138)$$

$$\alpha = \frac{I\Phi_0}{8\sqrt{2\pi} \lambda_2^{3/2}}, \quad (5.139)$$

$$\beta = -\frac{\hbar \omega_2}{16\lambda_2}, \quad (5.140)$$

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Consequently, \( a^* = a, \phi_\beta = \pi, \) and \( \phi_\nu = \pi. \) If we define \( \delta_2 = |\beta|/\hbar \omega = 1/(16\lambda_2 - 2) \cong 1/16\lambda_2, \) which is a very small quantity, the squeezing factor \( \xi_2 \) has the form

\[
\xi_2 = \delta_2 = \frac{1}{16\lambda_2},
\]

and the displacement \( d \) is

\[
d_2 = \frac{-\alpha^* - 2\alpha \delta e^{i(\phi_\nu - \phi_\mu)}}{\hbar \omega} = \frac{-\alpha_2}{\hbar \omega_2} (1 + 4\delta_2).
\]

(5.142)

In the case of \( H_3, \) which describes a Josephson junction in a superconducting ring with an external flux, the parameters are

\[
\omega = \omega_3 \left(1 - \frac{E_J}{8\lambda_3 E_{J3}} \cos \phi_{m3}\right),
\]

(5.143)

\[
\alpha = -\frac{\hbar \omega_3 E_J}{4\sqrt{2\lambda_3} E_{J3}} \sin \phi_{m3},
\]

(5.144)

\[
\beta = \frac{-\hbar \omega_3 E_J}{16\lambda_3 E_{J3}} \cos \phi_{m3},
\]

(5.145)

\[
\delta_3 = \frac{|\beta|}{\hbar \omega_3} \cong \frac{E_J \cos \phi_{m3}}{16\lambda_3 E_{J3}}.
\]

(5.146)

Here \( \delta_3 \) is again a very small quantity. The squeezing factor is now

\[
\xi_3 = \delta_3 = \frac{E_J \cos \phi_{m3}}{16\lambda_3 E_{J3}},
\]

(5.147)

and the displacement \( d \) now becomes

\[
d_3 = \frac{-\alpha^* - 2\alpha \delta e^{i(\phi_\nu - \phi_\mu)}}{\hbar \omega} = \frac{E_J \sin \phi_{m3}}{4\sqrt{2\lambda_3} E_{J3}} (1 + 4\delta_3) = d_3.
\]

(5.148)

5.5 Expectation Values and Fluctuations of the Number and Phase Operators

In the previous section we have obtained approximate ground state wavefunctions by neglecting the higher-order terms in the JJ interaction. Now we compute the quantum fluctuations for the phase and charge number in these approximate ground and excited states.
5.5.1 Squeezed Vacuum State

Let us first consider $H_1$ and $H_4$, for which the ground state can be written in the form $|\text{ground}\rangle = S(-\xi)|0\rangle$, which is a squeezed vacuum state. Recall that

\begin{align*}
S^{-1}(\xi) a S(\xi) &= a \cosh s - a^\dagger e^{i\theta} \sinh s, \\
S^{-1}(-\xi) a S(-\xi) &= a \cosh s + a^\dagger e^{i\theta} \sinh s,
\end{align*}

where $\xi = se^{i\theta}$. In a squeezed vacuum state $|0,-\xi\rangle = S(-\xi)|0\rangle$, we can calculate the fluctuation

\begin{align*}
\langle [\Delta (a + a^\dagger)]^2 \rangle &= \langle (a + a^\dagger)^2 \rangle - \langle (a + a^\dagger) \rangle^2 \\
&= \langle 0| S^{-1}(-\xi)(a^{\dagger 2} + a^2 + 2a^\dagger a + 1)S(-\xi)|0\rangle \\
&= e^{i\theta} \sinh s \cosh s + e^{-i\theta} \sinh s \cosh s + 1 + 2 \sinh^2 s \\
&= e^{-2s} \sin^2 \frac{\theta}{2} + e^{2s} \cos^2 \frac{\theta}{2}. 
\end{align*}

On the other hand,

\begin{align*}
\langle [\Delta (a - a^\dagger)]^2 \rangle &= \langle (a - a^\dagger)^2 \rangle - \langle (a - a^\dagger) \rangle^2 \\
&= \langle 0| S^{-1}(\xi)(a^{\dagger 2} + a^2 - 2a^\dagger a - 1)S(\xi)|0\rangle \\
&= e^{i\theta} \sinh s \cosh s + e^{-i\theta} \sinh s \cosh s - 1 - 2 \sinh^2 s \\
&= -e^{-2s} \cos^2 \frac{\theta}{2} - e^{2s} \sin^2 \frac{\theta}{2}. 
\end{align*}

For an isolated single junction, the phase and number operators are related to $a$ and $a^\dagger$ by

\begin{align*}
\phi &= \frac{1}{\sqrt{2}} \left( \frac{E_C}{E_J} \right)^{1/4} (a + a^\dagger) = \frac{1}{\sqrt{2\lambda}} (a + a^\dagger), \\
n &= \frac{1}{i\sqrt{2}} \left( \frac{E_J}{E_C} \right)^{1/4} (a - a^\dagger) = -i \sqrt{\frac{\lambda}{2}} (a - a^\dagger).
\end{align*}

Therefore, the ground state fluctuations for an isolated single junction are

\begin{align*}
\langle [\Delta \phi_{\text{local}}]^2 \rangle &= \frac{1}{2\lambda_1} \langle [\Delta (a + a^\dagger)]^2 \rangle \\
&= \frac{1}{2\lambda_1} \left( e^{-2s_1} \cos^2 \frac{\theta_1}{2} + e^{2s_1} \sin^2 \frac{\theta_1}{2} \right) \\
&= \sqrt{\frac{E_C}{4E_J}} e^{2s_1}, \\
\langle (\Delta n)^2 \rangle &= -\frac{\lambda_1}{2} \langle [\Delta (a - a^\dagger)]^2 \rangle
\end{align*}
\begin{align*}
\left( \frac{\lambda_1}{2} \left( e^{-2s_1} \sin^2 \frac{\theta_1}{2} + e^{2s_1} \cos^2 \frac{\theta_1}{2} \right) \right) \\
= \sqrt{\frac{E_J}{4E_C}} \ e^{-2s_1},
\end{align*}
\quad (5.158)

where

\begin{align*}
\lambda_1 &= \sqrt{\frac{E_J}{E_C}}, \\
\theta_1 &= 0, \\
s_1 &= \frac{1}{16\lambda_1}.
\end{align*}
\quad (5.159, 5.160, 5.161)

For a Josephson junction in a superconducting ring, with no external flux and at the
global potential minimum, the local phase and number operators are

\begin{align*}
\phi_{\text{local}} &= \left( \frac{E_C}{4E_J} \right)^{1/4} (a + a^\dagger) \\
n &= -i \left( \frac{E_J}{E_C} \right)^{1/4} (a - a^\dagger).
\end{align*}
\quad (5.162)

The Hamiltonian of this system is \( H_4 \), which has the same ground state as \( H_1 \) for the
isolated single junction case. Thus the ground state fluctuations are

\begin{align*}
\langle \Delta \phi_{\text{local}} \rangle^2 &= \frac{1}{2\lambda_4} \langle [\Delta (a + a^\dagger)]^2 \rangle \\
&= \sqrt{\frac{E_C}{4E_J}} \left( e^{-2s_1} \cos^2 \frac{\theta_1}{2} + e^{2s_1} \sin^2 \frac{\theta_1}{2} \right) \\
&= \sqrt{\frac{E_C}{4E_J}} \ e^{2s_1},
\end{align*}
\quad (5.163)

\begin{align*}
\langle \Delta n \rangle^2 &= \frac{\lambda_4}{2} \langle [\Delta (a - a^\dagger)]^2 \rangle \\
&= \sqrt{\frac{E_J}{4E_C}} \left( e^{-2s_1} \sin^2 \frac{\theta_1}{2} + e^{2s_1} \cos^2 \frac{\theta_1}{2} \right) \\
&= \sqrt{\frac{E_J}{4E_C}} \ e^{-2s_1},
\end{align*}
\quad (5.164)

where

\begin{align*}
E_{J4} &= E_J + \frac{\Phi_0^2}{4\pi^2L}, \\
\theta_4 &= 0, \\
s_4 &= \frac{1}{16} \sqrt{\frac{E_J^3}{E_{J4}}}.
\end{align*}
\quad (5.165, 5.166, 5.167)

Figure 5.3(a) schematically illustrates the uncertainty area of a squeezed vacuum state.
Both the average values of charge number \( n \) and phase difference \( \phi \) vanish. The uncertainty
area in the \( n \cdot \phi \) phase space is squeezed along the \( n \)-direction, which indicates smaller fluctuations for the charge number \( n \) compared to the zeroth-order approximation where the \( \cos \phi \) term is expanded to the second-order in \( \phi \).

