Tripartite optomechanical entanglement via optical-dark-mode control

Deng-Gao Lai,[1,*] Ye-Hong Chen,[1,2,†] Wei Qin,[1,‡] Adam Miranowicz,[1,3,§] and Franco Nori[1,2,4,‖]

[1]Theoretical Quantum Physics Laboratory, RIKEN Cluster for Pioneering Research, Wako-shi, Saitama 351-0198, Japan
[2]RIKEN Center for Quantum Computing (RQC), 2-1 Hirosawa, Wako-shi, Saitama 351-0198, Japan
[3]Institute of Spintronics and Quantum Information, Faculty of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland
[4]Physics Department, The University of Michigan, Ann Arbor, Michigan 48109-1040, USA

(Received 20 February 2022; revised 1 May 2022; accepted 3 May 2022; published 10 August 2022)

We propose how to generate a tripartite light-vibration entanglement by controlling an optical dark mode (ODM), which is induced by the coupling of two optical modes to a common vibrational mode. This ODM is decoupled from the vibration, and it can be controlled on demand by employing a synthetic gauge field, which can enable efficient switching between the ODM-unbreaking and ODM-breaking regimes. We find that the tripartite optomechanical entanglement is largely suppressed in the ODM-unbreaking regime, but it is significantly enhanced in the ODM-breaking regime. In particular, the noise robustness of quantum entanglement in the ODM-breaking regime can be more than twice than that in the ODM-unbreaking regime. This study offers a method for protecting and enhancing fragile quantum resources and for constructing noise-tolerant and dark-mode-immune quantum processors and entangled networks.

DOI: 10.1103/PhysRevResearch.4.033112

I. INTRODUCTION

Quantum entanglement, one of the most striking properties of quantum mechanics, corresponds to inseparable quantum correlations shared between distant parties [1]. It is the main resource for many future applications ranging from quantum metrology to computation, and communication [2]. Optomechanical systems [3–5], owing to a great progress in single-phonon manipulation [6–9], quantum ground-state refrigeration [10–18], and quantum squeezing [19–23], have become a powerful platform for achieving bipartite quantum entanglement between, e.g., two cavity-field modes, the mechanical and cavity-field modes, and two mechanical modes [24–37]. Particularly, remarkable advances in engineering bipartite quantum effects have recently been reported. For instance, the quantum entanglement between a cavity field and a mechanical system [38] and between two massive mechanical resonators [39–42] was generated and controlled experimentally.

Beyond bipartite quantum entanglement, multipartite entangled states can offer a more fundamental resource for a wide range of quantum information processing tasks [43–45], and thus, they can be used for future quantum technologies, such as quantum internet and programmable quantum networks [43–45]. However, based on cavity optomechanics, the generation of multipartite optomechanical entanglement (e.g., quantum entanglement of multiple cavity-field modes and a common vibrational mode), and, in particular, its protection in practical devices have not yet been fully revealed.

In recent years, much attention has been drawn to optical dark modes (ODMs) [46–50], which are special coherent superpositions of, e.g., two optical modes coupled to a common vibration in optomechanical systems. Owing to their unique characteristics in decoupling from the mechanical vibration, the ODM can be employed for protecting systems from mechanical dissipation [46], enable high fidelity conversion of optical fields or quantum states [47–49], and realize efficient routing and switching of photons of different wavelengths [50]. In light of the above potential applications and unique characteristics, it is natural to study the influence of the ODM on light-vibration quantum effects, e.g., optomechanical entanglement.

In this paper, we propose how to generate the tripartite light-vibration entanglement by controlling the ODM in a three-mode closed-loop optomechanical system, and reveal its strong robustness against thermal noise. This is realized by utilizing a synthetic gauge field, which provides a possibility of fully controllable switching between the optical-dark-mode-unbreaking (ODMU) and optical-dark-mode-breaking (ODMB) regimes. Specifically, driving the two cavities with phase-correlated lasers results in a synthetic magnetic flux, which induces a synthetic gauge field. This synthetic gauge field with a topological phase has, respectively, been theoretically proposed [51–54] and experimentally demonstrated [55–61] in closed-loop optomechanical systems.

*Corresponding author: denggaolai@foxmail.com
†yehong.chen@riken.jp
‡wei.qin@riken.jp
§miran@amu.edu.pl
‖fnori@riken.jp

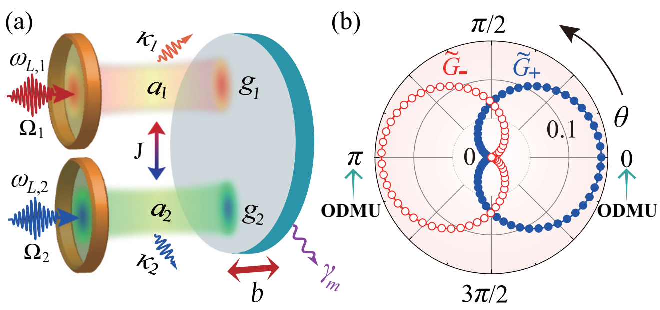

FIG. 1. (a) Schematics of a three-mode closed-loop optomechanical system. Two cavities $a_{j=1,2}$ (resonance frequencies $\omega_{c,j}$ and decay rates κ_j) are coupled to a common vibration b (resonance frequency ω_m and decay rate γ_m) via radiation-pressure couplings g_j. The two cavities a_1 and a_2 are coupled to each other through a photon-tunneling interaction with strength J. A monochromatic driving field, with frequency $\omega_{L,j}$ and field amplitude Ω_j, is introduced to drive the jth cavity. (b) In the polar coordinates, the redefined coupling strengths $\tilde{G}_\pm$ is shown versus the modulation phase θ. The ODM of the system emerges at $\theta = n\pi$ (for an integer n), corresponding to the ODMU regime, and it can be broken ($\tilde{G}_\pm \neq 0$) by tuning $\theta \neq n\pi$, corresponding to the ODMB regime. The parameters used here are $G_{0j}/\omega_m = 0.1$ and $J/\omega_m = 0.1$.

We find that in the ODMU regime, tripartite light-vibration entanglement is strongly suppressed by the ODM; while in the ODMB regime, it is significantly enhanced. The physical origin behind these phenomena is that the ODM is decoupled from the vibrational mode, which leads to the suppression of tripartite optomechanical entanglement. However, the breaking of the ODM enhances tripartite entanglement. These results show a clear inspiration for enhancing tripartite optomechanical entanglement by controlling the ODM. In particular, the resulting ODMB entanglement is noise-tolerant, which is up to twice as strong as that in the ODMU regime. Our findings pave a way towards engineering and protecting fragile quantum correlations from the ODM and thermal noise, and offer a possibility of integrating quantum processors.

The rest of the paper is organized as follows. In Sec. II, we present the Hamiltonian of a three-mode closed-loop optomechanical system, derive the corresponding Langevin equations, and obtain their solutions. In Sec. III, we derive the minimum residual contangle, which can be used to quantify the tripartite quantum entanglement, and analyze how to control an ODM. In Sec. IV, we study the tripartite light-motion entanglement by controlling the ODM. In Sec. V, we present some discussions. Finally, we give a brief conclusion in Sec. VI.

II. HAMILTONIAN, LANGEVIN EQUATIONS, AND THEIR SOLUTIONS

We consider a three-mode closed-loop optomechanical system, where two optical modes are optomechanically coupled to a common mechanical resonator, as shown in Fig. 1(a). The two cavity-field modes are coupled to each other via a photon-tunneling interaction with strength J. To manipulate the optical and mechanical degrees of freedom in the system, we assume that the jth cavity is subjected to a monochromatic driving field, with frequency $\omega_{L,j}$ and amplitude $\Omega_j = \sqrt{2\kappa_j P_{L,j}/\hbar\omega_{L,j}}$ where κ_j is the cavity-field decay rate and $P_{L,j}$ is the laser power. In the interaction picture, shutting all the time dependence due to $H_0 = \sum_{j=1}^{2} \omega_{L,j} a_j^\dagger a_j$, the Hamiltonian of the system is (with $\hbar = 1$)

$$H_I = \sum_{j=1}^{2} [\Delta_{c,j} a_j^\dagger a_j + g_j a_j^\dagger a_j (b + b^\dagger)] + \omega_m b^\dagger b + J(a_1 a_2^\dagger + a_2 a_1^\dagger) + \sum_{j=1}^{2} \Omega_j (a_j + a_j^\dagger), \tag{1}$$

where $\Delta_{c,j} = \omega_{c,j} - \omega_{L,j}$ is the driving detuning of the jth cavity. The operators a_j ($a_j^\dagger$) and b ($b^\dagger$) are the annihilation (creation) operators of the jth optical mode and the vibrational mode, respectively. The g_j term describes the light-vibration interactions between the jth optical mode and the vibrational mode. The cavity-field driving, which are applied to the jth cavity, is denoted by the Ω_j terms.

