Supplemental Material:
Nonmesonic Quantum Many-Body Scars in a 1D Lattice Gauge Theory

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I. QUANTUM MANY-BODY SCARS

A. Proof of $\hat{H}_K \left|\Psi_{n,l}\right\rangle = 0$

In the main text, we show that the wave function

$$\left|\Psi_{n,l}\right\rangle = \mathcal{N}_{n,l} \sum P_{\{\mathcal{S}_{k_j,\ell_j}\}_n^l} \left|\{\mathcal{S}_{k_j,\ell_j}\}_n^l\right\rangle, \tag{S1}$$

is an exact eigenstate of $\hat{H}$. Here we present details for proving this result. It is not difficult to find that $\hat{H}_E \left|\Psi_{n,l}\right\rangle = h(2l - L)\left|\Psi_{n,l}\right\rangle$, so we only need to prove $\hat{H}_K \left|\Psi_{n,l}\right\rangle = 0$. Since the action of $\hat{H}_K$ is increasing or reducing the total string length by one, while keeping n invariant, we have

$$\hat{H}_K \left|\Psi_{n,l}\right\rangle = \sum c_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{l-1}} \left|\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{l-1}\right\rangle + \sum c_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{l+1}} \left|\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{l+1}\right\rangle. \tag{S2}$$

Here, the factors have forms

$$\begin{aligned}
c_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{l-1}} = \mathcal{N}_{n,l} \sum &\big[(1-\delta_{k'_1+\ell'_1+1,k'_2})(P_{\mathcal{S}_{k'_1,\ell'_1+1},\mathcal{S}_{k'_2,\ell'_2}\dots} + P_{\mathcal{S}_{k'_1,\ell'_1},\mathcal{S}_{k'_2-1,\ell'_2+1}\dots})\\
&+(1-\delta_{k'_2+\ell'_2+1,k'_3})(P_{\dots,\mathcal{S}_{k'_2,\ell'_2+1},\mathcal{S}_{k'_3,\ell'_3}\dots} + P_{\dots,\mathcal{S}_{k'_2,\ell'_2},\mathcal{S}_{k'_3-1,\ell'_3+1}\dots}) + \dots\\
c_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{l+1}} = \mathcal{N}_{n,l} \sum &\big[(1-\delta_{\ell'_1,1})(P_{\mathcal{S}_{k'_1-1,\ell'_1-1},\mathcal{S}_{k'_2,\ell'_2}\dots} + P_{\mathcal{S}_{k'_1,\ell'_1-1},\mathcal{S}_{k'_2,\ell'_2}\dots})\\
&+(1-\delta_{\ell'_2,1})(P_{\dots,\mathcal{S}_{k'_2-1,\ell'_2-1},\mathcal{S}_{k'_3,\ell'_3}\dots} + P_{\dots,\mathcal{S}_{k'_2,\ell'_2-1},\mathcal{S}_{k'_3,\ell'_3}\dots}) + \dots.
\end{aligned} \tag{S3}$$

Since the parity satisfies $P_{\mathcal{S}_{k'_1,\ell'_1},\dots,\mathcal{S}_{k'_j,\ell'_1},\dots,\mathcal{S}_{k'_n,\ell'_n}} = \exp\left(i\pi \sum_j k'_j\right)$, we have

$$\begin{aligned}
P_{\dots\mathcal{S}_{k'_{j-1},\ell'_{j-1}},\mathcal{S}_{k'_j,\ell'_j+1},\mathcal{S}_{k'_{j+1},\ell'_{j+1}}\dots} &= -P_{\dots\mathcal{S}_{k'_{j-1},\ell'_{j-1}},\mathcal{S}_{k'_j,\ell'_j},\mathcal{S}_{k'_{j+1}-1,\ell'_{j+1}+1}\dots}\\
P_{\dots\mathcal{S}_{k'_{j-1},\ell'_{j-1}},\mathcal{S}_{k'_j-1,\ell'_j-1},\mathcal{S}_{k'_{j+1},\ell'_{j+1}}\dots} &= -P_{\dots\mathcal{S}_{k'_{j-1},\ell'_{j-1}},\mathcal{S}_{k'_j,\ell'_j-1},\mathcal{S}_{k'_{j+1},\ell'_{j+1}}\dots}.
\end{aligned} \tag{S4}$$

Therefore, $c_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{l-1}} = c_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{l+1}} = 0$, i.e., $\hat{H}_K \left|\Psi_{n,l}\right\rangle = 0$.