### 5.5.2 Squeezed Coherent State

In the case of the current-biased junction, the ground state takes the form of \( |\text{ground}\rangle_B \)

\[
|\text{ground}\rangle_2 = D_a \left( -\frac{\alpha_2(1 + 2\delta_2)}{\hbar \omega_2} \right) S_a(-\xi_2)|0\rangle_a , \quad (5.168)
\]

which is a squeezed coherent state. We can calculate the expectations and fluctuations:

\[
\langle (a_2 + a_2^\dagger) \rangle = -\frac{2\alpha_2(1 + 2\delta_2)}{\hbar \omega_2} , \quad (5.169)
\]

\[
\langle [\Delta(a_2 + a_2^\dagger)]^2 \rangle = e^{-2s_2} \sin^2 \frac{\theta_2}{2} + e^{2s_2} \cos^2 \frac{\theta_2}{2} , \quad (5.170)
\]

\[
\langle (a_2 - a_2^\dagger) \rangle = 0 , \quad (5.171)
\]

\[
\langle [\Delta(a_2 - a_2^\dagger)]^2 \rangle = -e^{-2s_2} \cos^2 \frac{\theta_2}{2} - e^{2s_2} \sin^2 \frac{\theta_2}{2} . \quad (5.172)
\]

Here \(-\xi_2 = s_2 e^{i(\theta_2 + \pi)}\) is the squeezing factor.

The local phase and number operators now have the following properties

\[
\langle \phi_{\text{local}} \rangle = \frac{1}{\sqrt{2}} \left( \frac{E_C}{E_{J2}} \right)^{1/4} \langle (a + a^\dagger) \rangle
\]

\[
= -\frac{A\alpha_2}{E_{J2}} \frac{\alpha_2 - \alpha_2^*}{\hbar \omega_2} (1 + 2\delta_2) , \quad (5.173)
\]

\[
\langle (\Delta \phi_{\text{local}})^2 \rangle = \frac{1}{2\lambda_2} \langle [\Delta (a + a^\dagger)]^2 \rangle
\]

\[
= \sqrt{\frac{E_C}{4E_{J2}}} \left( e^{-2s_2} \sin^2 \frac{\theta_2}{2} + e^{2s_2} \cos^2 \frac{\theta_2}{2} \right)
\]

\[
= \sqrt{\frac{E_C}{4E_{J2}}} \left( 1 - (I\Phi_0/2\pi E_J)^2 \right) e^{2s_2} , \quad (5.174)
\]

\[
\langle n \rangle = \frac{1}{i\sqrt{2}} \left( \frac{E_{J2}}{E_C} \right)^{1/4} \langle (a - a^\dagger) \rangle
\]

\[
= -\frac{1}{i\sqrt{2}} \left( \frac{E_{J2}}{E_C} \right)^{1/4} \frac{\alpha_2 - \alpha_2^*}{\hbar \omega_2} = 0 , \quad (5.175)
\]

\[
\langle (\Delta n)^2 \rangle = -\frac{\lambda_2}{2} \langle [\Delta (a - a^\dagger)]^2 \rangle
\]

\[
= \sqrt{\frac{E_{J2}}{4E_C}} \left( e^{-2s_2} \cos^2 \frac{\theta_2}{2} + e^{2s_2} \sin^2 \frac{\theta_2}{2} \right)
\]

\[
= \sqrt{\frac{E_C}{4E_{J2}}} \left( 1 - (I\Phi_0/2\pi E_J)^2 \right) e^{2s_2} , \quad (5.176)
\]

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Figure 5.3: Schematic diagram of the uncertainty area in the phase difference and charge number ($\phi$, $n$) phase space of (a) the ground states of $H_1$ and $H_4$ (which are squeezed vacuum states), and (b) the ground state of $H_2$ and $H_3$ (which are squeezed coherent states). The units for $\phi$ and $n$ are $(E_C/E_J)^{1/4}$ and $(E_J/E_C)^{1/4}$, respectively. It can be seen from the diagram (a) that the uncertainty of the charge number $n$ is squeezed by a factor of $1/16\lambda$, where $\lambda = \sqrt{E_J/E_C}$. In (b) the uncertainty area has been displaced away from the origin by an amount $d$, because linear driving terms are present in both $H_2$ and $H_3$. Again, the uncertainty of the charge number $n$ is squeezed by a factor $s = 1/16\lambda$, with $\lambda = \sqrt{E_J/E_C}$. The displacement $d$ from the origin is $-\sqrt{2}\alpha(1+2\delta)/\hbar\omega$, where $\alpha$ is the coefficient of the linear driving term, and $\delta$ reflects the relative strength of the nonlinear correction terms.
\[ E_{J2} = E_J \sqrt{1 - \left( \frac{I \Phi_0}{2 \pi E_J} \right)^2}, \quad (5.177) \]
\[ \hbar \omega_2 = \sqrt{E_{J2} E_C}, \quad (5.178) \]
\[ \lambda_2 = \sqrt{\frac{E_{J2}}{E_C}}, \quad (5.179) \]
\[ \alpha_2 = \frac{I \Phi_0}{8 \sqrt{2} \pi \lambda_2^{3/2}}, \quad (5.180) \]
\[ \theta_2 = 0, \quad (5.181) \]
\[ s_2 = \frac{1}{16 \lambda_2^2}. \quad (5.182) \]

In the case of Josephson junction in a superconducting ring with external flux, the ground state is similar to that of the current-biased junction. The parameters are different, so that the expectation values and fluctuations of the local phase and number operators become

\[ \langle \phi_{\text{local}} \rangle = \frac{1}{\sqrt{2}} \left( \frac{E_C}{E_{J3}} \right)^{1/4} \langle a_3 + a_3^\dagger \rangle \]
\[ = - \left( \frac{A E_C}{E_{J3}} \right)^{1/4} \frac{\alpha_3}{\hbar \omega_3} (1 + 2 \delta_3), \quad (5.183) \]

\[ \langle (\Delta \phi_{\text{local}})^2 \rangle = \frac{1}{2} \left( \frac{E_C}{E_{J3}} \right)^{1/4} \left( \left[ \Delta (a_3 + a_3^\dagger) \right]^2 \right) \]
\[ = \frac{1}{2} \left( \frac{E_C}{E_{J3}} \right)^{1/4} \left( e^{-2s_3} \sin^2 \frac{\theta_3}{2} + e^{2s_3} \cos^2 \frac{\theta_3}{2} \right) \]
\[ = \frac{1}{2} \left( \frac{E_C}{E_{J3}} \right)^{1/4} e^{2s_3}, \quad (5.184) \]

\[ \langle n \rangle = \frac{1}{i \sqrt{2}} \left( \frac{E_{J3}}{E_C} \right)^{1/4} \langle a_3 - a_3^\dagger \rangle \]
\[ = - \frac{1}{i \sqrt{2}} \left( \frac{E_{J3}}{E_C} \right)^{1/4} \frac{(1 + 2 \delta_3)(\alpha_3^* - \alpha_3)}{\hbar \omega_3} \]
\[ = 0, \quad (5.185) \]

\[ \langle (\Delta n)^2 \rangle = - \frac{1}{2} \sqrt{\frac{E_{J3}}{E_C}} \langle (\Delta (a_3 - a_3^\dagger))^2 \rangle \]
\[ = \frac{1}{2} \left( \frac{E_{J3}}{E_C} \right)^{1/4} \left( e^{-2s_3} \cos^2 \frac{\theta_3}{2} + e^{2s_3} \sin^2 \frac{\theta_3}{2} \right) \]
\[ = \frac{1}{2} \left( \frac{E_{J3}}{E_C} \right)^{1/4} e^{2s_3}, \quad (5.186) \]

where \( L \) is the inductance of the superconducting ring, \( \phi_{m3} \) is the phase of the local mini-
Figure 5.3(b) schematically illustrates the uncertainty area of a squeezed coherent state. The finite \( \langle \phi \rangle \) in this state represents the asymmetry in the potential energy. The uncertainty area has the same shape and orientation as that of the squeezed vacuum state considered in the last section, because these two cases share the same nonlinear potential of \( \cos \phi \).