By considering damping and noise effects in the system, the evolution of this system can be described by the quantum Langevin equations

$$\dot{a}_1 = -(\kappa_1 + i\Delta_{c,1})a_1 - ig_1 a_1 (b + b^\dagger) - iJa_2 - i\Omega_1 + \sqrt{2\kappa_1} a_{1,\text{in}}, \tag{2a}$$

$$\dot{a}_2 = -(\kappa_2 + i\Delta_{c,2})a_2 - ig_2 a_2 (b + b^\dagger) - iJa_1 - i\Omega_2 + \sqrt{2\kappa_2} a_{2,\text{in}}, \tag{2b}$$

$$\dot{b} = -(\gamma_m + i\omega_m)b - ig_1 a_1^\dagger a_1 - ig_2 a_2^\dagger a_2 + \sqrt{2\gamma_m} b_{\text{in}}, \tag{2c}$$

where $a_{j,\text{in}}$ and b_{in} are the noise operators of the jth cavity mode and the vibrational mode, respectively. These noise operators have zero mean values and satisfy the standard correlation functions:

$$\langle a_{j,\text{in}}(t) a_{j,\text{in}}^\dagger(t') \rangle = \delta(t - t'), \quad \langle a_{j,\text{in}}^\dagger(t) a_{j,\text{in}}(t') \rangle = 0, \tag{3a}$$

$$\langle b_{\text{in}}(t) b_{\text{in}}^\dagger(t') \rangle = (\bar{n} + 1)\delta(t - t'), \tag{3b}$$

$$\langle b_{\text{in}}^\dagger(t) b_{\text{in}}(t') \rangle = \bar{n}\delta(t - t'), \tag{3c}$$

where $\bar{n} = [\exp(\hbar\omega_m/k_B T) - 1]^{-1}$ is the thermal phonon number of the vibration, T denotes the temperature of the thermal reservoir associated with the vibration, and k_B is the Boltzmann constant.

We consider the strong-driving regime, so that the average photon numbers in the two cavities are sufficiently large. Then, a linearization procedure can be used to simplify the physical model. To this end, we express the operators in Eq. (2) as a sum of their steady-state mean values and quantum fluctuations, namely: $o = \langle o \rangle_{\text{ss}} + \delta o$, for operators $o = a_j, a_j^\dagger$, b, and $b^\dagger$. By separating the classical motion from quantum fluctuations, the linearized equations of motion for quantum fluctuations can be written as

$$\begin{aligned}\delta\dot{a}_1 &= (-\kappa_1 - i\Delta_1)\delta a_1 - iG_1(\delta b + \delta b^\dagger)\\ &\quad - iJ\delta a_2 + \sqrt{2\kappa_1}a_{1,\mathrm{in}}, \quad (4a)\\ \delta\dot{a}_2 &= (-\kappa_2 - i\Delta_2)\delta a_2 - iG_2(\delta b + \delta b^\dagger)\\ &\quad - iJ\delta a_1 + \sqrt{2\kappa_2}a_{2,\mathrm{in}}, \quad (4b)\\ \delta\dot{b} &= (-\gamma_m - i\omega_m)\delta b - iG_1^*\delta a_1 - iG_1\delta a_1^\dagger\\ &\quad - iG_2^*\delta a_2 - iG_2\delta a_2^\dagger + \sqrt{2\gamma_m}b_{\mathrm{in}}, \quad (4c)\end{aligned}$$

where $\Delta_j = \Delta_{c,j} + g_j(\beta + \beta^*)$ is the normalized driving detuning of the jth cavity field, and the linearized optomechanical coupling strength G_j, with phase θ_j, between the jth cavity-field mode and the vibrational mode is given by

$$G_j = G_{0j}e^{i\theta_j}, \quad \text{for} \quad G_{0j} = g_j|\alpha_j|, \tag{5}$$

where the steady-state average values of the dynamical variables are $\alpha_1 = -i(J\alpha_2 + \Omega_1)/(\kappa_1 + i\Delta_1)$, $\alpha_2 = -i(J\alpha_1 + \Omega_2)/(\kappa_2 + i\Delta_2)$, and $\beta = -i(g_1|\alpha_1|^2 + g_2|\alpha_2|^2)/(\gamma_m + i\omega_m)$. Note that a synthetic gauge field with a modulation phase has recently been realized using three-mode closed-loop optomechanical systems [61], in which the appropriate spatially dependent hopping phases are realized through external drives [55–61].

By defining the mechanical and optical quadratures $\delta X_o = (\delta o^\dagger + \delta o)/\sqrt{2}$ and $\delta Y_o = i(\delta o^\dagger - \delta o)/\sqrt{2}$, and the corresponding Hermitian input-noise operators $X_o^{\mathrm{in}} = (o_{\mathrm{in}}^\dagger + o_{\mathrm{in}})/\sqrt{2}$ and $Y_o^{\mathrm{in}} = i(o_{\mathrm{in}}^\dagger - o_{\mathrm{in}})/\sqrt{2}$, we obtain a compact form of the linearized equations of quantum fluctuations

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{N}(t), \tag{6}$$

where $\mathbf{u}(t) = [\delta X_{a_1}, \delta Y_{a_1}, \delta X_{a_2}, \delta Y_{a_2}, \delta X_b, \delta Y_b]^T$ is the fluctuation operator vector, $\mathbf{N}(t) = \sqrt{2}[\sqrt{\kappa_1}X_{a_1}^{\mathrm{in}}, \sqrt{\kappa_1}Y_{a_1}^{\mathrm{in}}, \sqrt{\kappa_2}X_{a_2}^{\mathrm{in}}, \sqrt{\kappa_2}Y_{a_2}^{\mathrm{in}}, \sqrt{\gamma_m}X_b^{\mathrm{in}}, \sqrt{\gamma_m}Y_b^{\mathrm{in}}]^T$ is the noise operator vector, and

$$\mathbf{A} = \begin{pmatrix} -\kappa_1 & \Delta_1 & 0 & J & iG_{1-} & 0\\ -\Delta_1 & -\kappa_1 & -J & 0 & -G_{1+} & 0\\ 0 & J & -\kappa_2 & \Delta_2 & iG_{2-} & 0\\ -J & 0 & -\Delta_2 & -\kappa_2 & -G_{2+} & 0\\ 0 & 0 & 0 & 0 & -\gamma_m & \omega_m\\ -G_{1+} & -iG_{1-} & -G_{2+} & -iG_{2-} & -\omega_m & -\gamma_m \end{pmatrix}, \tag{7}$$

is the coefficient matrix, where $G_{j\pm} = G_j^* \pm G_j$. The formal solution of the linearized Langevin equations in (6) is given by

$$\mathbf{u}(t) = \mathbf{M}(t)\mathbf{u}(0) + \int_0^t \mathbf{M}(t-s)\mathbf{N}(s)ds, \tag{8}$$

where $\mathbf{M}(t) = \exp(\mathbf{A}t)$.

III. MINIMUM RESIDUAL CONTANGLE AND DARK-MODE CONTROL

A. Minimum residual contangle and entanglement monogamy

For studying the tripartite quantum entanglement, we focus on calculating the steady-state value of the covariance matrix $\mathbf{V}$, which is defined by the matrix elements $\mathbf{V}_{kl} = \frac{1}{2}[\langle \mathbf{u}_k(\infty)\mathbf{u}_l(\infty)\rangle + \langle \mathbf{u}_l(\infty)\mathbf{u}_k(\infty)\rangle]$, for $k, l = 1, \ldots, 6$. Under the stability condition, the covariance matrix $\mathbf{V}$ fulfills the Lyapunov equation

$$\mathbf{A}\mathbf{V} + \mathbf{V}\mathbf{A}^T = -\mathbf{Q}, \tag{9}$$

where $\mathbf{Q} = \mathrm{diag}\{\kappa_1, \kappa_1, \kappa_2, \kappa_2, \gamma_m(2\bar{n}+1), \gamma_m(2\bar{n}+1)\}$. The *minimum* residual contangle $E_\tau^{r|s|t}$ [62,63] can be used to quantify a tripartite quantum entanglement, defined as

$$E_\tau^{r|s|t} \equiv \min_{(r,s,t)} \left[E_\tau^{r|(st)} - E_\tau^{r|s} - E_\tau^{r|t}\right], \tag{10}$$

and is referred here as a tripartite entanglement measure, where $E_\tau^{u|v}$ is the contangle of subsystems of u and v (v contains one or two modes), and $r, s, t \equiv (d_1, d_2, c)$ denotes all the permutations of the three mode indexes [62]. Note that $E_\tau^{u|v}$ is defined as the *squared* logarithmic negativity [62–65], and it is a proper entanglement monotone.

The residual contangle satisfies the monogamy of quantum entanglement

$$E_\tau^{r|(st)} \geqslant E_\tau^{r|s} + E_\tau^{r|t}, \tag{11}$$

which is based on the Coffman-Kundu-Wootters monogamy inequality [64]. Entanglement monogamy is the fundamental principle that quantum entanglement cannot be freely shared between three subsystems (in general, among arbitrarily many systems).

