SCIENTIFIC REPORTS

natureresearch

Check for updates

OPEN Dissipation-induced bistability in the two-photon Dicke model

Louis Garbe^{1,2^{\infty}}, Peregrine Wade¹, Fabrizio Minganti², Nathan Shammah^{2,3}, Simone Felicetti^{4,5} & Franco Nori^{2,6}

The Dicke model is a paradigmatic quantum-optical model describing the interaction of a collection of two-level systems with a single bosonic mode. Effective implementations of this model made it possible to observe the emergence of superradiance, i.e., cooperative phenomena arising from the collective nature of light-matter interactions. Via reservoir engineering and analogue quantum simulation techniques, current experimental platforms allow us not only to implement the Dicke model but also to design more exotic interactions, such as the two-photon Dicke model. In the Hamiltonian case, this model presents an interesting phase diagram characterized by two quantum criticalities: a superradiant phase transition and a spectral collapse, that is, the coalescence of discrete energy levels into a continuous band. Here, we investigate the effects of both qubit and photon dissipation on the phase transition and on the instability induced by the spectral collapse. Using a mean-field decoupling approximation, we analytically obtain the steady-state expectation values of the observables signaling a symmetry breaking, identifying a first-order phase transition from the normal to the superradiant phase. Our stability analysis unveils a very rich phase diagram, which features stable, bistable, and unstable phases depending on the dissipation rate.

The Dicke model¹ describes the interaction of a collection of two-level systems with a single bosonic mode. In the thermodynamic limit, this model exhibits a superradiant phase transition at zero temperature¹⁻⁵. Namely, the ground-state number of photons changes non analytically from zero to finite values as the light-matter coupling strength is increased across a critical value. The relevance of the Dicke model to capture the physics of light-matter coupling near the critical point is the object of an ongoing debate. In particular, an obstacle to the observation of a superradiant phase transition arises due to the presence of the so-called diamagnetic term. In this regard, resolutions of gauge ambiguities have been recently proposed, such as employing modified unitary transformations or going beyond the two-level system description⁶⁻¹

It is possible, however, to circumvent this controversy entirely by using driven systems and bath engineering to simulate effective Hamiltonians. Thanks to this approach, it was possible to observe the superradiant phase transition in several platforms, in particular atomic systems in cavity¹⁰⁻¹², and trapped ions^{13,14}. Other proposals have been put forward, using NV centers array¹⁵, or superconducting circuits¹⁶. In general, the implementation of analogue quantum simulations^{17,18} provides an ideal playground to test driven-dissipative physics in a controlled setting. Their experimental feasibility has motivated increasing research efforts devoted to the study of driven-dissipative quantum optical models, as it is known that noise and dissipation can drastically change the properties of the steady-state phase diagrams and the emergence of phase transitions^{19,20}. Analytical studies²¹⁻²⁷ are extremely challenging, as the intrinsic non-equilibrium nature of driven-dissipative systems does not allow a determination of the stationary state of the system via a free energy analysis²⁸; for Dicke-like models, it is especially true when local spin dissipation is considered^{25,26,29}.

Effective implementations offer others significant advantages. First, in a cavity-QED setting, driving the system allows turning virtual excitations inside the cavity into real photons which can exit the system³⁰, thus giving immediate access to the intra-cavity dynamics. Second, by manipulating the exchanges between a system and its environment, reservoir-engineering techniques allow us to realise previously inaccessible quantum

¹Laboratoire Matériaux et Phénomènes Quantiques, Sorbonne Paris Cité, CNRS UMR 7162, Université de Paris, 75013 Paris, France. ²Theoretical Quantum Physics Laboratory, RIKEN Cluster for Pioneering Research, Wako-shi, Saitama 351-0198, Japan. ³Present address: Unitary Fund, 340 S Lemon Ave. 7770, Walnut, CA 91789, USA. ⁴Departamento de Física Teórica de la Materia Condensada and Condensed Matter Physics Center (IFIMAC), Universidad Autónoma de Madrid, 28049 Madrid, Spain. ⁵Istituto di Fotonica e Nanotecnologie, Consiglio Nazionale delle Ricerche (IFN-CNR), Piazza Leonardo da Vinci 32, 20133 Milano, Italy. ⁶Physics Department, The University of Michigan, Ann Arbor, MI 48109-1040, USA. [™]email: louis.garbe@univ-paris-diderot.fr

phases of matter^{31–37}. For instance, it is possible to stabilize phases without an equilibrium counterpart^{38, 39}, and reservoir engineering methods for complex many-body phases have been thoroughly explored in different contexts^{32,40}. Several experiments and theoretical proposals have applied these ideas to study generalized Dicke models^{11,15,41–45} and the ultrastrong coupling regime (USC)^{46,47}, i.e., the regime of parameters where the coupling constant becomes a sizable fraction of both the qubit and bosonic frequencies.

Among the variety of phenomena that are made accessible by analogue quantum simulations^{17,18}, a particularly interesting one is the possibility to engineer a coupling involving the simultaneous exchange of several photons. In superconducting circuits⁴⁸, this is possible with nonlinear virtual processes in the USC regime^{49–53}, and experimentally a photon-pair driving mechanism has been realized³⁷, leading to the generation of so-called photonic Schrödinger cats^{54–56}. The possibility to control and protect such states is promising for the implementation of quantum computation protocols^{57–59}.

In the two-photon Dicke model^{42,44,60}, it is the light-matter interaction that creates or annihilates one pair of bosonic excitations per qubit flip. This exotic exchange leads to several unusual properties in the USC regime^{46,47}. In particular, it was shown very early on^{61,62} that there exists a critical value of the coupling strength for which the discrete spectrum collapses into a continuous band⁴². For higher values of the coupling, the two-photon Dicke Hamiltonian is no longer bounded from below, indicating the breaking down of the model itself. It was recently shown that, in spite of the spectral collapse, a superradiant-like phase transition can take place also in the two-photon Dicke model⁶⁰. This transition has been characterized also for other two-photon interaction models^{63,64} but, so far, only the Hamiltonian case has been considered.

In this paper, we investigate the fate of the spectral collapse in a driven-dissipative case. Specifically, we consider the *N*-body two-photon Dicke model connected to an engineered Markovian bath. The Lindblad formalism is used to introduce different incoherent processes, such as photon loss κ , individual qubit decay Γ_{\downarrow} , and local qubit dephasing Γ_{ϕ} . To analyze the system dynamical properties, we resort to a mean-field decoupling of the equation of motion, allowing us to determine (semi-)analytically the steady-state of the system. We show the emergence of a first-order phase transition from a normal to a superradiant phase, for a critical value of the light-matter coupling *g*. A numerical study of the stability of the different phases unveils a very rich phase diagram: depending on the strength of the atomic dissipation, the system may be stable, bistable, or unstable. In particular, the spectral collapse can disappear altogether. Interestingly, we find that both atomic decay and atomic dephasing can be beneficial to the stabilization of the superradiant phase. We found no evidence of supression and restoration of the transition, contrary to the phenomenology of the standard Dicke model²⁵.

This paper is organized as follows: first, we introduce the one- and two-photon Dicke model, and briefly describe some of their properties using heuristic arguments. Secondly, we discuss dissipation in the two-photon model. Thirdly, we present the phase diagram of the system obtained via a decoupling mean-field approximation, and show the existence of two different regimes of dissipation. Finally, we summarize our conclusions and present some perspectives for future work.

One-and two-photon Dicke models. The standard Dicke model was originally used to describe the behaviour of a collection of atoms with the electromagnetic field inside a high-quality-factor cavity. It can be derived by making several assumptions about the system. For instance, the atomic size must be small compared to the field wavelength, making the atoms insensitive to the field modulation^{1,65}. The atoms must couple to a single mode of the field. Finally, the atomic energy level structure must be highly anharmonic, so that only one transition is resonant with the field, allowing us to approximate the atoms by two-level systems (or qubits). This so-called two-level approximation, when handled improperly, can lead to a gauge ambiguity in the USC regime. Solutions to this problem have been proposed only recently^{8,9}.