### 5.5.3 The Excited States

Recall that the excited states for \( H_A \) are

\[
|n, \xi\rangle = S(-\xi)|n\rangle_a.
\]

It can be shown that

\[
S^{-1}(-\xi)aS(\xi) = a \cosh s + a^\dagger e^{i\theta} \sinh s,
\]

\[
S^{-1}(-\xi)a^\dagger S(\xi) = a^\dagger \cosh s + ae^{-i\theta} \sinh s,
\]

where \( \xi = se^{i\theta} \). Therefore, the matrix elements can be calculated as

\[
\langle a + a^\dagger \rangle = 0,
\]

\[
\langle a + a^\dagger \rangle^2 = |n, \xi\rangle (a + a^\dagger)^2 |n, \xi\rangle
\]

\[
= |n\rangle S^{-1}(-\xi) (a + a^\dagger)^2 S(\xi) |n\rangle
\]

\[
= |n\rangle \left\{ a \left( \cosh s + e^{-i\theta} \sinh s \right) + a^\dagger \left( \cosh s + e^{i\theta} \sinh s \right) \right\}^2 |n\rangle
\]

\[
= (2n + 1) \left( e^{-2s} \sin^2 \frac{\theta}{2} + e^{2s} \cos^2 \frac{\theta}{2} \right),
\]

\[
\langle a - a^\dagger \rangle = |n, \xi\rangle (a - a^\dagger)^2 |n, \xi\rangle
\]

\[
= -(2n + 1) \left( e^{-2s} \cos^2 \frac{\theta}{2} + e^{2s} \sin^2 \frac{\theta}{2} \right).
\]
We can then compute the fluctuations of the charge-number and phase operators:

\[
\langle (\Delta \phi)^2 \rangle = \frac{\lambda}{2} \langle (a + a^\dagger)^2 \rangle \\
= \frac{\lambda}{2} (2n + 1) \left( e^{-2s \sin^2 \frac{\theta}{2}} + e^{2s \cos^2 \frac{\theta}{2}} \right),
\]

\[
\langle (\Delta n)^2 \rangle = -\frac{\lambda}{2} \langle (a - a^\dagger)^2 \rangle \\
= \frac{\lambda}{2} (2n + 1) \left( e^{-2s \cos^2 \frac{\theta}{2}} + e^{2s \sin^2 \frac{\theta}{2}} \right).
\]

On the other hand, in a pure number state (again, not the charge-number eigenstates) in the \(a\)-representation, the fluctuations of the number and phase are

\[
\langle (\Delta \phi)^2 \rangle = \frac{\lambda}{2} \langle n |(a + a^\dagger)^2| n \rangle \\
= \frac{\lambda}{2} (2n + 1),
\]

\[
\langle (\Delta n)^2 \rangle = -\frac{\lambda}{2} \langle n |(a - a^\dagger)^2| n \rangle \\
= \frac{\lambda}{2} (2n + 1).
\]

Therefore, in the excited states of \(H_A\), the fluctuations in the charge number and phase are modulated because of the correction in the harmonic oscillator potential. Whether the noises will be larger or smaller depends on the phase angle \(\theta\) of the squeezing operator \(\xi\).

In the excited states of \(H_B\), the expectation values such as \(\langle (a + a^\dagger) \rangle\) do not vanish because the states are displaced. For example,

\[
\langle n_c | (a + a^\dagger) | n_c \rangle = \langle n_a | S^{-1}_a (-\xi) D_a^{-1} (-\zeta) (a + a^\dagger) D_a (-\zeta) S(-\xi)| n_a \rangle \\
= \langle n_a | S^{-1}_a (-\xi) (a + a^\dagger - \zeta - \zeta^*) S(-\xi)| n_a \rangle \\
= - (\zeta + \zeta^*).
\]

On the other hand, the fluctuations of \(a + a^\dagger\) in the excited states of \(H_B\) take the same form as that in the excited states of \(H_A\):

\[
\langle n_c | (a + a^\dagger)^2 | n_c \rangle = \langle n_a | S^{-1}_a (-\xi) D_a^{-1} (-\zeta) (a + a^\dagger)^2 D_a (-\zeta) S(-\xi)| n_a \rangle \\
= \langle n_a | S^{-1}_a (-\xi) (a + a^\dagger - \zeta - \zeta^*)^2 S_a (-\xi)| n_a \rangle \\
= \langle n_a | S^{-1}_a (-\xi) (a + a^\dagger)^2 S_a (-\xi)| n_a \rangle + (\zeta + \zeta^*)^2,
\]

so that

\[
\langle n_c | [\Delta (a + a^\dagger)]^2 | n_c \rangle = \langle n_a | S^{-1}_a (-\xi) (a + a^\dagger)^2 S_a (-\xi)| n_a \rangle \\
= (2n + 1) \left( e^{-2s \sin^2 \frac{\theta}{2}} + e^{2s \cos^2 \frac{\theta}{2}} \right),
\]
where $-\xi = se^{i(\theta+\pi)}$ is the squeezing factor. It is thus evident that the displacement operation in the excited states of $H_B$ is irrelevant to the fluctuation properties.

5.6 Time Evolution Operators of the Various Hamiltonians

In the previous sections we have obtained the eigenstates of a Josephson junction in various configurations. With the help of these eigenstates we can expand any initial state in terms of these eigenstates and study its time-evolution. However, this approach can be quite complicated for an arbitrary initial state. An easier way is to first find the time-evolution operator $U(t, t_0)$ of the particular Hamiltonian, and then calculate the transformation $U(t, t_0)f(a, a^\dagger)U(t, t_0)$.

5.6.1 Time Evolution Operator of $H_A$

The Hamiltonian $H_A$ encompasses both $H_1$ (for an isolated Josephson junction) and $H_A$ (for a Josephson junction in a superconducting ring without external flux and at the global potential minimum):

\[ H_A = \hbar \omega a^\dagger a + \beta a^2 + \beta^* a^{\dagger 2} \]

\[ = H_0 + V_A, \quad (5.206) \]

where

\[ H_0 = \hbar \omega a^\dagger a, \quad (5.207) \]
\[ V_A = \beta a^2 + \beta^* a^{\dagger 2}. \quad (5.208) \]

To obtain a factorized time-evolution operator, so that each term has a special meaning (such as the displacement and squeezing operators), the general practice is to first change to the interaction picture to eliminate the free oscillator term in the Hamiltonian, and then solve the Schrödinger equation for $U_I(t, t_0)$ in the interaction picture. The free-oscillator time-evolution operator $U_0(t)$ is

\[ U_0(t) = e^{-iH_0 t/\hbar} = e^{-i\omega a^\dagger a}. \quad (5.209) \]

In the interaction picture, the Schrödinger equation becomes

\[ i\hbar \frac{\partial}{\partial t} U_I(t) = V_I(t) U_I(t), \quad (5.210) \]
\[ |\psi(t)\rangle_I = U_I(t)|\psi(0)\rangle_I, \quad (5.211) \]
\[ |\psi(t)\rangle_S = U_0(t) |\psi(0)\rangle_t, \]  
\[ V_I(t) = U_0^{-1}(t) V_A U_0(0) \]
\[ = e^{i\omega t a^\dagger a} \beta a^2 e^{-i\omega t a^\dagger a} + h.c., \]  
(5.212)  
(5.213)

where \( U_I(t) \) is the time-evolution operator in the interaction picture, and the subscript \( S \) refers to the Schrödinger picture. Recall that for a single mode

\[ a e^{\beta t a^\dagger a} = a \sum_{n=0}^{\infty} \frac{\beta^n}{n!} (a^\dagger a)^n = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} (a^\dagger a + 1)^n a \]
\[ = e^{\beta (a^\dagger a + 1)} a = e^{\beta a^\dagger a} a e^{\beta}, \]  
(5.214)

\( V_I \) can thus be simplified to

\[ V_I(t) = \beta a^2 e^{-2i\omega t} + \beta^* a^\dagger a e^{2i\omega t}. \]  
(5.215)

Formally, the time-evolution operator can be written as

\[ U_I(t) = T \exp \left\{-\frac{i}{\hbar} \int_{t_0}^{t} V_I(\tau) d\tau \right\}, \]  
(5.216)

where \( T \) is the time-ordering operator. This expression cannot be easily simplified because \( V_I(t) \) is time-dependent; thus, in general, \( V_I(t) \)'s at different times do not commute with each other. One way to circumvent this problem is to divide the time interval \([t_0, t]\) into a large amount of small intervals, so that in each of the intervals \( V_I(t) \) is approximately a constant. Thus the Schrödinger equation can be integrated in each of the intervals and the time-evolution operator during that period can be obtained by direct integration. Since the time-evolution operator \( U(t_0, t) \) satisfies \( U(t_0, t) = U(t_0, t_1) U(t_1, t) \), we can therefore obtain

\[ U_I(t_0, t) = U_I(t_0, t_1) U_I(t_1, t_2) \cdots U_I(t_n, t). \]

Such an approach may be useful when a numerical computation is the sole purpose; although the efficiency of such an algorithm might not be very high. Analytically, this approach does not simplify the problem.