Here, $E_\tau^{r|s|t} > 0$ indicates that a tripartite quantum entanglement is generated. Note that all the parameters in the following calculations satisfy the stability condition, which is derived using the Routh-Hurwitz criterion [66]. In addition, we have confirmed that only a single stable solution exists and our system has no bistability in all our simulations.

B. Optical-dark-mode control

Actually, the phase θ_j of the effective optomechanical coupling G_j can be absorbed into the new definitions of the operator δa_j, and only the phase difference $\theta \equiv \theta_1 - \theta_2$ has physical effects. Note that to conveniently study the ODM effect, we fix the phase difference θ together with the photon-tuneling interaction J. Then, according to Eqs. (4) and (5), a linearized optomechanical Hamiltonian, under the rotating-wave approximation (RWA), can be approximately written in the following form (discarding the noise terms)

$$\begin{aligned}H_{\mathrm{RWA}} &= \sum_{j=1,2}[\Delta_j\delta a_j^\dagger\delta a_j + G_{0j}(\delta b\delta a_j^\dagger + \delta a_j\delta b^\dagger)]\\ &\quad + \omega_m\delta b^\dagger\delta b + J(e^{i\theta}\delta a_1^\dagger\delta a_2 + e^{-i\theta}\delta a_2^\dagger\delta a_1). \quad (12)\end{aligned}$$

In recent years, both theoretical [51–54] and experimental [55–61] works have demonstrated that a phase-dependent synthetic gauge field can be induced in closed-loop optomechanical platforms. In particular, employing a three-mode loop-coupling optomechanical system of Ref. [61] has led to the creation of a synthetic gauge field with appropriate spatially dependent hopping phases, which can be tuned through external drives [55–61].

In light of the above achievements in realizing the synthetic gauge field using the three-mode closed-loop optomechanical platforms [61], we below employ this synthetic gauge field to control the ODM in these systems, so that the tripartite light-vibration entanglement can be flexibly controlled. Specifically, we here consider the following two cases:

(i) In the absence of the synthetic gauge field, i.e., $J = 0$, the two optical modes can form an optical bright mode A_+ and an optical dark mode A_-, which are given by

$$A_+ = \frac{G_{01}\delta a_1 + G_{02}\delta a_2}{G_0}, \quad \text{optical } \textit{bright} \text{ mode}, \tag{13a}$$

$$A_- = \frac{G_{02}\delta a_1 - G_{01}\delta a_2}{G_0}, \quad \text{optical } \textit{dark} \text{ mode}, \tag{13b}$$

where $G_0 = \sqrt{G_{01}^2 + G_{02}^2}$. Then, the Hamiltonian in Eq. (12) can be rewritten as

$$H_{\rm RWA} = \sum_{j=\pm} \Delta_j A_j^\dagger A_j + \omega_m \delta b^\dagger \delta b + G_+(\delta b A_+^\dagger + A_+ \delta b^\dagger) + G_-(A_+^\dagger A_- + A_-^\dagger A_+), \tag{14}$$

where $\Delta_{+(-)} = [G_{01(02)}^2 \Delta_1 + G_{02(01)}^2 \Delta_2]/G_0^2$ and the coupling strengths are

$$G_- = \frac{G_{01}G_{02}(\Delta_1 - \Delta_2)}{G_0^2}, \quad G_+ = G_0. \tag{15}$$

We can see from Eqs. (14) and (15) that when $\Delta_1 = \Delta_2$, the mode A_- is decoupled from the system due to $G_- = 0$, and it becomes an ODM. Owing to the decoupling of the ODM from the vibrational mode, the generation of the tripartite light-vibration entanglement is largely suppressed.

(ii) Since the ODM leads to the suppression of the tripartite entanglement generation, it is natural to ask the question whether one can break this ODM to further generate purer tripartite optomechanical entanglement. To this end, a synthetic magnetism is introduced in the system for controlling the ODM, i.e., $J \neq 0$. Specifically, by introducing two superposition-optical modes associated with the synthetic magnetism $\tilde{A}_+$ and $\tilde{A}_-$, defined by

$$\tilde{A}_+ = f\delta a_1 - e^{i\theta} h \delta a_2, \tag{16a}$$

$$\tilde{A}_- = e^{-i\theta} h \delta a_1 + f \delta a_2, \tag{16b}$$

the Hamiltonian in Eq (12) becomes

$$H_{\rm RWA} = \sum_{j=\pm} (\tilde{\Delta}_j \tilde{A}_j^\dagger \tilde{A}_j + \tilde{G}_j^* \delta b \tilde{A}_j^\dagger + \tilde{G}_j \tilde{A}_j \delta b^\dagger) + \omega_m \delta b^\dagger \delta b, \tag{17}$$

where $\tilde{\Delta}_\pm = \frac{1}{2}(\Delta_1 + \Delta_2 \pm \sqrt{(\Delta_1 - \Delta_2)^2 + 4J^2})$, and the redefined coupling strengths are given by

$$\tilde{G}_+ = fG_{01} - e^{-i\theta} h G_{02}, \quad \tilde{G}_- = e^{i\theta} h G_{01} + f G_{02}, \tag{18}$$

with $f = \frac{|\tilde{\Delta}_- - \Delta_1|}{\sqrt{(\tilde{\Delta}_- - \Delta_1)^2 + J^2}}$, and $h = \frac{Jf}{\tilde{\Delta}_- - \Delta_1}$.

In Fig. 1(b), the redefined coupling strengths $\tilde{G}_\pm$ are plotted as a function of the modulation phase θ. It shows that only at $\theta = n\pi$ (i.e., in the ODMU regime), $\tilde{A}_+$ [for an odd n, $\tilde{G}_+ = 0$ (see blue symbols)] or $\tilde{A}_-$ [for an even n, $\tilde{G}_- = 0$ (see red symbols)] becomes the ODM. A counterintuitive coupling of the ODM to the vibrational mode can be achieved by tuning $\theta \neq n\pi$ (i.e., in the ODMB regime), which means the breaking of the ODM [see Fig. 1(b)]. The underlining physical mechanism is that the constructive or destructive interference between the two coupling paths, by modulating the phase θ, enables a flexible switch between the ODMU and ODMB regimes.

To provide a clearer interpretation of our results, we here derive various approximate analytical expressions for the redefined coupling strengths in the symmetric-detuning and symmetric-coupling regimes, as well as in the resolved-sideband regime. In the symmetric-detuning ($\Delta_{j=1,2} = \Delta$) and symmetric-coupling ($G_{0j=1,2} = G$) cases, the redefined-coupling strengths $\tilde{G}_\pm$ in Eq. (18) become $\tilde{G}_\pm \approx G(1 \pm e^{\mp i\theta})/\sqrt{2}$. One can clearly see that the redefined-coupling strengths $\tilde{G}_\pm$ strongly depend on the modulation phase θ. Furthermore, by considering the case of $\Omega_{j=1,2} = \Omega$, $\kappa_{j=1,2} = \kappa$, $g_{01} = g_{02} = g$, and $J \ll \omega_m$, Δ, in the resolved-sideband regime (i.e., $\kappa \ll \omega_m$), we obtain the approximate analytical expressions of the redefined coupling strengths as: $\tilde{G}_\pm \approx \frac{g\Omega}{\Delta}(1 \pm e^{\mp i\theta})/\sqrt{2}$. It can be intuitively seen that in the resolved-sideband regime, (i) the DMB is governed by the modulation phase θ and (ii) the strength of the redefined-coupling strengths can be enhanced by tuning the amplitude Ω of the driving lasers applied to the optical cavities. These observations based on our analytical approximate results are well matched with the numerical simulations based on the exact results.

Here it is worth presenting some discussions on the reason for choosing the values of the system parameters. In realistic systems, the strength of the linearized effective optomechanical coupling should be much smaller than mechanical resonance frequency ω_m. In addition, the photon-tunneling interaction between the two cavities can be realized by optical backscattering [34,59–61]. This backscattering of the photons is induced by the surface roughness and material defects in practical devices. Therefore the value of the photon-tunneling coupling strength used in our simulations should be of the same order of the decay rates of the cavity-field modes. These assumed values of parameters should be accessible under the near-future experimental conditions. In this work, to obtain a significative amount of quantum entanglement, we have made a careful analysis in a wide parameter range by combining theoretical [24] and experimental [38–42] works, and we have assumed experimentally feasible conditions in our manuscript.

IV. TRIPARTITE LIGHT-MOTION ENTANGLEMENT

In the above section, we have derived the minimum residual contangle, and have analyzed how to control the ODM in the three-mode optomechanical system. Now we study in detail the tripartite optomechanical entanglement by controlling the ODM.