When these assumptions hold, the system is described by the one-photon Dicke Hamiltonian (here $\hbar = 1$),

$$\hat{H}_{1} = \omega_{c} \hat{a}^{\dagger} \hat{a} + \omega_{0} \sum_{j=1}^{N} \hat{\sigma}_{z}^{j} + \frac{g}{\sqrt{N}} \sum_{j=1}^{N} \hat{\sigma}_{x}^{j} (\hat{a} + \hat{a}^{\dagger}),$$
(1)

where \hat{a} (\hat{a}^{\dagger}) is the annihilation (creation) operator of the bosonic mode, N is the number of qubits, and $\sigma_{x,y,z}^{j}$ are the Pauli matrices describing the *j*-th qubit. This Hamiltonian exhibits a \mathbb{Z}_2 symmetry, corresponding to the simultaneous exchange

$$\begin{cases} \hat{a} \to -\hat{a}, \\ \hat{\sigma}_x^j \to -\hat{\sigma}_x^j \quad \forall j. \end{cases}$$
(2)

For low values of the coupling, the ground state of this Hamiltonian is given by the product state of the field vacuum and the atoms individual ground states. When the coupling constant enters the USC regime, however, the system experiences a second-order phase transition in the thermodynamic limit which breaks the \mathbb{Z}_2 symmetry. The system enters the so-called superradiant phase, in which the bosonic field is described by a coherent state, while the qubits are collectively rotated³. The possibilities offered by quantum simulations have brought this model far beyond the atom-cavity setting. For instance, the cavity electromagnetic field may be replaced by microwave resonators in superconducting circuits, or by vibrational motion in atomic platforms. These effective implementations made it possible to circumvent the problems raised by gauge ambiguities and to observe the superradiant phase transition^{10–12,14}.

These platforms also pave the way to the experimental exploration of novel forms of light-matter interactions. In particular, quantum simulation schemes make it possible to implement two-photon interaction models both in the SC and in the USC regime^{42,66-68}. For instance, in trapped-ions experiments, laser-induced interactions

can be used to couple the internal state of the ions to their motional degrees of freedom. Let us assume that the properties of the trap allow us to single out a single vibrational mode with frequency ν . If the detuning between the laser and the internal transition is close to -2ν (red-detuned laser), then the laser can excite a process in which two phonons are destroyed and one qubit excitation is created. If the detuning is close to 2ν (blue-detuned laser), then the energy brought by the laser can be used to simultaneously create one qubit excitation and two phonons. Therefore, by using both a red-detuned and a blue-detuned lasers, one can engineer a qubit-boson coupling similar to the Dicke model, but where the standard one-boson interaction term is replaced by a two-boson term, which is generically called in the literature two-photon or two-phonon coupling term. For simplicity, we will only use the term "two-photon" and the cavity-QED terminology in the following. Furthermore, the modulation of photonic states by the laser pump permits to effectively renormalize both the bosonic frequency and the coupling constant, thus allowing to bring the two-photon coupling to the USC regime.

Similarly, it has recently been shown^{44,45} that two-photon interactions can also be implemented in superconducting circuits, engineering an intrinsic nondipolar coupling between a superconducting artificial atom and superconducting quantum interference device (SQUID). In this case, the standard linear coupling is suppressed, while the two-photon coupling terms emerge as the natural light-matter interaction in an undriven system and not as the result of a quantum simulation scheme.

These various possibilities of implementing the two-photon coupling term motivates the study of the *two-photon* Dicke model, whose Hamiltonian reads (setting $\hbar = 1$),

$$\hat{H} = \omega_c \hat{a}^{\dagger} \hat{a} + \omega_0 \sum_{j=1}^N \hat{\sigma}_z^j + \frac{g}{\sqrt{N}} \sum_{j=1}^N \hat{\sigma}_x^j [\hat{a}^2 + \hat{a}^{\dagger 2}].$$
(3)

This Hamiltonian exhibits a four-folded symmetry, stemming from the simultaneous exchange of

$$\begin{cases} \hat{a} \to i\hat{a}, \\ \hat{\sigma}_x^j \to -\hat{\sigma}_x^j \quad \forall j. \end{cases}$$

$$\tag{4}$$

In the USC limit, this model exhibits an instability known as spectral collapse⁴², where the discrete spectrum collapses into a continuous band for a critical value of *g*. Some intuition about this effect can be gained through the following reasoning. When the coupling constant *g* in Eq. (3) becomes large, the interaction term dominates the physics. Since this term commutes with the $\hat{\sigma}_x^i$, we can study the qubits domains $\hat{\sigma}_x^i = -\frac{1}{2}$ and $\hat{\sigma}_x^i = +\frac{1}{2}$ independently. Let us consider $\hat{\sigma}_x^i = -\frac{1}{2}$ for all *i*. Then we have an effective boson dynamics described by this Hamiltonian

$$\omega_c \hat{a}^{\dagger} \hat{a} - \frac{g\sqrt{N}}{2} \left[\hat{a}^2 + \hat{a}^{\dagger 2} \right] = \left(\frac{\omega_c}{4} - \frac{g\sqrt{N}}{4} \right) \hat{x}^2 + \left(\frac{\omega_c}{4} + \frac{g\sqrt{N}}{4} \right) \hat{p}^2, \tag{5}$$

which is a quadratic potential for the field quadratures $\hat{x} = \hat{a}^{\dagger} + \hat{a}$ and $\hat{p} = i(\hat{a}^{\dagger} - \hat{a})$. When g is large enough, this potential becomes almost flat, shrinking the gap between the different energy levels. Ultimately, these levels coalesce into a continuous band, causing the so-called spectral collapse⁴². When g is increased even further, the potential becomes an upside-down harmonic well, and is unbounded from below for $\hat{x} \to \infty$. Therefore, the dynamics of the system will become unstable, signaling the breaking down of the model. By contrast, in the one-photon Dicke model (1), the interaction term adds only a linear correction, meaning that the Hamiltonian can never be unbounded from below.

Very recently it has been shown⁶⁰ that the two-photon Dicke model can also display a second-order quantum phase transition very similar to the superradiant transition of the one-photon Dicke model. Instead of a coherent state, however, the bosonic field here will be described by a squeezed state for high values of the coupling. The ground-state phase diagram has been analyzed with different numerical and analytical techniques also for other two-photon coupling models^{63,64}. However, two-photon light-matter interaction models have so far never been considered from an open quantum system perspective.

Effect of dissipation. The physics of the one-photon Dicke model changes drastically once dissipation is taken into account. It was shown in Refs.^{25,69} that in the presence of qubit decay and dephasing, the transition of the Dicke model could be modified, suppressed, or restored. Similarly, the presence of dissipative processes in the two-photon Dicke model raises intriguing questions.

Assuming a Markovian environment and performing the Born approximation, the dissipation may be described by a Lindblad master equation²⁸.

In our analysis, we will assume that the Hamiltonian part of the evolution remains that of Eq. (3), and we will include three dissipation channels, that is, individual qubit decay and dephasing, and photon loss. We obtain the following Lindblad equation (we recall $\hbar = 1$),

$$\partial_t \hat{\rho} = -i[\hat{H}, \rho] + \kappa \mathcal{D}[\hat{a}](\hat{\rho}) + \sum_{j=1}^N \Gamma_{\downarrow} \mathcal{D}[\hat{\sigma}_{-}^j](\hat{\rho}) + \Gamma_{\phi} \mathcal{D}[\hat{\sigma}_{z}^j](\hat{\rho}), \tag{6}$$

where $\hat{\rho}(t)$ is the density matrix of the system at time $t, \hat{\sigma}_{-}^{j} = \hat{\sigma}_{x}^{j} - i\hat{\sigma}_{y}^{j}, [\hat{H}, \hat{\rho}]$ indicates the commutator, and $\mathcal{D}[\hat{A}]$ are the Lindblad dissipation superoperators defined as²⁸

$$\mathcal{D}[\hat{A}]\hat{\rho} = \hat{A}\hat{\rho}\hat{A}^{\dagger} - \frac{1}{2}\hat{\rho}\hat{A}^{\dagger}\hat{A} - \frac{1}{2}\hat{A}^{\dagger}\hat{A}\hat{\rho}.$$
(7)

Even if, in general, the symmetries of the Lindblad equation cannot be directly obtained from those of the Hamiltonian⁷⁰, Eq. (6) remains identical under the transformation given in Eq. (4). In this regard, also the Lindblad master equation presents a four-folded symmetry similar to that of the Hamiltonian case. Let us note that this equation is a purely phenomenological one and, while it is the most appropriate in a quantum simulation framework, it would fail to describe the true evolution for genuine implementations of the model in the USC regime. Indeed, in the presence of bare local dephasing and qubit decay processes, the system would not tend towards the dressed ground state. In fact, these processes would effectively pump energy into the system, forcing it away from the true polaritonic ground state and toward a different steady-state; while considering dressedoperator incoherent processes would lead to the polaritonic ground state²⁹. Moreover, a microscopic theory needs to be developed for arbitrary strengths of the dissipative couplings, which has shown that USC effects can be robust in loss-dominated systems⁷¹. However, our analysis is meant to describe effective implementations of the model, where the considered decoherence and dissipation processes can themselves be implemented via bath-engineering techniques⁶⁷. For instance, this Lindbladian dynamics could be observed in a strongly driven atomic cloud, where Raman processes effectively engineer the wanted processes as demonstrated in several experiments¹⁰⁻¹². Note also that, since quantum simulation allows to renormalize both the effective frequencies ω_c and ω_0 and the coupling constant g^{42} , the dissipation constant may be large compared to ω_c , ω_0 and g, while remaining small compared to the actual frequencies of the system.