To obtain an analytical expression for any function of the creation and annihilation operators, we can first transform into the \( b \)-representation, in which \( H_A \) is diagonalized: \( H_A = \hbar \omega_b b^\dagger b \). Here we have dropped a constant. Thus the time-evolution operator of the system in the \( b \)-representation is simply

\[ U_b(t_0, t) = e^{-i\omega_b t^b b^\dagger b} = R_b(\theta), \]  
(5.217)

where \( \theta = \omega_b t \). Therefore, the expectation value of a polynomial \( f(a, a^\dagger) \) of \( a \) and \( a^\dagger \) is

\[ \langle \Phi(t)|f(a, a^\dagger)|\Phi(t)\rangle = \langle \Phi(t_0)|U^\dagger(t_0, t) f(a, a^\dagger) U(t_0, t)|\Phi(t_0)\rangle \]
\[ = \langle \Phi(t_0)|e^{i\omega bt^b b^\dagger} f(a, a^\dagger) e^{-i\omega bt^b b^\dagger} |\Phi(t_0)\rangle \]
\[ = \langle \Phi(t_0)|R_b^\dagger(\theta) f(a, a^\dagger) R_b(\theta)|\Phi(t_0)\rangle. \]  
(5.218)  
(5.219)  
(5.220)
Let us now consider how to simplify the expression in between the state vectors. Recall that
\[ a = S_b^\dagger(s, \phi) b S_b(s, \phi) \]  
\[ a^\dagger = S_b^\dagger(s, \phi) b^\dagger S_b(s, \phi), \]  
where
\[ S_b(s, \phi) = \exp \left\{ \frac{s}{2} \left( e^{-2i\phi} b^2 - e^{2i\phi} b^\dagger)^2 \right) \right\} \]
is a single-mode squeezing operator in the \( b \)-representation, \( 0 \leq s < \infty \) and \( -\pi/2 \leq \phi \leq \pi/2 \). We can thus express the function \( f \) in terms of \( b \) and \( b^\dagger \). More specifically,
\[ f(a, a^\dagger) = f \left( S_b^\dagger(s, \phi) b S_b(s, \phi), S_b^\dagger(s, \phi) b^\dagger S_b(s, \phi) \right) \]
\[ = S_b^\dagger(s, \phi) f(b, b^\dagger) S_b(s, \phi), \]  
since \( S_b^\dagger(s, \phi) S_b(s, \phi) = 1 \). Now we have
\[ U^\dagger(t_0, t) f(a, a^\dagger) U(t_0, t) = R_b^\dagger(\theta) S_b^\dagger(s, \phi) R_b(\theta) f(b, b^\dagger) S_b(s, \phi) R_b(\theta) \]
\[ = R_b^\dagger(\theta) S_b^\dagger(s, \phi) R_b(\theta) R_b^\dagger(\theta) f(b, b^\dagger) R_b(\theta) R_b^\dagger(\theta) S_b(s, \phi) R_b(\theta) \]
\[ = \left\{ R_b^\dagger(\theta) S_b^\dagger(s, \phi) R_b(\theta) \right\} \left\{ R_b^\dagger(\theta) f(b, b^\dagger) R_b(\theta) \right\} \]
\[ \times \left\{ R_b^\dagger(\theta) S_b(s, \phi) R_b(\theta) \right\}. \]  
(5.225)

Since operators \( S \) and \( R \) satisfy [92]
\[ R_b^\dagger(\theta) S_b(s, \phi) R_b(\theta) = S_b(s, \phi + \theta) \]
\[ R_b^\dagger(\theta) b R_b(\theta) = b e^{-i\theta}, \]  
(5.226)
(5.227)
so that
\[ R_b^\dagger(\theta) f(b, b^\dagger) R_b(\theta) = f(R_b^\dagger(\theta) b R_b(\theta), R_b^\dagger(\theta) b^\dagger R_b(\theta)) \]
\[ = f(b e^{-i\theta}, b^\dagger e^{i\theta}), \]  
(5.228)
(5.229)
we can therefore simplify the original expression Eq. (5.225) as
\[ U^\dagger(t_0, t) f(a, a^\dagger) U(t_0, t) = \left\{ R_b^\dagger(\theta) S_b^\dagger(s, \phi) R_b(\theta) \right\} \left\{ R_b^\dagger(\theta) f(b, b^\dagger) R_b(\theta) \right\} \]
\[ \times \left\{ R_b^\dagger(\theta) S_b(s, \phi) R_b(\theta) \right\} \]
\[ = S_b^\dagger(s, \phi + \theta) f(be^{-i\theta}, b^\dagger e^{i\theta}) S_b(s, \phi + \theta) \]
\[ = f \left[ S_b^\dagger(s, \phi + \theta) b e^{-i\theta} S_b(s, \phi + \theta), \right. \]
\[ S_b^\dagger(s, \phi + \theta) b^\dagger e^{i\theta} S_b(s, \phi + \theta) \]  
. \]  
(5.230)
Since the parameters of the squeezing operator $S$ have been changed ($\phi$ becomes $\phi + \theta$), now $S^\dagger(s, \phi + \theta) b S_b(s, \phi + \theta)$ is not equal to $a$ anymore. However, it can still be expressed as a linear combination of $a$ and $a^\dagger$:

\[
S^\dagger_b(s, \phi + \theta) b S_b(s, \phi + \theta) = b \cosh s - b^\dagger e^{2i(\phi + \theta)} \sinh s \\
= \left(a \cosh s - a^\dagger e^{2i\phi} \sinh s \right) \cosh s \\
+ \left(a^\dagger \cosh s - ae^{-2i\phi} \sinh s \right) e^{2i(\phi + \theta)} \sinh s \\
= a \cosh^2 r - a^\dagger e^{2i\phi} \sinh s \cosh s \\
+ a^\dagger e^{2i(\phi + \theta)} \sinh s \cosh s - ae^{2i\theta} \sinh^2 r \\
= a \left( \cosh^2 r - e^{2i\theta} \sinh^2 s \right) \\
+ a^\dagger e^{2i\phi} \sinh s \cosh s \left(1 - e^{2i\theta}\right) . \quad (5.231)
\]

In particular, if $\theta = 0$,

\[
a \left( \cosh^2 s - e^{2i\theta} \sinh^2 s \right) + a^\dagger e^{2i\phi} \sinh s \cosh s \left(1 - e^{2i\theta}\right) = a . \quad (5.232)
\]

Therefore, $U^1(t_0, t)f(a, a^\dagger)U(t_0, t)$ now becomes a polynomial of $a$ and $a^\dagger$ and is easier to handle. Taking $H_1$ as an example, we obtain

\[
s = \frac{1}{16\lambda_1} \quad (5.233)
\]

\[
\phi = 0 \quad (5.234)
\]

\[
\theta = \omega_b t \cong \omega_1 t \left(1 - \frac{1}{16\lambda_1}\right) \left(1 - 2\delta^2\right) \\
\cong \omega_1 t \left(1 - \frac{1}{16\lambda_1}\right) . \quad (5.235)
\]

Here we have only kept terms up to first order in $\delta = 1/16\lambda_1$. Substituting these parameters into expressions [5.230] and [5.231], the expectation value of function $f$ at any time $t$ can then be calculated with an arbitrary initial state.

### 5.6.2 Time Evolution Operator of $H_B$

The time-evolution operator for $H_B$, which describes both $H_2$ (for a current-biased junction) and $H_3$ (for a Josephson junction in a superconducting ring driven by external magnetic flux), can be obtained similarly. We first separate the Hamiltonian $H_B$ into a free oscillator part $H_0$ and a correction $V_B$,

\[
H_B = \hbar \omega a^\dagger a + aa^* + \alpha a^\dagger + \beta a^2 + \beta^* a^\dagger^2 \\
= H_0 + V_B , \quad (5.236)
\]
where

\[ H_0 = \hbar \omega a^\dagger a, \]  
\[ V_B = \alpha a + \alpha^* a^\dagger + \beta a^2 + \beta^* a^\dagger^2. \]  

(5.237)  
(5.238)

Again, if we transform into the interaction picture, we will only be able to obtain a time-evolution operator that is either a formal time-ordered expression, or a product of time-evolution operators at each small time increment. To achieve a more analytical expression, we will now follow the approach we have taken for \( H_A \). In other words, we will transform into a representation in which \( H_B \) is diagonalized (that is, the \( c \)-representation we discussed in Section 5.4), obtain the time-evolution operator in that representation, do the calculations there, and then transform back to the \( a \)-representation. This procedure is presented below.

In the \( c \)-representation, \( H_B \) is diagonalized \( H_B = \hbar \omega_c c^\dagger c \); here we have dropped a constant. The time-evolution operator of the system in the \( c \)-representation is then simply

\[ U_c(t_0, t) = e^{-i\omega_c t c^\dagger c} = R_c(\theta), \]  
(5.239)

where \( \theta = \omega_c t \).