In Fig. 2(a), we plot the tripartite light-vibration entanglement, quantified by the minimum residual contangle $E_\tau^{r|s|t}$, as a function of the driving detunings $\Delta_j = \Delta$ of the two cavities, when the system operates in both ODMU ($J = 0$, see the blue curve) and ODMB ($J/\omega_m = 0.2$ and $\theta = \pi/2$, see the red curve) regimes. We find that in the ODMB regime, the tripartite entanglement is nearly twice as large as that in the ODMU regime. The underlying physical mechanism is that the ODM is decoupled from the vibration, and this results in

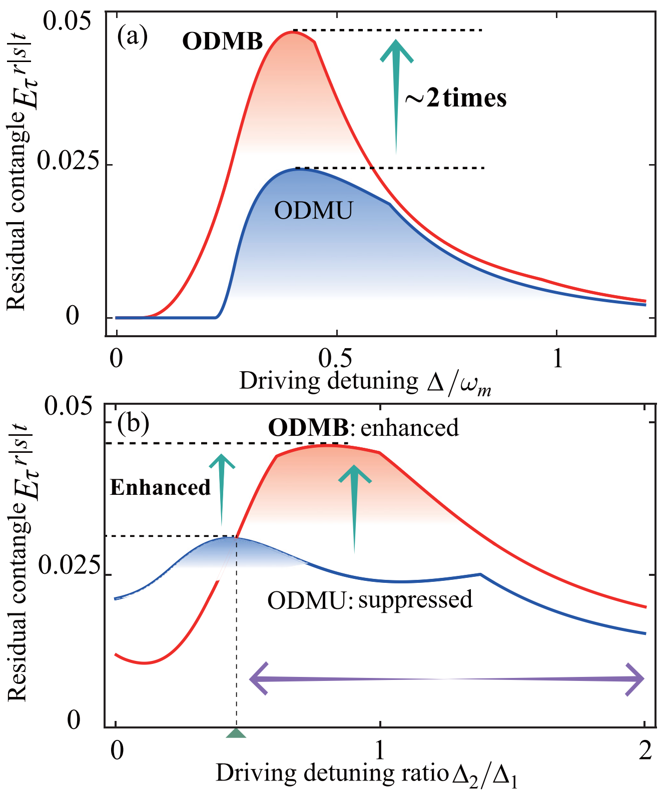

FIG. 2. (a) Tripartite optomechanical entanglement, quantified by the minimum residual contangle $E_\tau^{r|s|t}$, versus the effective driving detunings $\Delta_{j=1,2} = \Delta$ of the two cavities in the ODMU ($J = 0$, blue curves) and ODMB ($J/\omega_m = 0.2$ and $\theta = \pi/2$, red curves) regimes. (b) Minimum residual contangle $E_\tau^{r|s|t}$ as a function of Δ_2/Δ_1 in both ODMU and ODMB regimes, when $\Delta_1/\omega_m = 0.45$. Other parameters are: $G_{0j}/\omega_m = 0.2$, $\kappa_j/\omega_m = 0.2$, $\gamma_m/\omega_m = 10^{-5}$, and $\bar{n} = 0$.

the suppression of the tripartite optomechanical entanglement. However, the breaking of the ODM leads to the enhancement in quantum entanglement. These results show a clear inspiration for enhancing tripartite optomechanical entanglement via the control of the ODM.

Moreover, we find from Fig. 2(b) that, when turning off the synthetic gauge field (i.e., $J = 0$, see the blue curve), the tripartite optomechanical entanglement is strongly suppressed by the ODM, corresponding to the emergence of the dip [see the blue curve in Fig. 2(b)]. In particular, this ODM effect can work in a wider driving detuning range, i.e., $\Delta_2/\Delta_1 > 0.5$. In contrast to this, once turning on the synthetic gauge field (i.e., $J \neq 0$ and $\theta \neq n\pi$), the dip becomes a peak [see the red curve in Fig. 2(b)], and this indicates that the tripartite light-vibration entanglement is significantly enhanced due to breaking the ODM. In particular, the maximum amount of the resulting entanglement in the ODMB regime is larger than that in the ODMU case.

To determine how significant enhancement in the tripartite optomechanical entanglement can be reached, we here introduce an entanglement-amplification factor Λ, which is defined as

$$\Lambda = \frac{E_{\tau,\mathrm{ODMB}}^{r|s|t}}{E_{\tau,\mathrm{ODMU}}^{r|s|t}}. \tag{19}$$

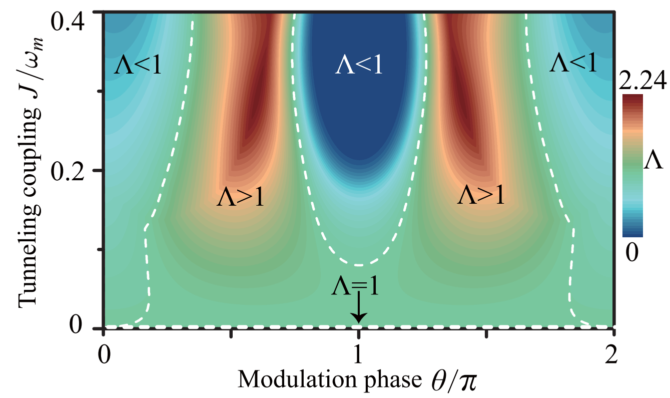

FIG. 3. Amplification factor Λ of the tripartite optomechanical entanglement versus the photon-tunneling coupling J and the modulation phase θ under the optimal driving detunings $\Delta_{j=1,2}/\omega_m = 0.45$. Here the white dashed curve denotes the case of $J = 0$. Other parameters are the same as those in Fig. 2.

Based on Eq. (19), we show the effect of the parameters J and θ of the synthetic magnetism on the entanglement-amplification factor Λ in Fig. 3. We find that the tripartite optomechanical entanglement can be significantly amplified by tuning the synthetic magnetism parameters. For example, when the synthetic magnetism is off (i.e., when $J = 0$), no amplification for the tripartite entanglement can be observed (i.e., $\Lambda = 1$, see the white horizontal dashed line). However, when the synthetic magnetism is on, the amplification of the tripartite entanglement emerges, and even the entanglement amplification factor can increase up to $\Lambda = 2.24$. When $J/\omega_m \to 0.4$ and $\theta \to n\pi$ for an integer n, we obtain $0 < \Lambda < 1$. This is because the optical backscattering losses in practical devices [34,61] strongly suppress the entanglement generation. Physically, the backscattering of photons can be induced by various imperfections of devices, such as surface roughness and material defects, as described by, e.g., the photon-tunneling coupling J [34,61].

Moreover, we can see from Fig. 3 that the tripartite light-vibration entanglement is significantly enhanced, i.e., $\Lambda > 1$, in the regions $0 < \theta < \pi$ and $\pi < \theta < 2\pi$, and the maximal amount can be observed around $\theta = \pi/2$ and $3\pi/2$, corresponding to maximal quantum interference. However, the tripartite optomechanical entanglement is strongly suppressed, i.e., $0 < \Lambda < 1$ at $\theta = n\pi$, which is due to the presence of the ODM.

In the above section, we have showed that a significant enhancement in the tripartite optomechanical entanglement can be achieved via the ODM control. It is natural to ask the question: can we further improve the tripartite light-vibration entanglement by tuning the other parameters of the system?

For further elucidating this problem, we plot the tripartite light-vibration entanglement in Fig. 4(a), quantified by the minimum residual contangle $E_\tau^{r|s|t}$, as functions of the optomechanical coupling strengths G_{01} and G_{02} in the ODMB regime. We see that the tripartite optomechanical entanglement can be significantly enhanced by increasing G_{0j}, and that the maximal amount of the tripartite entanglement can be observed for $0.2 \leqslant G_{0j}/\omega_m \leqslant 0.3$. Moreover, in Fig. 4(b), we

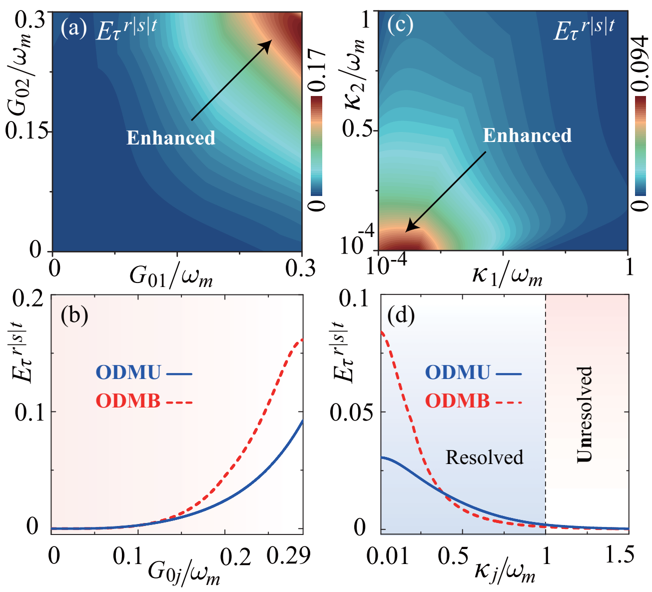

FIG. 4. (a) Tripartite light-vibration entanglement measure $E_\tau^{r|s|t}$ vs the optomechanical coupling strengths G_{01} and G_{02} when $\kappa_{j=1,2}/\omega_m = 0.2$, in the ODMB regime. (b) $E_\tau^{r|s|t}$ vs G_{0j} when $\kappa_j/\omega_m = 0.2$, in both ODMU (blue solid curve) and ODMB (red dashed curve) regimes. (c) $E_\tau^{r|s|t}$ vs the cavity-field decay rates κ_1 and κ_2 when $G_{0j}/\omega_m = 0.2$, in the ODMB regime. (d) $E_\tau^{r|s|t}$ vs κ_j when $G_{0j}/\omega_m = 0.2$, in both ODMU (blue solid curve) and ODMB (red dashed curve) regimes. Other parameters are the same as those in Fig. 2.