We expect that this model will have very different behavior from its Hamiltonian counterpart. On the one hand, the critical behaviour and the properties of the superradiant phase can drastically change, as for the one-photon model^{5,25}. On the other hand, the spectral collapse may be modified or avoided due to the presence of dissipation. Indeed, the photon loss term in Eq. (6) acts like a stabilizing quadratic term which can balance the effect of the Hamiltonian unstable potential.

Results

Symmetry breaking. When spin dissipation is absent^{3,21,23,27,69} or acts collectively on all qubits at once, one can significantly simplify the problem by treating the qubits as a single, collective spin, which allows to obtain the quantum state of the qubits. This is not possible here, since the dissipation acts on each qubit individually^{25,26,29}. However, we can still obtain meaningful results by focusing on some specific, relevant observables. We have focused our analysis on the following quantities: $\hat{j}_{u=x,y,z} = \frac{1}{N} \sum_{j=1}^{N} \hat{\sigma}_{u}^{j}$, $\hat{X} = \hat{a}^{2} + \hat{a}^{\dagger 2}$, $\hat{Y} = \hat{a}^{2} - \hat{a}^{\dagger 2}$, $\hat{a}^{\dagger} \hat{a}$. The Hamiltonian treatment⁶⁰ predicts a symmetry breaking during which the bosonic field becomes squeezed, which is captured by the second-order moment of the bosonic field. Therefore, we expect that the observables \hat{X} , \hat{Y} and $\hat{a}^{\dagger}\hat{a}$ will be valid order parameters also in the presence of dissipation. We have studied the evolution of these quantities using a mean-field decoupling approximation (see the Methods section). We found that the dynamics of these quantities has three possible solutions. The first one corresponds to $\langle \hat{a}^{\dagger} \hat{a} \rangle = \langle \hat{X} \rangle = \langle \hat{Y} \rangle = \langle \hat{f}_x \rangle = \langle \hat{f}_x \rangle = \langle \hat{f}_y \rangle = 0$ and $\langle \hat{I}_z \rangle = -1$. The other two phases have $\langle \hat{a}^{\dagger} \hat{a} \rangle, \langle \hat{X} \rangle = \pm X_s, \langle \hat{Y} \rangle = \pm Y_s \neq 0$ (complete expressions in the Methods section, Eq. (10)). In accordance with previous results in the one- and two-photon Dicke model^{3,5,25,60}, we can identify the first solution as the "normal phase", as it corresponds to the product state of the individual ground states of the field and the atoms. The other two solutions correspond to the "superradiant phase"; that is, they contain a macroscopic number of atomic and photonic excitations. Since the "superradiant" solutions have $\langle \hat{X} \rangle \neq 0$, the four-folded symmetry of the model is at least partially broken. The stability of each phase has been analyzed numerically (see Methods). Let us now illustrate the properties of the superradiant phase and the complex driven-dissipative phase diagram of the model considered.

Nature of the phase transition. In Fig. 1 we show the value of the steady-state photon number in the superradiant phase as a function of the coupling strength g, $\omega_0 = \omega_c$, and $\kappa = \omega_c$. For now, we have set $\Gamma_{\downarrow} = \Gamma_{\phi} = \Gamma$, and $\Gamma = 3\omega_c$. For small values of g, the superradiant phase yields nonphysical complex values for $\langle \hat{a}^{\dagger} \hat{a} \rangle$, showing that the system can only reach the normal phase $\langle \hat{a}^{\dagger} \hat{a} \rangle = 0$ until the critical value of the coupling strength is achieved. When g is increased, the superradiant phase becomes physical, and the stability analysis reveals it is stable as well.

Therefore, a nonzero number of bosonic excitations can appear in the system. Analyzing the average photon number in the system steady-state, we can already identify two qualitative differences compared to the ground state in the Hamiltonian case. First, in the driven-dissipative case, the number of photons in the superradiant phase does not go to zero when one approaches the limit of stability from above. Second, the point at which the normal phase becomes unstable and the superradiant phase becomes stable do not coincide. Therefore, the driven-dissipative two-photon Dicke model exhibits bistability at the mean-field level.

The emergence of bistability in mean-field models is well-known in open quantum systems. A typical example is that of the Kerr resonator, where the semiclassical solution obtained via the Gross-Pitaevskii mean-field has three different solutions: two which are stable and one unstable. As soon as one considers the quantum steady-state, however, only one solution is found⁷². This apparent contradiction can be solved by considering the full Liouvillian spectrum, where the onset of bistability is in close relation to the emergence of criticality^{19,73,74}. Indeed, several models presenting bistable behaviour at the mean-field level proved to display a genuine first-order phase transition in the thermodynamic limit of a full quantum model^{56,75-82}.

These results show that the Hamiltonian and dissipative versions of this model are strikingly different. In the equilibrium case, a second-order phase transition is predicted to occur, and only in the far-detuned regime⁶⁰



Figure 1. Photon number in the superradiant phase versus normalised light-matter coupling, for several qubit numbers *N*. We have set $\kappa = \omega_c$ and $\Gamma_{\downarrow} = \Gamma_{\phi} = 3\omega_c$. For small *g*, the superradiant phase becomes unphysical. The horizontal black line is a guide to the eye indicating the value $\langle \hat{a}^{\dagger} \hat{a} \rangle = 0$, which is the value the field adopts in the normal phase. The vertical dashed line indicates the point where the normal phase becomes unstable (this point is independent of *N*). In all cases, the superradiant phase becomes stable before the normal phase becomes unstable, indicating bistability.



Figure 2. Phase diagram of the model. **a**) N = 10 qubits, $\Gamma/\omega_c = 1.5$, **b**) N = 50 qubits, $\Gamma/\omega_c = 1.5$, **c**) N = 100 qubits, $\Gamma/\omega_c = 1.5$, **d**) N = 10 qubits, $\Gamma/\omega_c = 3$, **e**) N = 50 qubits, $\Gamma/\omega_c = 3$, and **f**) N = 100 qubits, $\Gamma/\omega_c = 3$. For all plots, $\kappa = \omega_c$. N: normal phase, S: superradiant phase, B: bistable phase, I: instability. For $\Gamma/\omega_c = 1.5$, the region of stability for the superradiant phase shrinks when the number of qubits increases. For $\Gamma/\omega_c = 3$, a bistable behavior is observed, and the phase diagram is almost invariant when the number of qubits is increased beyond a few dozens.

 $\omega_0 \ll \omega_c$. In the nonequilibrium case, a first-order phase transition takes place in the resonant regime $\omega_0 = 2\omega_c$, a condition that strongly simplifies possible experimental implementations.

Phase diagram. Having established the existence of a phase transition, we can produce the phase diagram of the model by studying the stability of both phases for a broad range of parameters. The analysis of these diagrams revealed the existence of two regimes of dissipation. In Fig. 2, we display the phase diagram in the g- ω_0 plane,



Figure 3. Stability of the superradiant phase versus coupling and dissipation. **a**) $\omega_0/\omega_c = 0.5$, **b**) $\omega_0/\omega_c = 2$, **c**) $\omega_0/\omega_c = 5$. For all plots, $\kappa = \omega_c$ and N = 100 qubits. S,B: superradiant or bistable phase; N,I: normal phase or instability. Here we do not distinguish between the normal and the unstable phase or between superradiant and bistable phase because we want to focus on the stability of the superradiant phase only. Except for small values of ω_0/ω_c or g, the value of Γ/ω_c for which the transition occurs is almost the same for all parameters, around 1.6. This means the phase diagram as a whole changes drastically when Γ goes across this value.

for two values of $\Gamma: \Gamma = 1.5\omega_c$ and $\Gamma = 3\omega_c$, and for various number of qubits *N*. For $\Gamma = 1.5\omega_c$ and the smaller value N = 10 qubits, we observe that the mean-field equations predict the existence of a zone where the superradiant phase is stable. However, the size of this zone shrinks when *N* increases. Since the mean-field description becomes correct only for $N \to \infty$, *no phase transition can happen in the mean-field limit for this value of dissipation*. The system will either reach the normal steady-state or be unstable. For $\Gamma = 3\omega_c$, however, we observe that bistability becomes possible, similarly to what happens in the driven-dissipative one-photon Dicke model⁸³. In the thermodynamic limit, the region of stability becomes independent of the number of qubits, meaning that *a phase transition can take place in the* $N \to \infty$ *limit*.