The transformation from the \( a \)-representation to the \( c \)-representation is given by

\[ b = S_b^\dagger(s, \phi) a S_a(s, \phi) \]  
(5.240)

\[ c = D_b^\dagger(\alpha) b D_b(\alpha). \]  
(5.241)

The inverse transformations are

\[ a = S_b(s, \phi) b S_b^\dagger(s, \phi) \]  
(5.242)

\[ b = D_c^\dagger(-\alpha) c D_c(-\alpha). \]  
(5.243)

Therefore,

\[ a = S_b(s, \phi) D_c^\dagger(-\alpha) c D_c(-\alpha) S_b^\dagger(s, \phi). \]  
(5.244)

From \( b = c - \alpha \), we can express \( S_b(s, \phi) \) in terms of \( c \)

\[
S_b(s, \phi) = \exp \left\{ \frac{s}{2} \left( e^{-2i\phi} b^2 - e^{2i\phi} b^\dagger^2 \right) \right\} 
= \exp \left\{ \frac{s}{2} \left[ e^{-2i\phi} (c^2 - \alpha^2) - e^{2i\phi} (c^\dagger^2 - \alpha^* c^2) \right] \right\} 
= \exp \left\{ \frac{s}{2} \left[ e^{-2i\phi} c^2 - 2\alpha c + \alpha^2 \right] - e^{2i\phi} (c^\dagger^2 - 2\alpha^* c^\dagger + \alpha^*\alpha) \right\} 
= \exp \left\{ \frac{s}{2} \left[ e^{-2i\phi} c^2 - e^{2i\phi} c^\dagger^2 \right] + s \left[ e^{2i\phi} \alpha^* c^\dagger - e^{-2i\phi} \alpha c \right] + \frac{s}{2} \left[ e^{-2i\phi} \alpha^2 - e^{2i\phi} \alpha^* \alpha \right] \right\}. \]  
(5.245)
Using the result we derived in Appendix B.4, this operator can be factorized:

\[
S_b(s, \phi) = \exp \left\{ \frac{\alpha}{2} \left[ e^{-2i\phi} c^2 - e^{2i\phi} c^\dagger c^\dagger \right] + \gamma \left[ e^{2i\phi} \alpha^* c^\dagger - e^{-2i\phi} \alpha c \right] + \frac{\alpha^2}{2} \left[ e^{-2i\phi} \alpha^2 - e^{2i\phi} \alpha^* \right] \right\}
\]

\[= D_c(\beta) S_c(s, \phi) e^\gamma, \tag{5.246} \]

where \(\beta\) and \(\gamma\) are complex numbers determined from \(\alpha\), \(r\), and \(\phi\). In particular, \(\gamma\) is pure imaginary. Therefore,

\[
a = e^\gamma D_c(\beta) S_c(s, \phi) D_c^\dagger(-\alpha) c D_c(-\alpha) S_c^\dagger(s, \phi) D_c^\dagger(\beta) e^{-\gamma}
\]

\[= D_c(\beta) S_c(s, \phi) D_c^\dagger(-\alpha) c D_c(-\alpha) S_c^\dagger(s, \phi) D_c^\dagger(\beta). \tag{5.247} \]

To calculate the expectation value of \(f(a, a^\dagger)\), we have

\[\langle \Phi(t) | f(a, a^\dagger) | \Phi(t) \rangle = \langle \Phi(0) | U^\dagger(t_0, t) f(a, a^\dagger) U(t_0, t) | \Phi(0) \rangle \]

\[= \langle \Phi(0) | R_b^\dagger(\theta) f(a, a^\dagger) R_b(\theta) | \Phi(0) \rangle \]

\[= \langle \Phi(0) | R_b^\dagger(\theta) f \left[ D_c(\beta) S_c(s, \phi) D_c^\dagger(-\alpha) c D_c(-\alpha) S_c^\dagger(s, \phi), \right. \]

\[\left. D_c(\beta) S_c(s, \phi) D_c^\dagger(-\alpha) c D_c(-\alpha) S_c^\dagger(s, \phi) \right] R_b(\theta) | \Phi(0) \rangle \]

\[= \langle \Phi(0) | R_b^\dagger(\theta) D_c(\beta) S_c(s, \phi) D_c^\dagger(-\alpha) \]

\[\times f(\alpha, \alpha^\dagger) D_c(-\alpha) S_c^\dagger(s, \phi) D_c^\dagger(\beta) R_b(\theta) | \Phi(0) \rangle. \tag{5.249} \]

Since operators \(S\), \(D\), and \(R\) satisfy [92]

\[R_b^\dagger(\theta) S_c(s, \phi) R_b(\theta) = S_c(s, \phi + \theta) \tag{5.250} \]

\[R_b^\dagger(\theta) D_c(\alpha) R_b(\theta) = D_c(\alpha e^{i\theta}) \tag{5.251} \]

\[R_b^\dagger(\theta) b R_b(\theta) = b e^{-i\theta}, \tag{5.252} \]

the above matrix element can be further simplified to

\[\langle \Phi(t) | f(a, a^\dagger) | \Phi(t) \rangle \]

\[= \langle \Phi(0) | D_c(\beta e^{i\theta}) S_c(s, \phi + \theta) D_c^\dagger(-\alpha e^{i\theta}) \times f(\alpha e^{-i\theta}, \alpha^\dagger e^{i\theta}) \]

\[\times D_c(-\alpha e^{i\theta}) S_c^\dagger(s, \phi + \theta) D_c^\dagger(\beta e^{i\theta}) | \Phi(0) \rangle \]

\[= \langle \Phi(0) | f \left[ D_c(\beta e^{i\theta}) S_c(s, \phi + \theta) D_c^\dagger(-\alpha e^{i\theta}) c e^{-i\theta} D_c(-\alpha e^{i\theta}) S_c^\dagger(s, \phi + \theta) D_c^\dagger(\beta e^{i\theta}), \right. \]

\[\left. D_c(\beta e^{i\theta}) S_c(s, \phi + \theta) D_c^\dagger(-\alpha e^{i\theta}) c e^{i\theta} D_c(-\alpha e^{i\theta}) S_c^\dagger(s, \phi + \theta) D_c^\dagger(\beta e^{i\theta}) \right] | \Phi(0) \rangle. \tag{5.253} \]

Since

\[D_c(\beta e^{i\theta}) c D_c^\dagger(\beta e^{i\theta}) = c - \beta e^{i\theta} \tag{5.254} \]

\[S_c(s, \phi + \theta) c S_c^\dagger(s, \phi + \theta) = c \cosh s + c^\dagger e^{i(\phi + \theta)} \sinh s, \tag{5.255} \]
the kernel of the above quantum average can thus be expressed as a polynomial of $c$ and $c^\dagger$, which in turn are linear functions of $a$ and $a^\dagger$. In other words, the kernel of Eq. (5.253) is also a polynomial of $a$ and $a^\dagger$, whose matrix elements can be calculated in a straightforward manner.

### 5.7 Rotating Wave Approximation

In the previous sections, we solved for the approximate eigenstates of the various configurations involving Josephson junctions. In this section we study the problem using a very different approach. Here we use the standard (see, e.g., [6]) rotating wave approximation (RWA) which enforces energy conservation—since the contribution to the energy correction from the non-number-conserving terms vanishes. This approach gives us a complementary viewpoint on the fluctuation properties of a Josephson junction. The results obtained are consistent with our previous calculations. Again, we consider the small-phase approximation and focus on the bottom of a local potential minimum.

For an isolated Josephson junction, from Eq. (5.38), after dropping the constant terms which only produce unimportant energy shifts, we obtain the following Hamiltonian

$$H_1 = \hbar \omega_1 \left( 1 - \frac{1}{16\lambda_1} \right) a^\dagger a - \frac{\hbar \omega_1}{16\lambda_1} \left( a^\dagger a \right)^2.$$  \hspace{1cm} (5.256)

In Appendix G we demonstrate that the RWA is only valid when $\langle a^\dagger a \rangle$ is small. Since throughout this chapter the system is near its ground state, the above condition for the RWA is always satisfied. Similarly, the Hamiltonian $H_2$ for the current-biased junction can be simplified from Eq. (5.47) into the form

$$H_2 = \hbar \omega_2 \left( 1 - \frac{1}{16\lambda_2} \right) a^\dagger a - \frac{\hbar \omega_2}{16\lambda_2} \left( a^\dagger a \right)^2 + \gamma (a + a^\dagger),$$  \hspace{1cm} (5.257)

with

$$\gamma = - \frac{I \Phi_0}{8\sqrt{2}\pi \lambda_2^{3/2}}.$$  \hspace{1cm} (5.258)

From now on, let us drop the subscripts and focus on the general properties of the Hamiltonians.

#### 5.7.1 Factorization of the Time Evolution Operator (TEO)

**TEO of a Free Oscillator**

To simplify the notation, let us first introduce

$$\mu = \omega \left( 1 - \frac{1}{16\lambda} \right),$$  \hspace{1cm} (5.259)

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The time evolution operators for the free oscillators of cases (1) (an isolated Josephson junction) and (4) (a Josephson junction in a superconducting ring with no external flux, at the global potential minimum) are already in factorized form, making the calculations straightforward. For example, for $H_1$,

$$
\langle \Psi_1(t) \rangle_S = U_1(t)\langle \Psi_1(0) \rangle_S = \exp \left( -\frac{i}{\hbar} \int_0^t H_1 d\tau \right) \langle \Psi_1(0) \rangle_S
$$

$$
= e^{-i\mu a^\dagger a - \lambda a^\dagger a^2} \langle \Psi_1(0) \rangle_S ,
$$

(5.261)

where $\langle \Psi_1 \rangle_S$ is the Schrödinger-picture state and $U_1(t)$ is the time evolution operator for free oscillators [valid for both cases (1) and (4)].