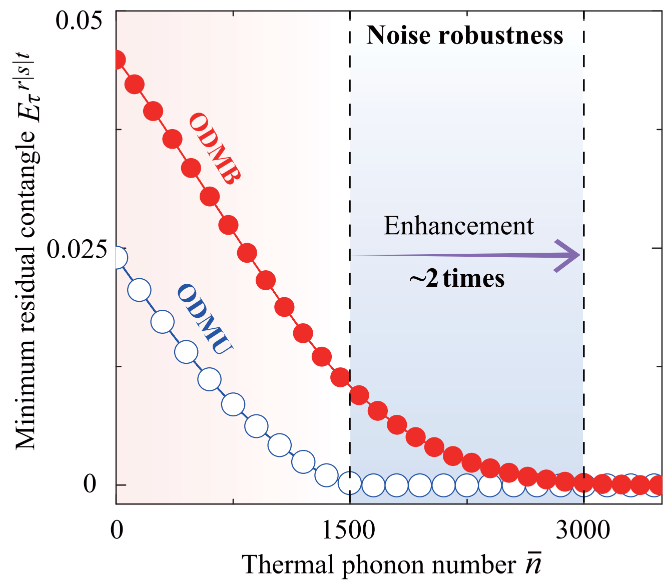

FIG. 5. Tripartite optomechanical entanglement measure $E_\tau^{r|s|t}$ as a function of the thermal phonon number $\bar{n}$ of the mechanical resonator in the ODMU (blue symbols) and ODMB (red symbols) regimes. Other parameters are the same as those in Fig. 2.

plot the tripartite optomechanical entanglement measure $E_\tau^{r|s|t}$ as a function of the light-motion coupling strengths G_{0j} in both ODMU (blue solid curve) and ODMB (red dashed curve) regimes. We find that in both ODMU and ODMB regimes, the tripartite optomechanical entanglement could be further enhanced by increasing the optomechanical coupling strengths G_{0j}. In particular, we show that the tripartite optomechanical entanglement in the ODMB regime is larger than that in the ODMU regime (i.e., $E_{\tau,\mathrm{ODMB}}^{r|s|t} > E_{\tau,\mathrm{ODMU}}^{r|s|t}$).

In Fig. 4(c), the tripartite optomechanical entanglement measure $E_\tau^{r|s|t}$ is plotted as functions of the decay rates κ_1 and κ_2 of the two cavities, when the system works in the ODMB regime. For clearly investigating the influence of the sideband-resolution condition on the tripartite optomechanical entanglement, the mechanical frequency ω_m is chosen as a frequency scale. This demonstrates that the phonon sideband can be well resolved from the cavity-emission spectrum when $\kappa_j/\omega_m \ll 1$, corresponding to the resolved-sideband limit [10–13]. In particular, in this resolved-sideband regime, i.e., $\kappa_j/\omega_m \ll 1$, We find that the tripartite optomechanical entanglement becomes much larger for a smaller cavity-field decay rate κ_j.

To further elucidate this observation, we plot the tripartite optomechanical entanglement measure $E_\tau^{r|s|t}$ as a function of the cavity-field decay rate κ_j, in both ODMU (blue solid curve) and ODMB (red dashed curve) regimes, as shown in Fig. 4(d). We can see that in these two regimes, the phonon sidebands can be well resolved (i.e., $\kappa_j/\omega_m \ll 1$, see the blue-shaded area), and that the tripartite quantum entanglement can be significantly improved with decreasing κ_j. However, in the unresolved-sideband regime $\kappa_j/\omega_m > 1$, the tripartite entanglement becomes much worse for a larger cavity-field decay rate κ_j (see the yellow-shaded area). Despite this, when $\kappa_j/\omega_m \ll 1$, the amount of the tripartite light-vibration entanglement in the ODMB regime is much larger than that in the ODMU regime (i.e., $E_{\tau,\mathrm{ODMB}}^{r|s|t} > E_{\tau,\mathrm{ODMU}}^{r|s|t}$). These findings offer a different method to enhance fragile quantum resources by appropriately designing the cavities.

Our ODMB mechanism paves a feasible way to enhance fragile quantum resources via the control of an optical dark mode, and it can be used for constructing the noise-tolerant quantum processors and quantum networks. To study the noise robustness of the resulting tripartite optomechanical entanglement, we plot the minimum residual contangle $E_\tau^{r|s|t}$ as a function of the thermal phonon number $\bar{n}$ of the vibration, when the system operates in both ODMB and ODMU regimes, as shown in Fig. 5. We find that for the ODMU regime, the tripartite optomechanical entanglement emerges only for the thermal phonon number $\bar{n} \ll 1500$ (see the blue symbols). However, for the ODMB regime, it can persist for the thermal phonon number near $\bar{n} = 3000$ (see the red symbols), which is twice that of the ODMU case. In particular, we find that with the increase of the thermal phonon number $\bar{n}$, the tripartite light-vibration entanglement in the ODMB regime is always much lager than that in the ODMU regime. Our findings provide a way for enhancing and protecting fragile quantum resources from thermal noise and ODM, and pave a way towards noise-tolerant large-scale entanglement networks.

V. DISCUSSIONS

In this section, we show some discussions on possible experimental realizations of our model, and on the tripartite entanglement measures and the importance of our measure of entanglement.

A. Discussions on possible experimental realizations

In this section, some discussions are presented for showing the experimental realization of the photon-tunneling coupling for our model and related ones. Currently, the photon-hopping interaction between the two cavity fields has been experimentally implemented in a microsphere optomechanical cavity system [59–61,67], where a microresonator supports a pair of degenerate counterclockwise and clockwise traveling-wave whispering-galley modes. The resonant frequencies of these modes are modulated by a radial breathing (vibrational) mode, which changes the circumference of a microsphere. Then, the optomechanical coupling is induced between the radial breathing mode and the counterclockwise (clockwise) mode. Note that the photon-tunneling interaction between the counterclockwise and clockwise optical modes is realized due to optical backscattering, and, in experiments, this backscattering of photons can be induced by the surface roughness and material defects in practical devices [59–61]. By utilizing counterclockwise and clockwise optical modes and a vibrational mode, we can induce a precisely controllable synthetic gauge field by just tuning the phases of the external driving lasers [61].

In addition, based on membrane-in-the-middle configuration optomechanical-cavity systems [68], we can realize the photon-tunneling coupling between the two cavity fields and, then, implement the loop-coupled optomechanical system in experiments in an optomechanical-cavity system. In the membrane-in-the-middle configuration, there exist two cavity-field modes (the left and right subcavity modes), which are coupled to each other via a photon-tunneling interaction. The vibration of the mechanical membrane modulates the resonant frequencies of the two subcavity modes and, as a result, the radiation-pressure coupling is induced between the left (right) subcavity mode and this vibrational mode.

Actually, this nontrivial phase can also be realized by using multiple tones based on cavity optomechanical systems [58,69,70]. For example, in a four-mode loop-coupled optomechanical configurations [58,69], two cavity modes are coupled indirectly via two nondegenerate vibrational modes, and the system is driven by four tones in the well-resolved sidebands. In this case, the nontrivial phase of the system can be induced by tuning the phases of the driving tones. In addition, using the standard optomechanical interactions can lead to a nontrivial phase in a three-mode optomechanical system [70], where two nondegenerate resonators are coupled to a common cavity. In that system, the cavity is driven by four tones, and this nontrivial phase can be tuned in situ simply via the phases of the driving tones applied to the cavity.

B. Discussions on tripartite entanglement measures and the importance of our measure of entanglement

Here we present some discussions on tripartite entanglement measures and on the importance of the measure of entanglement applied by us.