Values of Γ/ω_c lower that 1.5 or higher than 3 yield qualitatively similar results, which allows us to conclude that there are two regimes of dissipation: a large dissipation regime in which a phase transition is possible, and a low dissipation regime in which only the normal phase is stable in the thermodynamic limit.

Interestingly, the transition between these two regimes of parameters when Γ is increased is quite sharp, especially in the thermodynamic limit. To visualize this, we study the stability of the superradiant phase versus both *g* and Γ , for 100 qubits, and for various values of ω_0 , the other parameters being the same (this amounts to taking horizontal slices in Fig. 2 and study their evolution when Γ changes).

The results are displayed in Fig. 3: for $\Gamma/\omega_c \approx 1.6$, the instability disappears and the superradiant phase becomes stable for most values of ω_0 and g. Hence, the phase diagram as a whole changes drastically when Γ/ω_c goes across this threshold.

Hence, we have established that the presence of dissipation is instrumental in stabilizing the superradiant phase. If we compare this with the results obtained in the one-photon version of the driven-dissipative model²⁵, an instructive analogy can be made. Adding enough qubit dissipation appears to preserve the superradiant phase transition, which normally would be destroyed in the presence of noise. In the one-photon case, however, decay and dephasing can play antagonistic roles: adding an infinitesimal amount of qubit dephasing without decay destroys the transition, while adding both dephasing and decay stabilizes it. To see if such effect is also present in the two-photon model, we study the stability of the superradiant phase with respect to both Γ_{ϕ} and Γ_{\downarrow} . The results are displayed in Fig. 4. We see no evidence of suppression and restoration of the phase transition. Rather, these plots indicate that both dephasing and decay contribute positively to the stabilization of the superradiant phase.

Discussion

In this paper, we present the first analysis of the steady-state phase diagram of a two-photon interaction model in the driven-dissipative case. In particular, we have explored numerically the mean-field behavior of the *N*-body two-photon Dicke model. We have identified a rich behavior, including a superradiant phase transition of first order, a bistable phase, and an instability that is removed by dissipation. Although one may be tempted to interpret this instability as the dissipative counterpart of the spectral collapse, there are important differences between the two phenomena.

In the Hamiltonian case, the spectral collapse is expected to occur when g increases. In our case, for $\Gamma = 3\omega_c$, we have increased g towards higher values, up to 10^3 (not shown). We have found only a stable superradiant phase. To summarize, the spectral collapse is entirely controlled by the coupling, while the instability is controlled by a non-trivial interplay between dissipation and coupling. Another, perhaps even more important, difference is the scaling of the parameters with N. In the Hamiltonian case, the collapse occurs for $g \sim \frac{\omega_c t^{2,60}}{\sqrt{N}}$ (note that the definition of the coupling constant is not the same in our work and these references). The stable dynamics can only take place in the interval $0 \le g \le \frac{\omega_c}{\sqrt{N}}$, which becomes vanishingly small in the thermodynamic limit. As a consequence, a transition can take place only if the qubit frequency ω_0 is also allowed to scale like $1/\sqrt{N^{60}}$.



Figure 4. Stability of the superradiant phase versus qubit dephasing and decay. **a**) $\omega_0/\omega_c = 0.5$, **b**) $\omega_0/\omega_c = 2$, **c**) $\omega_0/\omega_c = 5$. For all plots, $\kappa/\omega_c = 1$, $g/\omega_c = 1$, and N = 100 qubits. S,B: stable or bistable phase, N,I: normal phase or instability. Here, both dephasing and decay contribute positively to the stabilization of the superradiant phase.

By contrast here, the instability, when present, only emerges for $g \sim \omega_c$, which allows a phase transition to take place for all values of ω_0 . Indeed, the phase diagram we have obtained here are quite different from the one of the Hamiltonian model published in⁶⁰. The fact that the spectral collapse and the instability scale differently with the number of atoms suggests that they could be qualitatively different processes.

Hence, the two main properties of the two-photon Dicke Hamiltonian, second-order phase transition and spectral collapse, are qualitatively modified in the presence of dissipation. This illustrates how the critical behavior of a given phase transition can change radically when one goes from the equilibrium case to the non-equilibrium one.

Furthermore, we have pointed out conceptual differences between the behavior of the one- and two-photon Dicke models. For the latter, both local dephasing and decay appear to help stabilize the transition, in contrast to what was found in the dissipative one-photon Dicke model. In the thermodynamic limit of the two-photon Dicke model, the entire phase diagram changes very abruptly when dissipation is moved across a very narrow range of parameters.

We note that while a mean-field approach can predict several solutions for the steady-state equation in the bistable region, and a unique solution in the mono-stable regions, in a finite-size quantum system there can only be a unique steady-state in each region. Indeed, the introduction of both quantum and classical fluctuations prevents the fields from remaining stationary around their mean-field values. Several works^{73,84} have illustrated these behaviours in the similar case of the two-photon Kerr resonator. In turn, the presence of multiple solutions at the mean-field level is translated into an observable bistable behavior of the critical parameters of the full quantum model in a quantum trajectory approach⁸⁵.

Indeed, for future perspectives, a full quantum treatment of the driven-dissipative two-photon Dicke model would be an interesting and yet challenging task. Extracting information on the thermodynamic limit from direct numerical simulations of the full dynamics is far from straightforward, due to the exponentially increasing Liouvillian space. Exploiting the permutational invariance of the Liouvillian²⁹ can exponentially reduce the computational overhead with regard to the qubit degrees of freedom, but the photonic subspace, approximated by a cut-off photon excitation number $n_{\rm ph}$, needs to be larger than in the case of the single-photon Dicke model to avoid spurious results induced by the finite approximation of the otherwise unbounded Hilbert space. With this regard, two possible solutions could be considered, also simultaneously: first, the inclusion a two-photon dissipation process, which would effectively reduce the highest excitation number explored, for appropriately large values of the two-photon decay rate; second, the use of quantum trajectories, which reduces the computational overhead from being that of the Liouvillian space to just that of an effective Hilbert space, at the cost of averaging over many runs. We point out though that the intermittent dynamics characterizing a bi-stable phase can be grasped even by single quantum trajectory simulations. Alternatively to these approaches, a qubit-only description of the system could be obtained at the cost of abandoning the Lindbladian formalism for a full Redfield theory⁸⁶.

From an experimental perspective, we have shown that in the driven-dissipative case the superradiant phase transition of the two-photon Dicke model can be observed also for resonant interactions, and does not require large detuning as in the static case.

Hence, this quantum phase transition could be observed in a more accessible regime of parameters than previously thought. The driven-dissipative two-photon Dicke model could be implemented with trapped ions⁴² or in a cold atoms setup, similar to what has been done already for the driven Rabi model^{87,88}. Superconducting circuits⁴⁸ provide another platform to simulate this dynamics, in which the two-photon interaction can be engineered between a flux qubit and a SQUID resonator^{44,45}.