**TEO of a Linearly-Driven Junction**

The time-evolution operators for the case (2) (linearly-driven Josephson junction) and (3) (Josephson junction in a superconducting ring with an external flux), $U_2(t)$ and $U_3(t)$, are much more complicated since these systems have interactions. We use the interaction picture to derive the factorized time evolution operator. The linear term is treated as the interaction. The full Hamiltonian $H_2$ is split as

$$
H_2 = H_1 + H_I ,
$$

(5.262)

$$
H_1 = \hbar \mu a^\dagger a - \hbar \nu (a^\dagger a)^2 ,
$$

(5.263)

$$
H_I = \gamma (a + a^\dagger) .
$$

(5.264)

The Schrödinger equation is

$$
i\hbar \frac{\partial}{\partial t} \langle \Psi_2 \rangle_S = H_2 \langle \Psi_2 \rangle_S ,
$$

(5.265)

where $\langle \Psi_2 \rangle_S$ is the Schrödinger-picture state for a linearly-driven junction. It can be proved that approximately (see Appendix B),

$$
\langle \Psi_2(t) \rangle_S = U_2(t) \langle \Psi_2(0) \rangle_S = \exp \left( -\frac{i}{\hbar} \int_0^t H_2 d\tau \right) \langle \Psi_2(0) \rangle_S
$$

$$
\cong e^{-iH_1 t/\hbar} e^{\eta a^\dagger - \eta^* a} \langle \Psi_2(0) \rangle_S ,
$$

(5.266)

where

$$
\eta = \frac{\gamma}{\hbar \omega} \frac{1 - e^{i(1 - a^\dagger a/8\lambda)\omega t}}{1 - a^\dagger a/8\lambda} .
$$

(5.267)

Notice that the operator $\eta$ does not commute with either $a$ or $a^\dagger$. Therefore, the transformation matrix (or displacement operator) $\exp(\eta a^\dagger - \eta^* a)$ cannot be directly factorized.
To simplify the problem of factorizing \( U_2(t) \), we first expand the numerator and denominator of \( \eta \) to zeroth order in \( a^\dagger a \) (i.e., we take \( a^\dagger a/8\lambda \sim 0 \)), and consider the small-time limit, in which \( \omega t < 1 \). Thus, to zeroth order in \( a^\dagger a/8\lambda \), the TEO becomes

\[
U_2^{(0)}(t) = \exp \left\{ -i\mu t a^\dagger a + i\nu t (a^\dagger a)^2 \right\} \exp \left( \eta_0 a^\dagger - \eta_0^* a \right), \tag{5.268}
\]

where

\[
\eta_0 = \frac{\gamma}{\hbar \omega} \left( 1 - e^{i\omega t} \right). \tag{5.269}
\]

To check the accuracy of the above approximation, we also expand \( \eta \) to the next higher order (i.e., first order) in \( a^\dagger a/8\lambda \), so that now the TEO becomes

\[
U_2^{(1)}(t) = \exp \left\{ i\omega t \left( 1 - \frac{1}{16\lambda} \right) a^\dagger a - i\frac{\omega t}{16\lambda} (a^\dagger a)^2 \right\} \times \exp \left\{ (\eta_0 + \eta_1) a^\dagger - (\eta_0 + \eta_1)^* a + \eta_1 a^t a - \eta_1^* a^\dagger a^2 \right\}, \tag{5.270}
\]

where

\[
\begin{align*}
\eta_0 &= \frac{\gamma}{\hbar \omega} (1 - e^{i\omega t}), \tag{5.271} \\
\eta_1 &= \frac{\gamma}{8\lambda \hbar \omega} [1 - e^{i\omega t} (1 - i\omega t)]. \tag{5.272}
\end{align*}
\]

According to the Baker-Hausdorff formula \[6\], \( e^{A+B} = e^A e^B e^{[A,B]/2} \), if \([A, [A, B]] = [B, [A, B]] = 0\). However, the later condition does not hold in this case, where \( A = (\eta_0 + \eta_1)a^\dagger - (\eta_0 + \eta_1)^* a \) and \( B = \eta_1 a^t a - \eta_1^* a^\dagger a^2 \). Nevertheless, higher-order terms, like \([A, [A, B]]\), \([A, [A, B]]\), etc., are at most \( O(\eta_0 \eta_1) \) and thus smaller than the terms that are \( O(\eta_0) \) or \( O(\eta_1) \), since \( |\eta_0| < 1 \) and \( |\eta_1| < 1 \). Therefore, we only consider the dominant terms, which are \( \sim O(\eta_0) \) and \( O(\eta_1) \), and factorize the exponential of \( U_2^{(1)}(t) \) as \( e^{A+B} \approx e^A e^B \). Now, \( U_2^{(1)}(t) \) becomes

\[
U_2^{(1)}(t) = \exp \left\{ i\omega t \left( 1 - \frac{1}{16\lambda} \right) a^\dagger a - i\frac{\omega t}{16\lambda} (a^\dagger a)^2 \right\} \exp \left\{ \eta_1 a^t a - \eta_1^* a^\dagger a^2 \right\}
\times \exp \left\{ (\eta_0 + \eta_1) a^\dagger - (\eta_0 + \eta_1)^* a \right\}, \tag{5.273}
\]

which has a more factorized form. These results, derived for \( H_2 \), also apply \textit{mutatis mutandis} to \( H_3 \).

### 5.7.2 Calculation of the Quantum Fluctuations

#### Free Oscillator Case

The state vector for a free oscillator can then be expressed as

\[
|\psi_1(t)\rangle = e^{-i\mu t a^\dagger a} e^{-i\nu t (a^\dagger a)^2} |\psi_1(0)\rangle. \tag{5.274}
\]
Thus, given an initial state, the state vector at any future time is determined. A special initial state is a coherent state, namely,

\[ |\psi_1(0)\rangle = |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \] (5.275)

A coherent state [6] is an eigenstate of the annihilation operator, \( a|\alpha\rangle = \alpha|\alpha\rangle \). It can also be generated by acting a displacement operator on a vacuum state, \( |\alpha\rangle = D(\alpha)|0\rangle \), where the displacement operator is \( D(\alpha) = \exp(a\alpha^* - a^*a) \). See Appendix A for further explanations.

Now the state vector of the free oscillator with an initial coherent state is

\[ |\psi_{coh}(t)\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\mu t n^2} |n\rangle. \] (5.276)

The canonical coordinate and momentum are defined as

\[ X = \frac{1}{\sqrt{2}} (a + a^\dagger), \] (5.277)

\[ P = \frac{1}{\sqrt{2}i} (a - a^\dagger). \] (5.278)

\( X \) and \( P \) are dimensionless quadrature operators. They are related to \( n \) and \( \phi \) by

\[ X = \left( \frac{E_J}{E_C} \right)^{1/4} \phi, \] (5.279)

\[ P = \left( \frac{E_C}{E_J} \right)^{1/4} n. \] (5.280)

So the fluctuations of the phase and charge number are

\[ \langle (\phi_{\text{local}})^2 \rangle = \sqrt{\frac{E_C}{E_J}} \langle (\Delta X)^2 \rangle = \frac{1}{\lambda} \langle (\Delta X)^2 \rangle, \] (5.281)

\[ \langle (\Delta n)^2 \rangle = \sqrt{\frac{E_J}{E_C}} \langle (\Delta P)^2 \rangle = \lambda \langle (\Delta P)^2 \rangle. \] (5.282)

Now that we have the state vector \( |\psi_1(t)\rangle \), we can calculate the fluctuations of \( X \) and \( P \):

\[ \langle (\Delta X)^2 \rangle_1 = \langle X^2 \rangle_1 - \langle X \rangle_1^2 \] (5.283)

\[ = \frac{1}{\lambda} \left( 1 + 2|\alpha|^2 + e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n+2}}{n!} \cos(2\mu t + 4(n + 1)\nu t) \right) + 2 \left( e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n+1}}{n!} \cos(\mu t + (2n + 1)\nu t) \right)^2. \] (5.284)
\[(\Delta P)^2_1 = \langle P^2 \rangle_1 - \langle P \rangle_1^2 \]  
\[= \frac{1}{2}(1 + 2|\alpha|^2) - e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n+2}}{n!} \cos(2\mu t + 4(n + 1)v t) \]
\[-2 \left\{ e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n+1}}{n!} \sin(\mu t + (2n + 1)\nu t) \right\}^2 \]  
This calculation can be numerically carried out when \(\alpha, \omega, \) and \(\lambda\) are given. We have done so and we conclude that there is a squeezing effect when \(\alpha \neq 0\), i.e., when the initial state is a coherent state; but there is no squeezing when the initial state is a vacuum state.