TABLE I. Five disjoint classes of three-mode (tripartite) states concerning their bipartite and tripartite entanglement based on the classification of Ref. [71]. Here "+" indicates that a given measure is nonzero. Examples of states $\rho \equiv \rho^{ABC}$ of each class can be generated by our optomechanical system for the parameters, which are specified in the last column. Here $\mathcal{A}$ (i.e., an example for class 1) includes the GHZ-like states of Ref. [72]; $\mathcal{B}$ (i.e., an example for class 2) is the two-mode squeezed vacuum in the first two modes and the vacuum in the third mode; $\mathcal{C}$ (i.e., an example for class 3) includes separable states with respect to two of the three bipartitions but inseparable with respect to the third bipartition; $\mathcal{D}$ (i.e., an example for class 4) includes separable states with respect to all three bipartitions but cannot be written as a mixture of tripartite product states; $\mathcal{E}$ (i.e., an example for class 5) includes the vacuum state in all three modes. Note that tripartite entanglement corresponds to three-mode biseparability (class 4).

Class	State ρ	$E_\rho^{A\vert B\vert C}$	$E_\rho^{A\vert (BC)}$	$E_\rho^{B\vert (CA)}$	$E_\rho^{C\vert (AB)}$	Examples
1	Fully entangled	+	+	+	+	$\mathcal{A}$
2	Single-mode biseparable	+	+	+	0	$\mathcal{B}$
		+	+	0	+	
		+	0	+	+	
3	Two-mode biseparable	+	+	0	0	$\mathcal{C}$
		+	0	+	0	
		+	0	0	+	
4	Three-mode biseparable	+	0	0	0	$\mathcal{D}$
5	Fully separable	0	0	0	0	$\mathcal{E}$

One can consider some (nonnegative) measures of tripartite ($E_\rho^{A|B|C}$) and bipartite ($E_\rho^{A|(BC)}$, $E_\rho^{B|(CA)}$, and $E_\rho^{C|(AB)}$) entanglement for a given three-mode state $\rho \equiv \rho^{ABC}$ such that they are zero if and only if ρ can be decomposed, respectively, as follows:

$$E_\rho^{A|(BC)} = 0 \quad \text{iff} \quad \rho = \sum_i \rho_i^A \otimes \rho_i^{BC}, \tag{20}$$

$$E_\rho^{B|(CA)} = 0 \quad \text{iff} \quad \rho = \sum_i \rho_i^B \otimes \rho_i^{CA}, \tag{21}$$

$$E_\rho^{C|(AB)} = 0 \quad \text{iff} \quad \rho = \sum_i \rho_i^C \otimes \rho_i^{AB}, \tag{22}$$

$$E_\rho^{A|B|C} = 0 \quad \text{iff} \quad \rho = \sum_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C. \tag{23}$$

These measures of entanglement can be chosen in various ways, and ours are specified below. Note that the above single-mode (ρ_i^A,...) and two-mode (ρ_i^{AB},...) mixed states can be replaced by pure states (as in, e.g., Ref. [71]) without loosing any generality of these decompositions.

Following the classification introduced in Ref. [71] (and then applied to many specific systems, including optomechanical ones in, e.g., Ref. [73]), one can distinguish five different *classes* of entanglement, which are listed in Table I for a system composed of three modes (or parties, say A, B, and C). Note class 4 of three-mode biseparable states, which are separable with respect to all three bipartitions, but cannot be written as a mixture of tripartite product states, given in Eq. (23). These are usually referred to as tripartite entangled states, and play a central role in this paper.

In particular, various measures of entanglement can be used to properly classify bipartite and tripartite entanglement of classes of states listed in Table I. Concerning measures of bipartite entanglement, one can choose among dozens of measures, but for quantifying tripartite entanglement the choice of measures is much more limited [74]. In addition to the minimum residual contangle introduced in Ref. [62], as studied by us, one could apply, in principle, a finite-dimensional multipartite concurrence introduced in Refs. [75,76] in the limit of large-dimensional modes. That measure has also an intuitive physical interpretation, but it is much more difficult to be calculated for systems of a large dimension (note ours is infinite-dimensional), because it requires finding numerically an infimum over all possible decompositions of a given mixed state.

We note that tripartite entanglement in optomechanical systems was studied already two decades ago (see, e.g., Refs. [73,77,78]). In particular, Ref. [78] discusses how to generate bipartite and tripartite entanglement in a system composed of three subsystems (modes, denoted here as A, B, C): a micromechanical resonator and two output optical fields. Specifically, (i) purely optical bipartite entanglement between two output modes (of the two-mode reduced system) and (ii) fully tripartite optomechanical entanglement (corresponding to class 1 in Table I). Bipartite entanglement is quantified by the (logarithmic) negativity corresponding to the negative eigenvalue of the partially transposed (PT) two-mode-reduced matrices. Tripartite entanglement is revealed and quantified analogously by the minimum eigenvalues for the PT three-mode matrices with respect to each of the three modes. This corresponds to applying bipartite PT-based entanglement measures for all the three different bipartitions of the ABC system, i.e., $A|(BC)$, $B|(CA)$, and $C|(AB)$. It is found, for some ranges of the system parameters, that these eigenvalues can be all negative, showing that their system can be fully tripartite entangled [73]. Clearly, tripartite entanglement (corresponding to Class 4) occurs if a tripartite state ρ exhibits both (i) $E_\rho^{A|(BC)} = E_\rho^{B|(CA)} = E_\rho^{C|(AB)} = 0$ and (ii) $E_\rho^{A|B|C} > 0$. By checking conditions (i) without testing conditions (ii) (i.e., without calculating $E_\rho^{A|B|C}$), one cannot say whether a given state ρ belongs to Class 4 or Class 5 of completely separable states. However, by calculating $E_\rho^{A|B|C}$ via the minimum residual contangle, we are able to distinguish these two cases for our model.

In fact, the contangle (and its absolute value) can be useful for various quantum information applications. Here some discussions are shown to clearly explain the importance of the measure of entanglement applied by us. In this work, the main reason for us of applying the minimum residual contangle, as a measure of a tripartite entanglement, is its relation to entanglement monogamy in addition to its simple computability. In fact, the measure was introduced in Ref. [62], as a continuous-variable analog of the famous Coffman-Kundu-Wootters monogamy inequality [64] designed for testing the monogamy of entanglement of discrete systems. Entanglement monogamy describes one of the most fundamental properties of quantum entanglement, which means that entanglement cannot be freely shared between arbitrarily many parties (in our case, three modes). The monogamy of entanglement has many fundamental applications (for a review see Ref. [79]). It is especially useful in proving the security of quantum cryptosystems (see, e.g., Refs. [80,81]). But the interest in multipartite entanglement and its monogamy is not limited to quantum cryptology, quantum communication, and quantum computing [82]. For example, an apparent violation of entanglement monogamy, which is referred to as polyamory and studied in, e.g., black holes [83], would lead to major paradoxes of quantum mechanics and cosmology.

VI. CONCLUSIONS

We have showed how to realize a tripartite optomechanical entanglement via the control of the ODM, which is formed by two optical modes coupled to a common vibrational mode. The tripartite light-vibration entanglement can be significantly enhanced via the ODMB mechanism, without which it is strongly suppressed. In particular, the noise robustness of the tripartite quantum entanglement in the ODMB regime can be even twice as large as that in the ODMU regime. This study can enable constructing large-scale entanglement networks with the dark-mode immunity and the noise tolerance, and also open up a range of exciting opportunities for quantum information processing and quantum metrology protected against the ODM.

ACKNOWLEDGMENTS

A.M. is supported by the Polish National Science Centre (NCN) under the Maestro Grant No. DEC-2019/34/A/ST2/00081. F.N. is supported in part by Nippon Telegraph and Telephone Corporation (NTT) Research, the Japan Science and Technology Agency (JST) [via the Quantum Leap Flagship Program (Q-LEAP) program, the Moonshot R&D Grant No. JPMJMS2061], the Japan Society for the Promotion of Science (JSPS) [via the Grants-in-Aid for Scientific Research (KAKENHI) Grant No. JP20H00134], the Army Research Office (ARO) (Grant No. W911NF-18-1-0358), the Asian Office of Aerospace Research and Development (AOARD) (via Grant No. FA2386-20-1-4069), and the Foundational Questions Institute Fund (FQXi) via Grant No. FQXi-IAF19-06.

[1] F. J. Duarte and T. Taylor, *Quantum Entanglement Engineering and Applications* (IOP Press, London, 2021).

[2] H. J. Kimble, The quantum internet, Nature (London) **453**, 1023 (2008).

[3] T. J. Kippenberg and K. J. Vahala, Cavity optomechanics: Back-action at the mesoscale, Science **321**, 1172 (2008).

[4] P. Meystre, A short walk through quantum optomechanics, Ann. Phys. (Berlin) **525**, 215 (2013).

[5] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Cavity optomechanics, Rev. Mod. Phys. **86**, 1391 (2014).

[6] A. D. O'Connell *et al.*, Quantum ground state and single-phonon control of a mechanical resonator, Nature (London) **464**, 697 (2010).