Methods

From the Lindblad master equation (6), we can obtain the field equation governing the dynamics of any operator $\partial_t \langle \hat{A} \rangle$. For the operators \hat{X} , \hat{Y} , $\hat{a}^{\dagger} \hat{a}$, \hat{J}_x , \hat{J}_y , \hat{J}_z that we have studied, this gives:

$$\begin{cases} \partial_t \langle X \rangle = -\kappa \langle X \rangle - 2i\omega_c \langle Y \rangle, \\ \partial_t \langle \hat{Y} \rangle = -\kappa \langle \hat{Y} \rangle - 2i\omega_c \langle \hat{X} \rangle - 4ig\sqrt{N} \langle \hat{J}_x \rangle - 8ig\sqrt{N} \langle \hat{J}_x \hat{a}^{\dagger} \hat{a} \rangle, \\ \partial_t \langle \hat{a}^{\dagger} \hat{a} \rangle = 2ig\sqrt{N} \langle \hat{J}_x \hat{Y} \rangle - \kappa \langle \hat{a}^{\dagger} \hat{a} \rangle, \\ \partial_t \langle \hat{J}_x \rangle = -2\omega_0 \langle \hat{J}_y \rangle - \Gamma' \langle \hat{J}_x \rangle, \\ \partial_t \langle \hat{J}_y \rangle = 2\omega_0 \langle \hat{J}_x \rangle - \Gamma' \langle \hat{J}_y \rangle - \frac{2g}{\sqrt{N}} \langle \hat{J}_z \hat{X} \rangle, \\ \partial_t \langle \hat{J}_z \rangle = \frac{2g}{\sqrt{N}} \langle \hat{J}_y \hat{X} \rangle - \Gamma_{\downarrow} \langle \hat{J}_z \rangle - \Gamma_{\downarrow}, \end{cases}$$
(8)

where we have defined $\Gamma' = 2\Gamma_{\phi} + \frac{\Gamma_{\downarrow}}{2}$. The solution of Eq. (8) is, in general, a formidable task. If one is interested in the properties of the steady-state, however, the time derivatives can be set to zero. This approximation is not sufficient to solve Eq. (8), since some operators are a function of higher-order correlation functions, thus resulting in an infinite hierarchy of coupled equations. In the normal phase (i.e., when no symmetry is broken), one can reduce the complexity of the problem by considering $\kappa \gg \Gamma_{\downarrow} \simeq \Gamma_{\phi}$, i.e., that the bosonic field reaches a steady-state long before the qubits do. Indeed, in this region the Liouvillian gap must be opend¹⁹, and in the absence of critical-slowing down the typical timescale is dictated by the dissipation rates. Using adiabatic elimination, one can easily find the behavior of the system close to the normal phase characterised by $\langle \hat{X} \rangle = \langle \hat{I}_x \rangle = \langle \hat{J}_y \rangle = 0$, while $\langle \hat{J}_z \rangle = -1$. The results of adiabatic elimination, however, fail to capture the superradiant phase: in this regime, the diverging timescale coming from the closure of the Liouvillian gap makes the photonic timescale comparable to the qubit one.

In order to truncate the hierarchy of equations stemming from a Liouvillian problem, in many-body quantum physics one often resorts to a Gutzwiller mean-field approximation. In this case, one assumes that the system density matrix can be factorised as a tensor product between the qubit and photonic part, decoupling all the high-order correlation in Eq. (8). For instance, we will assume that $\langle \hat{J}_x \hat{a}^{\dagger} \hat{a} \rangle = \langle \hat{J}_x \rangle \langle \hat{a}^{\dagger} \hat{a} \rangle$. Note that in general, the mean-field approximation can lead to incorrect predictions in the presence of strong quantum correlations. It is expected to be true for high-dimensional models (i.e., when the number of nearest neighbors is elevated) and in the thermodynamic limit $N \to \infty$. In Dicke-like models, all spins are coupled to the mode, which induces an effective all-to-all coupling. In the thermodynamic limit, the model is effectively infinite-dimensional, and the qubits act as a collective, classical spin. Furthermore, in the superradiant phase, the fluctuations of both the spins and the field are negligible compared to their mean-field values⁸⁹. As a consequence, it was shown that the mean-field approximation gives indeed correct results for Dicke-like models with large $N^{26,65}$. Under the mean-field approximation, Eq. (8) becomes

$$\begin{cases} \partial_t \langle \hat{X} \rangle = -\kappa \langle \hat{X} \rangle - 2i\omega_c \langle \hat{Y} \rangle, \\ \partial_t \langle \hat{Y} \rangle = -\kappa \langle \hat{Y} \rangle - 2i\omega_c \langle \hat{X} \rangle - 4ig\sqrt{N} \langle \hat{J}_x \rangle - 8ig\sqrt{N} \langle \hat{J}_x \rangle \langle \hat{a}^{\dagger} \hat{a} \rangle, \\ \partial_t \langle \hat{a}^{\dagger} \hat{a} \rangle = 2ig\sqrt{N} \langle \hat{J}_x \rangle \langle \hat{Y} \rangle - \kappa \langle \hat{a}^{\dagger} \hat{a} \rangle, \\ \partial_t \langle \hat{J}_x \rangle = -2\omega_0 \langle \hat{J}_y \rangle - \Gamma' \langle \hat{J}_x \rangle, \\ \partial_t \langle \hat{J}_y \rangle = 2\omega_0 \langle \hat{J}_x \rangle - \Gamma' \langle \hat{J}_y \rangle - \frac{2g}{\sqrt{N}} \langle \hat{J}_z \rangle \langle \hat{X} \rangle, \\ \partial_t \langle \hat{J}_z \rangle = \frac{2g}{\sqrt{N}} \langle \hat{J}_y \rangle \langle \hat{X} \rangle - \Gamma_{\downarrow} \langle \hat{J}_z \rangle - \Gamma_{\downarrow}. \end{cases}$$
(9)

One solution to this equation is $\langle \hat{X} \rangle = \langle \hat{Y} \rangle = \langle \hat{a}^{\dagger} \hat{a} \rangle = \langle \hat{J}_{x} \rangle = \langle \hat{J}_{y} \rangle = 0$, while $\langle \hat{J}_{z} \rangle = -1$. That is, the normal phase with no photons is always a solution to Eq. (9). However, there are other two solutions to this equation, where the bosonic field is populated. Namely:

$$\begin{cases} \langle \hat{X} \rangle_{\rm ss} = \frac{8g\omega_c \sqrt{N} \langle \hat{J}_x \rangle_{\rm ss}}{-(\kappa^2 + 4\omega_c^2) + 16g^2 N \langle \hat{J}_x \rangle_{\rm ss}^2}, \\ \langle \hat{Y} \rangle_{\rm ss} = \frac{i\kappa}{2\omega_c} \langle \hat{X} \rangle_{\rm ss}, \\ \langle \hat{a}^{\dagger} \hat{a} \rangle_{\rm ss} = \frac{2ig\sqrt{N}}{\kappa} \langle \hat{J}_x \rangle_{\rm ss} \langle \hat{Y} \rangle_{\rm ss}, \\ \langle \hat{J}_x \rangle_{\rm ss} = \pm \sqrt{\frac{\kappa^2 + 4\omega_c^2}{16g^2 N} + \frac{2\omega_0 \omega_c}{N(4\omega_0^2 + \Gamma'^2)}} \langle \hat{J}_z \rangle_{\rm ss}, \\ \langle \hat{J}_y \rangle_{\rm ss} = -\frac{\Gamma'}{2\omega_0} \langle \hat{J}_x \rangle_{\rm ss}, \\ \langle \hat{J}_z \rangle_{\rm ss} = -\frac{1 + \beta}{2} + \sqrt{\left(\frac{1 + \beta}{2}\right)^2 - \beta \frac{g_t^2}{g^2}}, \end{cases}$$
(10)

where we have introduced $\beta = \frac{\omega_c \Gamma'}{2\omega_0 N \Gamma_{\downarrow}}$ and $g_t = \sqrt{\left(2\omega_c + \frac{\kappa^2}{2\omega_c}\right)\left(2\omega_0 + \frac{\Gamma'^2}{2\omega_0}\right)/8}$.

Having identified the three possible solutions, we study their stability by considering a linear perturbation of the steady-state value:

$$\vec{\mathbf{A}} = \left\{ \langle \hat{X} \rangle, \langle \hat{Y} \rangle, \langle \hat{a}^{\dagger} \hat{a} \rangle, \langle \hat{J}_{x} \rangle, \langle \hat{J}_{y} \rangle, \langle \hat{J}_{z} \rangle \right\}^{\mathrm{T}} = \vec{\mathbf{A}}_{\mathrm{ss}} + \delta \vec{\mathbf{A}}.$$
(11)