According to the above results for \((\Delta X)^2_1\) and \((\Delta P)^2_1\), a large-area junction, which has large \(E_J\) and small \(E_C\), should have small fluctuations in phase and large fluctuations in particle number. On the other hand, a small-area junction has smaller Josephson coupling and smaller junction capacitance (which means a larger charging energy \(E_C\)); therefore it should have larger phase fluctuations but smaller number fluctuations. All these conclusions are qualitatively consistent with experiments. Further experimental studies, with thermal and environmental noise smaller than the intrinsic quantum noise, are needed to quantitatively verify the above results.

**Linearly-Driven Josephson Junctions**

In this section we derive some analytical results for the fluctuations of the phase and charge number of a linearly-driven Josephson junction. Recall that the corresponding state vector is approximately

\[|\psi_2(t)\rangle = e^{-i\mu t a^+_a} e^{-i\omega t (a^+_a)^2} e^{\eta_0 a^+_a - \eta a^+_a} |\psi_2(0)\rangle, \]  
where \(\eta_0 = \gamma (1 - e^{i\omega t})/\hbar \omega \), \(\lambda = \sqrt{E_J/E_C}\), and \(\gamma = -E_S/\sqrt{2}\lambda\). The latter parameter, \(\gamma\), reflects the relative strength of the external driving current, which acts like a “force”.

We calculate the fluctuations with two different initial states: vacuum state and coherent state. If the initial state is a vacuum state, the state vector is

\[|\psi_2^{\text{vac}}(t)\rangle = e^{-i\mu t a^+_a} e^{-i\omega t (a^+_a)^2} e^{\eta_0 a^+_a - \eta a^+_a}|0\rangle = e^{-i\mu t a^+_a} e^{-i\omega t (a^+_a)^2} |\eta_0\rangle. \]  

On the other hand, we can also use a coherent state as the initial state. As an example, we use an initial coherent state \(|\xi\rangle\) with \(\xi = 1\). The state vector now becomes

\[|\psi_2^{\text{coh}}(t)\rangle = e^{-i\mu t a^+_a} e^{-i\omega t (a^+_a)^2} e^{\eta_0 a^+_a - \eta a^+_a}|1\rangle. \]
Coherent states exhibit the following property

\[ D(\eta_0)|\xi\rangle = e^{i\delta}|\eta_0 + \xi\rangle, \]  

(5.290)

where \( \delta \) is a real number related to \( \eta_0 \) and \( \xi \). In other words, \( \delta \) is just a phase which will not affect the calculation of the expectation values. Thus we can drop this constant phase, and the state vector takes the form

\[ |\psi_{2{\text{coh}}}(t)\rangle = e^{-i\mu t a_{\text{a}}^\dagger e^{-i\omega t a_{\text{a}}^\dagger}}|1 + \eta_0\rangle. \]  

(5.291)

For both of these initial states, \( |\psi_{2{\text{vac}}}(t)\rangle \) and \( |\psi_{2{\text{coh}}}(t)\rangle \), the fluctuations are given by

\[
\langle(\Delta X)^2\rangle_2 = \frac{1}{2} + |\xi|^2 + e^{-|\xi|^2} \sum_{n=0}^{\infty} \frac{|\xi|^{2n+2}}{n!} \cos(2\phi - 2\mu t - 4\nu t(n + 1)) \\
-2 \left\{ e^{-|\xi|^2} \sum_{n=0}^{\infty} \frac{|\xi|^{2n+1}}{n!} \cos(\phi - \mu t - \nu t(2n + 1)) \right\}^2, \]  

(5.292)

\[
\langle(\Delta P)^2\rangle_2 = \frac{1}{2} + |\xi|^2 - e^{-|\xi|^2} \sum_{n=0}^{\infty} \frac{|\xi|^{2n+2}}{n!} \cos(2\phi - 2\mu t - 4\nu t(n + 1)) \\
-2 \left\{ e^{-|\xi|^2} \sum_{n=0}^{\infty} \frac{|\xi|^{2n+1}}{n!} \sin(\phi - \mu t - \nu t(2n + 1)) \right\}^2. \]  

(5.293)

When the initial state is a vacuum state, \( |0\rangle \), the parameters \( \zeta \) and \( \phi \) in \( \langle(\Delta X)^2\rangle_2 \) and \( \langle(\Delta P)^2\rangle_2 \) become

\[
\zeta_{\text{vac}} = \eta_0 = \frac{\gamma}{\hbar \omega}(1 - e^{i\omega t}), \]  

(5.294)

\[
\phi_{\text{vac}} = \arctan \left( \frac{\sin \omega t}{\cos \omega t - 1} \right) = \pi/2 + \omega t/2. \]  

(5.295)

However, if the initial state is the coherent state \( |1\rangle \), the parameters are,

\[
\zeta_{\text{coh}} = 1 + \eta_0 = 1 + \frac{\gamma}{\hbar \omega}(1 - e^{i\omega t}), \]  

(5.296)

\[
\phi_{\text{coh}} = \arctan \left( \frac{\sin \omega t}{\cos \omega t - 1 - \hbar \omega/\gamma} \right). \]  

(5.297)

These results, derived for case (2), also apply for case (3). We have carried out the numerical calculation of the above fluctuations but found no squeezing effect in either \( X \) or \( P \) when the initial state is a vacuum state. On the other hand, with an initial coherent state, we have found squeezing effects over a wide range of values of \( \lambda \). The results are presented in Fig. 5.4, where solid lines correspond to initial coherent states and dotted lines to an initial vacuum state.
Figure 5.4: Time evolution of the variances of the quadrature phase difference in the linearly-driven Josephson junctions, using the rotating wave approximation. The products are normal ordered and the quadrature phase is $X = (E_J/E_C)^{1/4} \phi$. The squeezing or reduction of the quadrature phase fluctuations is reached whenever its normal-ordered variance falls below zero. The solid lines represent results obtained with a zeroth-order approximation in $a^\dagger a/8\lambda$ and with an initial coherent state. The dotted lines are results obtained with the same approximation but with an initial vacuum state. The dashed lines use the first-order approximation in $a^\dagger a/8\lambda$ and with an initial vacuum state. The fluctuations $\langle (\Delta X)^2 \rangle(t)$ and $\langle (\Delta P)^2 \rangle(t)$ for the first-order approximation in $a^\dagger a/8\lambda$, and starting from an initial coherent state, are indistinguishable from the zeroth-order results (at least for small times). This indicates that our approximation is robust and works well when in the regime $0.5 < \lambda < 5$, where $\lambda = \sqrt{E_J/E_C}$. 
First order Correction for the Linearly-Driven Case

For the linearly-driven junction cases (2) and (3), and with an initial vacuum state, we have also calculated the fluctuations of the canonical conjugate variables by expanding $\eta$ to first order in $a^\dagger a/\hbar\omega$. In other words, now

$$\eta = \frac{\gamma}{\hbar\omega} \frac{1 - e^{i(1-a^\dagger a/\hbar\omega)\omega t}}{1 - a^\dagger a/\hbar\omega} \approx \eta_0 + \eta_1 a^\dagger a,$$

(5.298)

where

$$\eta_0 = \frac{\gamma}{\hbar\omega} (1 - e^{i\omega t}),$$

(5.299)

$$\eta_1 = \frac{\gamma}{8\lambda\hbar\omega} \left[ 1 - e^{i\omega t} (1 - i\omega t) \right].$$

(5.300)

The formal displacement operator can now be simplified as:

$$e^{\eta a^\dagger - \eta^\dagger a} \approx e^{(\eta_0 a^\dagger - \eta_0^\ast a) + \eta_1 a^\dagger a - \eta_1^\ast a^\dagger a} e^{(\eta_0 + \eta_1) a^\dagger - (\eta_0 + \eta_1)^\ast a}$$

$$\approx (1 + \eta_1 a^\dagger a - \eta_1^\ast a^\dagger a) D(\eta_0 + \eta_1).$$

(5.301)

Where in the last line we have kept the dominant terms of the first exponential factor.

If the initial state is a vacuum state, we will have a state vector

$$|\psi^{\text{vac}}(t)\rangle_S \approx e^{-i\eta_1 a^\dagger a} e^{-i\eta_0 t(a^\dagger a)^2} (1 + \eta_1 a^\dagger a - \eta_1^\ast a^\dagger a^2) D(\eta_0 + \eta_1) |0\rangle$$

$$= e^{-i\eta_1 a^\dagger a} e^{-i\eta_0 t(a^\dagger a)^2} (1 + \eta_1 a^\dagger a - \eta_1^\ast a^\dagger a^2) |\eta_0 + \eta_1\rangle.$$

(5.302)

The numerical evaluation of the fluctuations $\langle (\Delta X)^2 \rangle_2$ and $\langle (\Delta P)^2 \rangle_2$ does not show a qualitative difference from the zeroth-order approximation (see Fig. 5.4, in which these results are shown in dashed lines), which demonstrates that the approximation we make for $\eta$ is robust. The fluctuations for large values of $t$ show an abnormal increase in magnitude because of the limitations of the small-time approximation. Therefore, we will only focus on the zeroth-order approximation, keeping in mind that we should remain in the small-time regime ($\omega t \sim 1$).