[7] Y. Chu, P. Kharel, W. H. Renninger, L. D. Burkhart, L. Frunzio, P. T. Rakich, and R. J. Schoelkopf, Quantum acoustics with superconducting qubits, Science **358**, 199 (2017).

[8] S. Hong, R. Riedinger, I. Marinković, A. Wallucks, S. G. Hofer, R. A. Norte, M. Aspelmeyer, and S. Gröblacher, Hanbury Brown and Twiss interferometry of single phonons from an optomechanical resonator, Science **358**, 203 (2017).

[9] W. Qin, A. Miranowicz, G. L. Long, J. Q. You, and F. Nori, Proposal to test quantum wave-particle superposition on massive mechanical resonators, npj Quantum Inf. **5**, 58 (2019).

[10] I. Wilson-Rae, N. Nooshi, W. Zwerger, and T. J. Kippenberg, Theory of Ground State Cooling of a Mechanical Oscillator Using Dynamical Backaction, Phys. Rev. Lett. **99**, 093901 (2007).

[11] F. Marquardt, J. P. Chen, A. A. Clerk, and S. M. Girvin, Quantum Theory of Cavity-Assisted Sideband Cooling of Mechanical Motion, Phys. Rev. Lett. **99**, 093902 (2007).

[12] J. Chan, T. P. Alegre, A. H. Safavi-Naeini, J. T. Hill, A. Krause, S. Groeblacher, M. Aspelmeyer, and O. Painter, Laser cooling of a nanomechanical oscillator into its quantum ground state, Nature (London) **478**, 89 (2011).

[13] J. D. Teufel, T. Donner, D. Li, J. W. Harlow, M. S. Allman, K. Cicak, A. J. Sirois, J. D. Whittaker, K. W. Lehnert, and R. W. Simmonds, Sideband cooling of micromechanical motion to the quantum ground state, Nature (London) **475**, 359 (2011).

[14] D.-G. Lai, F. Zou, B. P. Hou, Y. F. Xiao, and J. Q. Liao, Simultaneous cooling of coupled mechanical resonators in cavity optomechanics, Phys. Rev. A **98**, 023860 (2018).

[15] D.-G. Lai, J.-F. Huang, X.-L. Yin, B.-P. Hou, W. Li, D. Vitali, F. Nori, and J.-Q. Liao, Nonreciprocal ground-state cooling of multiple mechanical resonators, Phys. Rev. A **102**, 011502(R) (2020).

[16] D.-G. Lai, J. Huang, B.-P. Hou, F. Nori, and J.-Q. Liao, Domino cooling of a coupled mechanical-resonator chain via cold-damping feedback, Phys. Rev. A **103**, 063509 (2021).

[17] D.-G. Lai, W. Qin, B.-P. Hou, A. Miranowicz, and F. Nori, Significant enhancement in refrigeration and entanglement in auxiliary-cavity-assisted optomechanical systems, Phys. Rev. A **104**, 043521 (2021).

[18] D.-G. Lai, W. Qin, A. Miranowicz, and F. Nori, Efficient optomechanical refrigeration of two vibrations via an auxiliary feedback loop: Giant enhancement in mechanical susceptibilities and net cooling rates, Phys. Rev. Research **4**, 033102 (2022).

[19] E. E. Wollman, C. U. Lei, A. J. Weinstein, J. Suh, A. Kronwald, F. Marquardt, A. A. Clerk, and K. C. Schwab, Quantum squeezing of motion in a mechanical resonator, Science **349**, 952 (2015).

[20] X.-Y. Lü, Y. Wu, J. R. Johansson, H. Jing, J. Zhang, and F. Nori, Squeezed Optomechanics with Phase-Matched Amplification and Dissipation, Phys. Rev. Lett. **114**, 093602 (2015).

[21] W. Qin, A. Miranowicz, P.-B. Li, X.-Y. Lü, J. Q. You, and F. Nori, Exponentially Enhanced Light-Matter Interaction, Cooperativities, and Steady-State Entanglement Using Parametric Amplification, Phys. Rev. Lett. **120**, 093601 (2018).

[22] Y.-H. Chen, W. Qin, X. Wang, A. Miranowicz, and F. Nori, Shortcuts to Adiabaticity for the Quantum Rabi Model: Efficient Generation of Giant Entangled Cat States via Parametric Amplification, Phys. Rev. Lett. **126**, 023602 (2021).

[23] W. Qin, A. Miranowicz, H. Jing, and F. Nori, Generating Long-Lived Macroscopically Distinct Superposition States in Atomic Ensembles, Phys. Rev. Lett. **127**, 093602 (2021).

[24] D. Vitali, S. Gigan, A. Ferreira, H. R. Böhm, P. Tombesi, A. Guerreiro, V. Vedral, A. Zeilinger, and M. Aspelmeyer, Optomechanical Entanglement between a Movable Mirror and a Cavity Field, Phys. Rev. Lett. **98**, 030405 (2007).

[25] S. Mancini, V. Giovannetti, D. Vitali, and P. Tombesi, Entangling Macroscopic Oscillators Exploiting Radiation Pressure, Phys. Rev. Lett. **88**, 120401 (2002).

[26] D. Vitali, S. Mancini, and P. Tombesi, Stationary entanglement between two movable mirrors in a classically driven Fabry-Perot cavity, J. Phys. A: Math. Theor. **40**, 8055 (2007).

[27] M. Paternostro, D. Vitali, S. Gigan, M. S. Kim, C. Brukner, J. Eisert, and M. Aspelmeyer, Creating and Probing Multipartite Macroscopic Entanglement with Light, Phys. Rev. Lett. **99**, 250401 (2007).

[28] M. J. Hartmann and M. B. Plenio, Steady State Entanglement in the Mechanical Vibrations of Two Dielectric Membranes, Phys. Rev. Lett. **101**, 200503 (2008).

[29] Y.-D. Wang and A. A. Clerk, Reservoir-Engineered Entanglement in Optomechanical Systems, Phys. Rev. Lett. **110**, 253601 (2013).

[30] S. Huang and G. S. Agarwal, Entangling nanomechanical oscillators in a ring cavity by feeding squeezed light, New J. Phys. **11**, 103044 (2009).

[31] R. Riedinger, S. Hong, R. A. Norte, J. A. Slater, J. Shang, A. G. Krause, V. Anant, M. Aspelmeyer, and S. Gröblacher, Non-classical correlations between single photons and phonons from a mechanical oscillator, Nature (London) **530**, 313 (2016).

[32] M. Ho, E. Oudot, J.-D. Bancal, and N. Sangouard, Witnessing Optomechanical Entanglement with Photon Counting, Phys. Rev. Lett. **121**, 023602 (2018).

[33] S. Barzanjeh, E. S. Redchenko, M. Peruzzo, M. Wulf, D. P. Lewis, G. Arnold, and J. M. Fink, Stationary entangled radiation from micromechanical motion, Nature (London) **570**, 480 (2019).

[34] Y.-F. Jiao, S.-D. Zhang, Y.-L. Zhang, A. Miranowicz, L.-M. Kuang, and H. Jing, Nonreciprocal Optomechanical Entanglement against Backscattering Losses, Phys. Rev. Lett. **125**, 143605 (2020).

[35] J. Chen, M. Rossi, D. Mason, and A. Schliesser, Entanglement of propagating optical modes via a mechanical interface, Nat. Commun. **11**, 943 (2020).

[36] H. Yu, L. McCuller, M. Tse, N. Kijbunchoo, L. Barsotti, N. Mavalvala, and L. S. Collaboration, Quantum correlations between light and the kilogram-mass mirrors of LIGO, Nature (London) **583**, 43 (2020).

[37] D.-G. Lai, J.-Q. Liao, A. Miranowicz, and F. Nori, Noise-Tolerant Optomechanical Entanglement via Synthetic Magnetism, Phys. Rev. Lett. **129**, 063602 (2022).

[38] T. A. Palomaki, J. D. Teufel, R. W. Simmonds, and K. W. Lehnert, Entangling mechanical motion with microwave fields, Science **342**, 710 (2013).