For the normal phase, we have

$$\partial_t \vec{\mathbf{A}} = \partial_t (\delta \vec{\mathbf{A}}) = M_{\rm N} \delta \vec{\mathbf{A}} = \begin{bmatrix} -\kappa & -2i\omega_c & 0 & 0 & 0 & 0 \\ -2i\omega_c & -\kappa & 0 & -4ig\sqrt{N} & 0 & 0 \\ 0 & 0 & -\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Gamma' & -2\omega_0 & 0 \\ \frac{2g}{\sqrt{N}} & 0 & 0 & 2\omega_0 & -\Gamma' & 0 \\ 0 & 0 & 0 & 0 & 0 & -\Gamma_{\downarrow} \end{bmatrix} \delta \vec{\mathbf{A}},$$
(12)

while for the superradiant phase

$$\partial_{t}\vec{\mathbf{A}} = \partial_{t}(\delta\vec{\mathbf{A}}) = M_{S}\delta\vec{\mathbf{A}} = \begin{bmatrix} -\kappa & -2i\omega_{c} & 0 & 0 & 0 & 0 \\ -2i\omega_{c} & -\kappa & -8ig\sqrt{N}\langle\hat{J}_{x}\rangle_{s} & -4ig\sqrt{N}(1+2\langle\hat{a}^{\dagger}\hat{a}\rangle_{s}) & 0 & 0 \\ 0 & 2ig\sqrt{N}\langle\hat{J}_{x}\rangle_{s} & -\kappa & 2ig\sqrt{N}\langle\hat{Y}\rangle_{s} & 0 & 0 \\ 0 & 0 & 0 & -\Gamma' & -2\omega_{0} & 0 \\ -\frac{2g}{\sqrt{N}}\langle\hat{J}_{z}\rangle_{s} & 0 & 0 & 2\omega_{0} & -\Gamma' & -\frac{2g}{\sqrt{N}}\langle\hat{X}\rangle_{s} \\ \frac{2g}{\sqrt{N}}\langle\hat{J}_{y}\rangle_{s} & 0 & 0 & 0 & \frac{2g}{\sqrt{N}}\langle\hat{X}\rangle_{s} & -\Gamma_{\downarrow} \end{bmatrix} \delta\vec{\mathbf{A}}.$$
(13)

Only if all the eigenvalues of $M_N(M_S)$ are negative, the normal (superradiant) phase is stable. We have obtained these eigenvalues numerically to study the phase diagram of the system.

Note that in the thermodynamic limit $N \to \infty$, the solution (10) can be simplified as:

$$\begin{aligned} \langle \hat{X} \rangle_{\rm ss} &= \frac{-2\omega_c}{\beta \sqrt{\kappa^2 + 4\omega_c^2}} \\ \langle \hat{Y} \rangle_{\rm ss} &= \frac{i\kappa}{2\omega_c} \langle \hat{X} \rangle_{\rm ss}, \\ \langle \hat{a}^{\dagger} \hat{a} \rangle_{\rm ss} &= \frac{1}{2\beta}, \\ \langle \hat{J}_x \rangle_{\rm ss} &= \pm \sqrt{\frac{\kappa^2 + 4\omega_c^2}{16g^2 N}}, \\ \langle \hat{J}_y \rangle_{\rm ss} &= -\frac{\Gamma'}{2\omega_0} \langle \hat{J}_x \rangle_{\rm ss}, \\ \langle \hat{J}_z \rangle_{\rm ss} &= -\beta \frac{g_c^2}{g_c^2}, \end{aligned}$$
(14)

The stability matrix M_S can be correspondingly simplified. However, even in this case, the dependence in N does not disappear; furthermore, exact solutions are still out of reach. We have been able to check, however, that these exact results, for large N, showed consistency with the thermodynamic limit expressions (in particular, in both cases, the superradiant phase does disappear for small dissipation).

Received: 9 December 2019; Accepted: 13 July 2020 Published online: 07 August 2020

References

- Dicke, R. H. Coherence in spontaneous radiation processes. *Phys. Rev.* 93, 99–110. https://doi.org/10.1103/PhysRev.93.99 (1954).
 Hepp, K. & Lieb, E. H. Equilibrium statistical mechanics of matter interacting with the quantized radiation field. *Phys. Rev. A* 8, 2517–2525. https://doi.org/10.1103/PhysRevA.8.2517 (1973).
- Emary, C. & Brandes, T. Chaos and the quantum phase transition in the Dicke model. *Phys. Rev. E* 67, 066203. https://doi. org/10.1103/PhysRevE.67.066203 (2003).
- Lambert, N., Emary, C. & Brandes, T. Entanglement and the phase transition in single-mode superradiance. *Phys. Rev. Lett.* 92, 073602. https://doi.org/10.1103/PhysRevLett.92.073602 (2004).
- Kirton, P., Roses, M. M., Keeling, J. & Dalla Torre, E. G. Introduction to the Dicke model: from equilibrium to nonequilibrium, and vice versa. Adv. Quantum Technol. 2, 1800043. https://doi.org/10.1002/qute.201800043 (2019).
- Nataf, P. & Ciuti, C. No-go theorem for superradiant quantum phase transitions in cavity QED and counter-example in circuit QED. Nat. Commun. 1, 72. https://doi.org/10.1038/ncomms1069 (2010).
- De Liberato, S. Light-matter decoupling in the deep strong coupling regime: the breakdown of the Purcell effect. *Phys. Rev. Lett.* 112, 016401. https://doi.org/10.1103/PhysRevLett.112.016401 (2014).
- De Bernardis, D., Pilar, P., Jaako, T., De Liberato, S. & Rabl, P. Breakdown of gauge invariance in ultrastrong-coupling cavity QED. Phys. Rev. A 98, 053819. https://doi.org/10.1103/PhysRevA.98.053819 (2018).
- Di Stefano, O. et al. Resolution of gauge ambiguities in ultrastrong-coupling cavity quantum electrodynamics. Nat. Phys. https:// doi.org/10.1038/s41567-019-0534-4 (2019).
- Baumann, K., Guerlin, C., Brennecke, F. & Esslinger, T. Dicke quantum phase transition with a superfluid gas in an optical cavity. *Nature* 464, 1301–1306. https://doi.org/10.1038/nature09009 (2010).
- 11. Klinder, J., Keßler, H., Wolke, M., Mathey, L. & Hemmerich, A. Dynamical phase transition in the open Dicke model. *PNAS* **112**, 3290–3295. https://doi.org/10.1073/pnas.1417132112 (2015).
- 12. Zhiqiang, Z. *et al.* Nonequilibrium phase transition in a spin-1 Dicke model. *OPTICA* 4, 424–429. https://doi.org/10.1364/OPTIC A.4.000424 (2017).
- Aedo, I. & Lamata, L. Analog quantum simulation of generalized Dicke models in trapped ions. *Phys. Rev. A* 97, 042317. https:// doi.org/10.1103/PhysRevA.97.042317 (2018).
- Safavi-Naini, A. et al. Verification of a many-ion simulator of the Dicke Model through slow quenches across a phase transition. Phys. Rev. Lett. 121, 040503. https://doi.org/10.1103/PhysRevLett.121.040503 (2018).
- Zou, L. et al. Implementation of the Dicke lattice model in hybrid quantum system arrays. Phys. Rev. Lett. 113, 023603. https:// doi.org/10.1103/PhysRevLett.113.023603 (2014).
- Lamata, L. Digital-analog quantum simulation of generalized Dicke models with superconducting circuits. Sci. Rep. 7, 43768. https://doi.org/10.1038/srep43768 (2017).
- 17. Buluta, I. & Nori, F. Quantum simulators. Science 326, 108-111. https://doi.org/10.1126/science.1177838 (2009).
- Georgescu, I., Ashhab, S. & Nori, F. Quantum simulation. Rev. Mod. Phys. 86, 153–185. https://doi.org/10.1103/RevModPhys .86.153 (2014).