Physical Significance of the Initial States

Now let us look at the rotating wave approximation to this problem more closely. It is clear that when we take the small phase approximation, we have already implicitly assumed that
the states are localized. Thus our approach is better suited for the situations where $E_J$ is relatively large.

As shown above, with an initial vacuum state we cannot reach a squeezed state at any time. The reason can be seen from the squeezing mechanism considered here. The fourth-order terms in the creation and annihilation operators are responsible for the redistribution of phase, which in turn leads to a squeezing effect. To make this term more effective, a large number of quanta are needed, because each one of the fourth-order terms involves four quanta at a time. An initial vacuum state cannot satisfy this condition, thus preventing the formation of squeezed states. On the other hand, an initial coherent state does have a nonzero average number of quanta, therefore making it possible for squeezed states to be generated.

### 5.8 Discussions and Open Problems

As we mentioned before, a key parameter in a Josephson junction is $A = \sqrt{E_J/E_c}$. This parameter does not lead to squeezing directly, and so far cannot be tuned at will in most experiments. However, $\lambda$ does provide a powerful control over quantum noise. By adjusting the value of $\lambda$, it is possible to redistribute the noise in both $n$ (the tunneling Cooper pair number) and $\phi$ (the phase difference between the two sides of the junction). This is very important because the number $n$ and the phase $\phi$ are observable quantities.

Here we have focused on four model Hamiltonians describing different ways to couple a Josephson junction to its environment. Needless to say, this list is not exhaustive, although we believe that these basic cases constitute a first step towards the study of more complicated and hybrid-mode interactions with the environment. Other interactions with the environment can be considered. For example, a thermal reservoir can be represented by a series of harmonic oscillators and introduce a linear coupling. Such a model was used to discuss dissipation in tunneling events (see Ref. [93] and references therein).

Recently, another Hamiltonian was proposed [95] in which the coupling between a Josephson junction and an external electromagnetic mode is contained in the quasicharge. Such an interaction can give rise to modulation of supercurrent through the junction. However, in this approach the phase difference over a Josephson junction is treated as an exactly measurable classical quantity, whose fluctuation comes solely from the driving external electromagnetic field. A more complete treatment has to consider both the external noise and the intrinsic quantum fluctuation we have discussed in this paper.
We did not include a heat reservoir in any of the cases considered in this paper, assuming that the noise due to the exchange with the environment is relatively weak. With a reservoir, a better approach would be to solve a quantum Langevin equation for a relevant variable such as the phase difference. This is beyond the scope of this work.

Throughout this paper, we have used two major approximations: we treat Josephson junctions as ideal and consider their quantum states to be localized. These approximations are valid in the limit of \( T = 0, E_J \gg E_C \), and a very small biasing current. The states considered here should be either the ground states or the low-energy excited states. If the above conditions are not satisfied, other effects can be important.

Recall that the Josephson coupling energy is a sinusoidal function of the phase difference across the junction. In our treatment, we choose one of the potential minima and expand the whole potential around this minimum in a Taylor series up to the fourth-order terms. Such an expansion and truncation provides a potential that can localize wavefunctions. The expansion is a good representation of the original potential only around the potential minimum. Furthermore, by replacing a periodic potential with a localized one, we are using localized states to represent “snapshots” of the extended states. Therefore, to improve our results, we need to either change to an extended-state basis, or at least take into account the fluctuations caused by the tunneling events.

We can consider the effect of the periodicity from another point of view. From elementary quantum mechanics, a band structure will form in the energy spectrum as a result of a periodic potential. In Josephson junctions, the periodicity in the original sinusoidal potential transforms the discrete energy levels we obtained to a series of bands [85, 89]. The corresponding states are Bloch states. When \( E_J \gg E_C \), the lowest-energy bands are very narrow and highly degenerate, and the energy spectrum is very similar to a localized oscillator, which is what we obtained from our approximations.

Real junctions are not ideal. A commonly-used model for them is the resistively-shunted junction model [83], in which the junction is represented by an ideal junction in parallel with a capacitor and a resistor. The damping of a non-ideal junction comes from the resistor. Microscopically, this dissipation originates from the interaction between Cooper pairs and their environmental degrees of freedom and also the excitation of quasiparticles. Therefore, the real quantum noise of the junction also includes contributions from the quasiparticles and other environmental degrees of freedom. Whether our results will provide the most important contribution depends on the relative strength of the dissipation and the thermal
energy to the Josephson coupling constant $E_J$.

To go beyond the approximations used in this work, it would be interesting to study the interplay between the intra-potential-well fluctuation and the inter-potential-well tunneling. To relate the Bloch states to the localized states studied here would also be an interesting subject. Such a comparison would clearly show whether our results, which are based on an approximation, contain most of the relevant information in the Bloch states, which are the exact solution to the periodic potential problem. In addition, to describe the quantum noise in a more complete manner, it is also important to study the shot noise from the quasiparticle tunneling and the white noise (or colored noise, to be more general) of the environment. Another very interesting topic is the possible manipulation of the quantum fluctuations in $n$ and $\phi$. Notice that in this paper we have only discussed the variation of the quantum noises due to the nonlinearity of the Josephson coupling energy $\cos \phi$. No external tuning was considered. Therefore, an open problem would be to consider an external mechanism that would control the level of quantum noise in a given variable, similar to our work on phonons [18, 19, 20], where the incoming coherent light pumps optical phonons into a parametric amplification process through a Raman mechanism. These optically excited coherent optical phonons can then convert into squeezed acoustic phonons through an anharmonic interaction.

5.9 Conclusions

In this chapter, we investigate the quantum fluctuation properties of a Josephson junction in several different configurations. Specifically, we work in the limit of large Josephson coupling energy and small charging energy, so that the junctions are in the nearly-localized regime. This limit can be easily realized for the large S-I-S junctions. It is analogous to the tight-binding limit for electrons in a crystal. Furthermore, we expand the $E_J \cos \phi$ Josephson coupling energy around $\phi = 0$ to fourth order in $\Delta \phi$, since we work in the nearly-localized regime. Such an expansion enables us to solve for the eigenstates analytically.

We also obtain the approximate ground states of a Josephson junction in a variety of configurations. In particular, the ground state is a squeezed vacuum state for either an isolated junction in a potential minimum or a junction in a superconducting ring without external flux and in the global potential minimum. On the other hand, if the junction is current-biased, or there is an external flux through the superconducting ring, the ground state is a squeezed coherent state. In both of the above cases, we calculate the corresponding
fluctuations of the charge and phase difference over the junction in these states. The squeezing factors are determined by the parameter $\lambda = \sqrt{E_J/E_C}$, where $E_J$ is the Josephson coupling energy and $E_C$ is the charging energy of the junction. The squeezing effect is strong when $\lambda$ is small. Since our working limit is at large $\lambda$, a compromise should be reached in order to both preserve the effectiveness of our approximation and maximize the squeezing effect.

The excited states of a Josephson junction in different circumstances are also obtained. We show that these excited states are similar to the number states of a simple harmonic oscillator but with different fluctuation properties.

One can think of the squeezing effect intuitively in terms of the expanded $\cos \phi$ term. The second-order term in the expansion provides a harmonic potential, which has constant quantum fluctuations in its eigenstates. The fourth-order term introduces a time-dependent modulation to these states, thus also modulates the intrinsic quantum noise in the states.

We have also studied the time evolution of quantum fluctuations of a Josephson junction under the rotating wave approximation, which enforces energy conservation. When calculating the fluctuations of the canonical momentum and coordinate in the free oscillator cases (e.g., a Josephson junction coupled to a capacitor or a superconducting ring) and in the linearly driven cases (e.g., a current biased junction), we find that in the small-phase, small-time limit, there is squeezing when we have an initial coherent state, and no squeezing if we have an initial vacuum state.

According to the calculations for $\langle (\Delta \phi)^2 \rangle$ and $\langle (\Delta n)^2 \rangle$, a large-area junction, which has large $E_J$ and small $E_C$, should have small fluctuations in phase and large fluctuations in particle number. On the other hand, a small-area junction has smaller Josephson coupling and smaller junction capacitance (which means a larger charging energy $E_C$); thus it should have larger phase fluctuations but smaller number fluctuations. Therefore, our calculations of $\langle (\Delta \phi)^2 \rangle$ and $\langle (\Delta n)^2 \rangle$ provide results qualitatively consistent with experiments. Further experimental studies, with thermal and environmental noise smaller than the intrinsic quantum noise, are needed to quantitatively verify our results.