[39] R. Riedinger, A. Wallucks, I. Marinković, C. Löschnauer, M. Aspelmeyer, S. Hong, and S. Gröblacher, Remote quantum

entanglement between two micromechanical oscillators, Nature (London) **556**, 473 (2018).
[40] C. F. Ockeloen-Korppi, E. Damskägg, J.-M. Pirkkalainen, M. Asjad, A. A. Clerk, F. Massel, M. J. Woolley, and M. A. Sillanpää, Stabilized entanglement of massive mechanical oscillators, Nature (London) **556**, 478 (2018).
[41] S. Kotler *et al.*, Direct observation of deterministic macroscopic entanglement, Science **372**, 622 (2021).
[42] L. Mercier de Lépinay, C. F. Ockeloen-Korppi, M. J. Woolley, and M. A. Sillanpää, Quantum mechanics-free subsystem with mechanical oscillators, Science **372**, 625 (2021).
[43] S. Armstrong, J.-F. Morizur, J. Janousek, B. Hage, N. Treps, P. K. Lam, and H.-A. Bachor, Programmable multimode quantum networks, Nat. Commun. **3**, 1026 (2012).
[44] W. McCutcheon *et al.*, Experimental verification of multipartite entanglement in quantum networks, Nat. Commun. **7**, 13251 (2016).
[45] Y. Cai, J. Roslund, G. Ferrini, F. Arzani, X. Xu, C. Fabre, and N. Treps, Multimode entanglement in reconfigurable graph states using optical frequency combs, Nat. Commun. **8**, 15645 (2017).
[46] C. Dong, V. Fiore, M. C. Kuzyk, and H. Wang, Optomechanical Dark Mode, Science **338**, 1609 (2012).
[47] Y.-D. Wang and A. A. Clerk, Using Interference for High Fidelity Quantum State Transfer in Optomechanics, Phys. Rev. Lett. **108**, 153603 (2012).
[48] L. Tian, Adiabatic State Conversion and Pulse Transmission in Optomechanical Systems, Phys. Rev. Lett. **108**, 153604 (2012).
[49] L. Tian, Robust Photon Entanglement via Quantum Interference in Optomechanical Interfaces, Phys. Rev. Lett. **110**, 233602 (2013).
[50] D. P. Lake, M. Mitchell, B. C. Sanders, and P. E. Barclay, Two-colour interferometry and switching through optomechanical dark mode excitation, Nat. Commun. **11**, 2208 (2020).
[51] X.-W. Xu and Y. Li, Optical nonreciprocity and optomechanical circulator in three-mode optomechanical systems, Phys. Rev. A **91**, 053854 (2015).
[52] M. A. Miri, F. Ruesink, E. Verhagen, and A. AlÅ¡Å½, Optical nonreciprocity based on optomechanical coupling, Phys. Rev. Applied **7**, 064014 (2017).
[53] X.-F. Zhou, X.-W. Luo, S. Wang, G.-C. Guo, X. Zhou, H. Pu, and Z.-W. Zhou, Dynamically Manipulating Topological Physics and Edge Modes in a Single Degenerate Optical Cavity, Phys. Rev. Lett. **118**, 083603 (2017).
[54] M. Schmidt, S. Kessler, V. Peano, O. Painter, and F. Marquardt, Optomechanical creation of magnetic fields for photons on a lattice, Optica **2**, 635 (2015).
[55] Z. Shen, Y.-L. Zhang, Y. Chen, C.-L. Zou, Y.-F. Xiao, X.-B. Zou, F.-W. Sun, G.-C. Guo, and C.-H. Dong, Experimental realization of optomechanically induced non-reciprocity, Nat. Photonics **10**, 657 (2016).
[56] F. Ruesink, M.-A. Miri, A. Alù, and E. Verhagen, Nonreciprocity and magnetic-free isolation based on optomechanical interactions, Nat. Commun. **7**, 13662 (2016).
[57] K. Fang, J. Luo, A. Metelmann, M. H. Matheny, F. Marquardt, A. A. Clerk, and O. Painter, Generalized noneciprocity in an optomechanical circuit via synthetic magnetism and reservoir engineering, Nat. Phys. **13**, 465 (2017).
[58] N. R. Bernier, L. D. Tóth, A. Koottandavida, M. A. Ioannou, D. Malz, A. Nunnenkamp, A. K. Feofanov, and T. J. Kippenberg, Nonreciprocal reconfigurable microwave optomechanical circuit, Nat. Commun. **8**, 604 (2017).
[59] Z. Shen, Y.-L. Zhang, Y. Chen, F.-W. Sun, X.-B. Zou, G.-C. Guo, C.-L. Zou, and C.-H. Dong, Reconfigurable optomechanical circulator and directional amplifier, Nat. Commun. **9**, 1797 (2018).
[60] F. Ruesink, J. P. Mathew, M.-A. Miri, A. Alù, and E. Verhagen, Optical circulation in a multimode optomechanical resonator, Nat. Commun. **9**, 1798 (2018).
[61] Y. Chen, Y.-L. Zhang, Z. Shen, C.-L. Zou, G.-C. Guo, and C.-H. Dong, Synthetic Gauge Fields in a Single Optomechanical Resonator, Phys. Rev. Lett. **126**, 123603 (2021).
[62] G. Adesso and F. Illuminati, Entanglement in continuous-variable systems: recent advances and current perspectives, J. Phys. A: Math. Theor. **40**, 7821 (2007); Continuous variable tangle, monogamy inequality, and entanglement sharing in Gaussian states of continuous variable systems, New J. Phys. **8**, 15 (2006); Entanglement sharing: from qubits to Gaussian states, Int. J. Quantum Inform. **04**, 383 (2006).
[63] J. Li, S.-Y. Zhu, and G. S. Agarwal, Magnon-Photon-Phonon Entanglement in Cavity Magnomechanics, Phys. Rev. Lett. **121**, 203601 (2018).
[64] V. Coffman, J. Kundu, and W. K. Wootters, Distributed entanglement, Phys. Rev. A **61**, 052306 (2000).
[65] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002); M. B. Plenio, Logarithmic Negativity: A Full Entanglement Monotone That is not Convex, Phys. Rev. Lett. **95**, 090503 (2005).
[66] E. X. DeJesus and C. Kaufman, Routh-Hurwitz criterion in the examination of eigenvalues of a system of nonlinear ordinary differential equations, Phys. Rev. A **35**, 5288 (1987).
[67] H. Deng, F. Zou, J.-F. Huang, and J.-Q. Liao, Optical normal-mode-induced phonon-sideband splitting in the photon-blockade effect, Phys. Rev. A **104**, 033706 (2021).
[68] D. Lee, M. Underwood, D. Mason, A. B. Shkarin, S. W. Hoch, and J. G. E. Harris, Multimode optomechanical dynamics in a cavity with avoided crossings, Nat. Commun. **6**, 6232 (2015).
[69] X.-W. Xu, Y. Li, A.-X. Chen, and Y.-x. Liu, Nonreciprocal conversion between microwave and optical photons in electro-optomechanical systems, Phys. Rev. A **93**, 023827 (2016).
[70] H. Xu, L. Jiang, A. A. Clerk, and J. G. E. Harris, Nonreciprocal control and cooling of phonon modes in an optomechanical system, Nature (London) **568**, 65 (2019).
[71] W. Dür, J. I. Cirac, and R. Tarrach, Separability and Distillability of Multiparticle Quantum Systems, Phys. Rev. Lett. **83**, 3562 (1999); W. Dür and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, Phys. Rev. A **61**, 042314 (2000).
[72] P. van Loock and S. L. Braunstein, Greenberger-Horne-Zeilinger nonlocality in phase space, Phys. Rev. A **63**, 022106 (2001).
[73] S. Pirandola, S. Mancini, D. Vitali, and P. Tombesi, Continuous-variable entanglement and quantum-state teleportation between optical and macroscopic vibrational modes through radiation pressure, Phys. Rev. A **68**, 062317 (2003).
[74] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[75] Z.-H. Ma, Z.-H. Chen, and J.-L. Chen, Christoph Spengler, Andreas Gabriel, Marcus Huber, Measure of genuine multipartite entanglement with computable lower bounds, Phys. Rev. A **83**, 062325 (2011).

[76] W.-C. Qiang, G.-H. Sun, Q. Dong, and S.-H. Dong, Genuine multipartite concurrence for entanglement of Dirac fields in noninertial frames, Phys. Rev. A **98**, 022320 (2018).

[77] S. Pirandola, S. Mancini, D. Vitali, and P. Tombesi, Light reflection upon a movable mirror as a paradigm for continuous variable teleportation network, J. Mod. Opt. **51**, 901 (2004).

[78] C. Genes, A. Mari, P. Tombesi, and D. Vitali, Robust entanglement of a micromechanical resonator with output optical fields, Phys. Rev. A **78**, 032316 (2008).

[79] J. S. Kim, G. Gour, and B. C. Sanders, Limitations to sharing entanglement, Contemp. Phys. **53**, 417 (2012).

[80] M. Pawłowski, Security proof for cryptographic protocols based only on the monogamy of Bell's inequality violations, Phys. Rev. A **82**, 032313 (2010).

[81] S. Schauer, M. Huber, and B. C. Hiesmayr, Experimentally feasible security check for n-qubit quantum secret sharing, Phys. Rev. A **82**, 062311 (2010).

[82] R. Raussendorf and H. J. Briegel, A One-Way Quantum Computer, Phys. Rev. Lett. **86**, 5188 (2001).

[83] A. Grudka, M. J. W. Hall, M. Horodecki, R. Horodecki, J. Oppenheim, and J. A. Smolin, Do black holes create polyamory? J. High Energ. Phys. 11 (2018) 045.