- Minganti, F., Biella, A., Bartolo, N. & Ciuti, C. Spectral theory of Liouvillians for dissipative phase transitions. *Phys. Rev. A* 98, 042118. https://doi.org/10.1103/PhysRevA.98.042118 (2018).
- 20. Kessler, E. M. et al. Dissipative phase transition in a central spin system. Phys. Rev. A 86, 012116. https://doi.org/10.1103/PhysR evA.86.012116 (2012).
- Liu, N. *et al.* Light-shift-induced quantum phase transitions of a Bose-Einstein condensate in an optical cavity. *Phys. Rev. A* 83, 033601. https://doi.org/10.1103/PhysRevA.83.033601 (2011).
- Dalla Torre, E. G., Diehl, S., Lukin, M. D., Sachdev, S. & Strack, P. Keldysh approach for nonequilibrium phase transitions in quantum optics: beyond the Dicke model in optical cavities. *Phys. Rev. A* 87, 023831. https://doi.org/10.1103/PhysRevA.87.02383 1 (2013).
- Zhao, X., Liu, N. & Liang, J.-Q. Nonlinear atom-photon interaction induced population inversion and inverted quantum phase transition of bose-einstein condensate in an optical cavity. *Phys. Rev. A* 90, 023622. https://doi.org/10.1103/PhysRevA.90.023622 (2014).
- 24. Sieberer, L. M., Buchhold, M. & Diehl, S. Keldysh field theory for driven open quantum systems. *Rep. Prog. Phys.* **79**, 096001. https://doi.org/10.1088/0034-4885/79/9/096001 (2016).
- Kirton, P. & Keeling, J. Suppressing and restoring the Dicke superradiance transition by dephasing and decay. *Phys. Rev. Lett.* 118, 123602. https://doi.org/10.1103/PhysRevLett.118.123602 (2017).
- Kirton, P. & Keeling, J. Superradiant and lasing states in driven-dissipative Dicke models. New J. Phys. 20, 015009. https://doi.org/10.1088/1367-2630/aaa11d (2018).
- 27. Hwang, M.-J., Rabl, P. & Plenio, M. B. Dissipative phase transition in the open quantum Rabi model. *Phys. Rev. A* 97, 013825. https://doi.org/10.1103/PhysRevA.97.013825 (2018).
- 28. Breuer, H. & Petruccione, F. The theory of open quantum systems (Oxford University Press, Oxford, 2007).
- Shammah, N., Ahmed, S., Lambert, N., De Liberato, S. & Nori, F. Open quantum systems with local and collective incoherent processes: efficient numerical simulations using permutational invariance. *Phys. Rev. A* 98, 063815. https://doi.org/10.1103/PhysR evA.98.063815 (2018).
- 30. Marković, D. *et al.* Demonstration of an effective ultrastrong coupling between two oscillators. *Phys. Rev. Lett.* **121**, 040505. https://doi.org/10.1103/PhysRevLett.121.040505 (2018).
- Poyatos, J. F., Cirac, J. I. & Zoller, P. Quantum reservoir engineering with laser cooled trapped ions. *Phys. Rev. Lett.* 77, 4728–4731. https://doi.org/10.1103/PhysRevLett.77.4728 (1996).
- Verstraete, F., Wolf, M. M. & Cirac, J. I. Quantum computation and quantum-state engineering driven by dissipation. *Nat. Phys.* 5, 633. https://doi.org/10.1038/nphys1342 (2009).
- Tan, H., Li, G. & Meystre, P. Dissipation-driven two-mode mechanical squeezed states in optomechanical systems. *Phys. Rev. A* 87, 033829. https://doi.org/10.1103/PhysRevA.87.033829 (2013).
- Arenz, C., Cormick, C., Vitali, D. & Morigi, G. Generation of two-mode entangled states by quantum reservoir engineering. J. Phys. B At. Mol. Opt. Phys. 46, 224001. https://doi.org/10.1088/0953-4075/46/22/224001 (2013).
- Asjad, M. & Vitali, D. Reservoir engineering of a mechanical resonator: generating a macroscopic superposition state and monitoring its decoherence. J. Phys. B At. Mol. Opt. Phys. 47, 045502. https://doi.org/10.1088/0953-4075/47/4/045502 (2014).
- Roy, A., Leghtas, Z., Stone, A. D., Devoret, M. & Mirrahimi, M. Continuous generation and stabilization of mesoscopic field superposition states in a quantum circuit. *Phys. Rev. A* 91, 013810. https://doi.org/10.1103/PhysRevA.91.013810 (2015).
- 37. Leghtas, Z. et al. Confining the state of light to a quantum manifold by engineered two-photon loss. Science 347, 853-857. https://doi.org/10.1126/science.aaa2085 (2015).
- Lee, T. E., Gopalakrishnan, S. & Lukin, M. D. Unconventional magnetism via optical pumping of interacting spin systems. *Phys. Rev. Lett.* 110, 257204. https://doi.org/10.1103/PhysRevLett.110.257204 (2013).
- 39. Jin, J. et al. Cluster mean-field approach to the steady-state phase diagram of dissipative spin systems. *Phys. Rev. X* 6, 031011. https://doi.org/10.1103/PhysRevX.6.031011 (2016).
- Diehl, S. et al. Quantum states and phases in driven open quantum systems with cold atoms. Nat. Phys. 4, 878. https://doi.org/10.1038/nphys1073 (2008).
- Genway, S., Li, W., Ates, C., Lanyon, B. P. & Lesanovsky, I. Generalized Dicke nonequilibrium dynamics in trapped ions. *Phys. Rev. Lett.* 112, 023603. https://doi.org/10.1103/PhysRevLett.112.023603 (2014).
- Felicetti, S. *et al.* Spectral collapse via two-phonon interactions in trapped ions. *Phys. Rev. A* 92, 033817. https://doi.org/10.1103/ PhysRevA.92.033817 (2015).
- Wang, X., Miranowicz, A., Li, H.-R. & Nori, F. Multiple-output microwave single-photon source using superconducting circuits with longitudinal and transverse couplings. *Phys. Rev. A* 94, 053858. https://doi.org/10.1103/PhysRevA.94.053858 (2016).
- Felicetti, S., Rossatto, D. Z., Rico, E., Solano, E. & Forn-Díaz, P. Two-photon quantum Rabi model with superconducting circuits. *Phys. Rev. A* 97, 013851. https://doi.org/10.1103/PhysRevA.97.013851 (2018).
- Felicetti, S., Hwang, M.-J. & Le Boité, A. Ultrastrong-coupling regime of nondipolar light-matter interactions. *Phys. Rev. A* 98, 053859. https://doi.org/10.1103/PhysRevA.98.053859 (2018).
- Frisk Kockum, A., Miranowicz, A., De Liberato, S., Savasta, S. & Nori, F. Ultrastrong coupling between light and matter. *Nat. Rev. Phys.* 1, 19. https://doi.org/10.1038/s42254-018-0006-2 (2019).
- Forn-Díaz, P., Lamata, L., Rico, E., Kono, J. & Solano, E. Ultrastrong coupling regimes of light-matter interaction. *Rev. Mod. Phys.* 91, 025005. https://doi.org/10.1103/RevModPhys.91.025005 (2019).
- Gu, X., Kockum, A. F., Miranowicz, A., Liu, Y.-X. & Nori, F. Microwave photonics with superconducting quantum circuits. *Phys. Rep.* 718–719, 1–102. https://doi.org/10.1016/j.physrep.2017.10.002 (2017).
- Garziano, L. et al. Multiphoton quantum Rabi oscillations in ultrastrong cavity QED. Phys. Rev. A 92, 063830. https://doi. org/10.1103/PhysRevA.92.063830 (2015).
- Kockum, A. F., Miranowicz, A., Macri, V., Savasta, S. & Nori, F. Deterministic quantum nonlinear optics with single atoms and virtual photons. *Phys. Rev. A* 95, 063849. https://doi.org/10.1103/PhysRevA.95.063849 (2017).
- Kockum, A. F., Macri, V., Garziano, L., Savasta, S. & Nori, F. Frequency conversion in ultrastrong cavity QED. Sci. Rep. 7, 5313. https://doi.org/10.1038/s41598-017-04225-3 (2017).
- 52. Stassi, R. et al. Quantum nonlinear optics without photons. Phys. Rev. A 96, 023818. https://doi.org/10.1103/PhysRevA.96.02381 8 (2017).
- Settineri, A. *et al.* Dissipation and thermal noise in hybrid quantum systems in the ultrastrong-coupling regime. *Phys. Rev. A* 98, 053834. https://doi.org/10.1103/PhysRevA.98.053834 (2018).
- Kerckhoff, J., Armen, M. A. & Mabuchi, H. Remnants of semiclassical bistability in the few-photon regime of cavity QED. Opt. Express 19, 24468–24482. https://doi.org/10.1364/OE.19.024468 (2011).
- Minganti, F., Bartolo, N., Lolli, J., Casteels, W. & Ciuti, C. Exact results for Schrödinger cats in driven-dissipative systems and their feedback control. Sci. Rep. 6, 26987. https://doi.org/10.1038/srep26987 (2016).
- Savona, V. Spontaneous symmetry breaking in a quadratically driven nonlinear photonic lattice. *Phys. Rev. A* 96, 033826. https:// doi.org/10.1103/PhysRevA.96.033826 (2017).
- Ourjoumtsev, A., Tualle-Brouri, R., Laurat, J. & Grangier, P. Generating optical Schrödinger kittens for quantum information processing. *Science* 312, 83–86. https://doi.org/10.1126/science.1122858 (2006).

- Mirrahimi, M. et al. Dynamically protected cat-qubits: a new paradigm for universal quantum computation. N. J. Phys. 16, 045014. https://doi.org/10.1088/1367-2630/16/4/045014 (2014).
- Puri, S., Boutin, S. & Blais, A. Engineering the quantum states of light in a Kerr-nonlinear resonator by two-photon driving. NPJ Quantum Inf. 3, 18. https://doi.org/10.1038/s41534-017-0019-1 (2017).
- 60. Garbe, L. et al. Superradiant phase transition in the ultrastrong-coupling regime of the two-photon Dicke model. *Phys. Rev. Ahttps*://doi.org/10.1103/PhysRevA.95.053854 (2017).
- Ng, K. M., Lo, C. F. & Liu, K. L. Exact eigenstates of the intensity-dependent Jaynes-Cummings model with the counter-rotating term. Phys. A Stat. Mech. Appl. 275, 463–474. https://doi.org/10.1016/S0378-4371(99)00401-X (2000).
- Emary, C. & Bishop, R. F. Exact isolated solutions for the two-photon Rabi Hamiltonian. J. Phys. A: Math. Gen. 35, 8231–8241. https://doi.org/10.1088/0305-4470/35/39/307 (2002).
- Chen, X.-Y. & Zhang, Y.-Y. Finite-size scaling analysis in the two-photon Dicke model. *Phys. Rev. A* 97, 053821. https://doi. org/10.1103/PhysRevA.97.053821 (2018).
- Cui, S. et al. Two-photon Rabi-Hubbard and Jaynes-Cummings-Hubbard models: Photon-pair superradiance, mott insulator, and normal phases. Phys. Rev. A 100, 033608. https://doi.org/10.1103/PhysRevA.100.033608 (2019).
- Shammah, N., Lambert, N., Nori, F. & De Liberato, S. Superradiance with local phase-breaking effects. *Phys. Rev. A* 96, 023863. https://doi.org/10.1103/PhysRevA.96.023863 (2017).
- Hayn, M., Emary, C. & Brandes, T. Phase transitions and dark-state physics in two-color superradiance. *Phys. Rev. A* 84, 053856. https://doi.org/10.1103/PhysRevA.84.053856 (2011).
- Puebla, R., Hwang, M.-J., Casanova, J. & Plenio, M. B. Probing the dynamics of a superradiant quantum phase transition with a single trapped ion. *Phys. Rev. Lett.* 118, 073001. https://doi.org/10.1103/PhysRevLett.118.073001 (2017).
- 68. Cheng, X.-H. *et al.* Nonlinear quantum Rabi model in trapped ions. *Phys. Rev. A* 97, 023624. https://doi.org/10.1103/PhysR evA.97.023624 (2018).
- Dalla Torre, E. G., Shchadilova, Y., Wilner, E. Y., Lukin, M. D. & Demler, E. Dicke phase transition without total spin conservation. *Phys. Rev. A* 94, 061802. https://doi.org/10.1103/PhysRevA.94.061802 (2016).
- Albert, V. V. & Jiang, L. Symmetries and conserved quantities in Lindblad master equations. *Phys. Rev. A* 89, 022118. https://doi. org/10.1103/PhysRevA.89.022118 (2014).
- 71. De Liberato, S. Virtual photons in the ground state of a dissipative system. *Nat. Commun.* **8**, 1465. https://doi.org/10.1038/s4146 7-017-01504-5 (2017).
- Drummond, P. D. & Walls, D. F. Quantum theory of optical bistability. I. Nonlinear polarisability model. J. Phys. A Math. General 13, 725. https://doi.org/10.1088/0305-4470/13/2/034 (1980).
- Bartolo, N., Minganti, F., Casteels, W. & Ciuti, C. Exact steady state of a Kerr resonator with one- and two-photon driving and dissipation: Controllable wigner-function multimodality and dissipative phase transitions. *Phys. Rev. A* 94, 033841. https://doi. org/10.1103/PhysRevA.94.033841 (2016).
- 74. Landa, H., Schiró, M. & Misguich, G. Multistability of driven-dissipative quantum spins. arXiv:1905.10349 (2019).
- Biondi, M., Blatter, G., Türeci, H. E. & Schmidt, S. Nonequilibrium gas-liquid transition in the driven-dissipative photonic lattice. *Phys. Rev. A* 96, 043809. https://doi.org/10.1103/PhysRevA.96.043809 (2017).
- Biella, A. et al. Phase diagram of incoherently driven strongly correlated photonic lattices. Phys. Rev. A 96, 023839. https://doi. org/10.1103/PhysRevA.96.023839 (2017).
- Jin, J. et al. Phase diagram of the dissipative quantum Ising model on a square lattice. Phys. Rev. B 98, 241108. https://doi. org/10.1103/PhysRevB.98.241108 (2018).
- Foss-Feig, M. et al. Emergent equilibrium in many-body optical bistability. Phys. Rev. A 95, 043826. https://doi.org/10.1103/PhysR evA.95.043826 (2017).
- Vicentini, F., Minganti, F., Rota, R., Orso, G. & Ciuti, C. Critical slowing down in driven-dissipative Bose-Hubbard lattices. *Phys. Rev. A* 97, 013853. https://doi.org/10.1103/PhysRevA.97.013853 (2018).
- Rota, R., Minganti, F., Ciuti, C. & Savona, V. Quantum critical regime in a quadratically driven nonlinear photonic lattice. *Phys. Rev. Lett.* 122, 110405. https://doi.org/10.1103/PhysRevLett.122.110405 (2019).
- Le Boité, A., Orso, G. & Ciuti, C. Steady-state phases and tunneling-induced instabilities in the driven dissipative Bose-Hubbard model. *Phys. Rev. Lett.* 110, 233601. https://doi.org/10.1103/PhysRevLett.110.233601 (2013).
- Le Boité, Á., Orso, G. & Ciuti, C. Bose-Hubbard model: Relation between driven-dissipative steady states and equilibrium quantum phases. *Phys. Rev. A* 90, 063821. https://doi.org/10.1103/PhysRevA.90.063821 (2014).
- Gelhausen, J. & Buchhold, M. Dissipative dicke model with collective atomic decay: Bistability, noise-driven activation, and the nonthermal first-order superradiance transition. *Phys. Rev. A* 97, 023807. https://doi.org/10.1103/PhysRevA.97.023807 (2018).
- Bartolo, N., Minganti, F., Lolli, J. & Ciuti, C. Homodyne versus photon-counting quantum trajectories for dissipative Kerr resonators with two-photon driving. *Eur. Phys. J.* ST 226, 2705–2713. https://doi.org/10.1140/epist/e2016-60385-8 (2017).
- Minganti, F. *Out-of-Equilibrium Phase Transitions in Nonlinear Optical Systems*. PhD thesis, Université Sorbonne Paris Cité (2018).
 Damanet, F., Daley, A. J. & Keeling, J. Atom-only descriptions of the driven-dissipative Dicke model. *Phys. Rev. A* 99, 033845. https
- ://doi.org/10.1103/PhysRevA.99.033845 (2019). 87. Schneeweiss, P., Dareau, A. & Sayrin, C. Cold-atom-based implementation of the quantum Rabi model. *Phys. Rev. A* **98**, https://
- doi.org/10.1103/PhysRevA.98.021801 (2018).
 88. Dareau, A., Meng, Y., Schneeweiss, P. & Rauschenbeutel, A. Observation of ultrastrong spin-motion coupling for cold atoms in optical microtraps. *Phys. Rev. Lett.* https://doi.org/10.1103/PhysRevLett.121.253603 (2018).
- Larson, J. & Irish, E. K. Some remarks on 'superradiant' phase transitions in light-matter systems. J. Phys. A: Math. Theor. 50, 174002. https://doi.org/10.1088/1751-8121/aa65dc (2017).

Acknowledgements

N.S., S.F., and L.G. acknowledge hospitality by Marco Genoni and Matteo G.A. Paris in the Quantum Technology Lab at the University of Milan, Italy. F.M. is supported by the FY2018 JSPS Postdoctoral Fellowship for Research in Japan. F.N. is supported in part by: NTT Research, Army Research Office (ARO) (Grant No. W911NF-18-1-0358), Japan Science and Technology Agency (JST) (via the CREST Grant No. JPMJCR1676), Japan Society for the Promotion of Science (JSPS) (via the KAKENHI Grant No. JP20H00134, and the grant JSPS-RFBR Grant No. JPJSBP120194828), and the Grant No. FQXi-IAF19-06 from the Foundational Questions Institute Fund (FQXi), a donor advised fund of the Silicon Valley Community Foundation. S. F. acknowledges support from the European Research Council (ERC-2016-STG-714870).

Author Contributions

All authors contributed to the manuscript. L.G. and P.W. studied the stability of the equations and contributed equally to this work. N.S. and S.F. conceived the work. N.S. and F.M. wrote the codes for the simulations. L.G. produced the plots. F.N. provided funding and supervised the project. All authors reviewed the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to L.G.

Reprints and permissions information is available at www.nature.com/reprints.

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this license, visit http://creativecommons.org/licenses/by/4.0/.

© The Author(s) 2020