B. Proof of $\left|\Psi_{n,n+m}\right\rangle = \mathcal{D}_{n,m}\hat{L}_m^\dagger \left|\Psi_{n,n+m-1}\right\rangle$

Next we show the detail of proving Eq. (8) in the main text, i.e,

$$\left|\Psi_{n,n+m}\right\rangle = \mathcal{D}_{n,m}\hat{L}_m^\dagger \left|\Psi_{n,n+m-1}\right\rangle, \tag{S5}$$

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where $\mathcal{D}_{n,m}$ is a normalization factor, and

$$\hat{L}_m^\dagger = \sum_j \Big(\sum_{k\le m} \prod_{\ell\le k} \hat{\mathcal{P}}^-_{j+\frac{1}{2}-\ell} \Big) \hat{\sigma}_j^- \hat{\tau}^{\mathrm{M}}_{j+\frac{1}{2}} \hat{\sigma}^+_{j+1} \tag{S6}$$

It is not difficult to demonstrate

$$\Big(\sum_{k\le m} \prod_{\ell\le k} \hat{\mathcal{P}}^-_{j+\frac{1}{2}-\ell} \Big) \hat{\sigma}_j^- \hat{\tau}^{\mathrm{M}}_{j+\frac{1}{2}} \hat{\sigma}^+_{j+1} |\mathcal{S}_{k,\ell}\rangle = \begin{cases} \ell\delta_{j,k+\ell} |\mathcal{S}_{k,\ell+1}\rangle & \ell \le m \\ m\delta_{j,k+\ell} |\mathcal{S}_{k,\ell+1}\rangle & \ell > m. \end{cases} \tag{S7}$$

Thus, the action of $\hat{L}_m^\dagger$ is increasing the total string length of a basis without changing the parity and string number. Therefore,

$$\hat{L}_m^\dagger |\Psi_{n,n+m-1}\rangle = \sum \alpha_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{n+m}} |\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{n+m}\rangle . \tag{S8}$$

For the wave function $|\Psi_{n,n+m-1}\rangle = \mathcal{N}_{n,n+m-1} \sum P_{\{\mathcal{S}_{k_j,\ell_j}\}_n^{n+m-1}} |\{\mathcal{S}_{k_j,\ell_j}\}_n^{n+m-1}\rangle$, the length of each string satisfies $\ell_j \le m$. Hence, the factor has the form

$$\alpha_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{n+m}} = \mathcal{N}_{n,n+m-1} P_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{n+m}} [(1-\delta_{\ell'_1,1})(\ell'_1-1) + (1-\delta_{\ell'_2,1})(\ell'_2-1) + \ldots + (1-\delta_{\ell'_n,1})(\ell'_n-1)]. \tag{S9}$$

If $\ell'_j = 1$, then $(1-\delta_{\ell'_j,1})(\ell'_j-1) = (\ell'_j-1) = 0$, and if $\ell'_j \ne 1$, then $(1-\delta_{\ell'_j,1})(\ell'_j-1) = (\ell'_j-1)$. Thus

$$\alpha_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{n+m}} = \mathcal{N}_{n,n+m-1} P_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{n+m}} \sum_{j=1}^{n} (\ell'_j - 1) = \mathcal{N}_{n,n+m-1} P_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{n+m}} (m-1). \tag{S10}$$

Therefore, we have

$$\hat{L}_m^\dagger |\Psi_{n,n+m-1}\rangle = (m-1)\mathcal{N}_{n,n+m-1} \sum P_{\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{n+m}} |\{\mathcal{S}_{k'_j,\ell'_j}\}_n^{n+m}\rangle = \frac{(m-1)\mathcal{N}_{n,n+m-1}}{\mathcal{N}_{n,n+m}} |\Psi_{n,n+m}\rangle . \tag{S11}$$

That is, Eq. (8) is proved, and the normalization factor satisfies

$$\mathcal{D}_{n,m} = \frac{\mathcal{N}_{n,n+m}}{(m-1)\mathcal{N}_{n,n+m-1}}. \tag{S12}$$

II. INITIAL STATE

Here we discuss the initial state $|\psi_2\rangle$ in Eq. (10b) of the main text, where it reads

$$|\psi_2\rangle = \frac{1}{2^{L/2}} \sum_{n,l} \sum_{\{k_j,\ell_j\}} P_{\{\mathcal{S}_{k_j,\ell_j}\}_n^l} |\{\mathcal{S}_{k_j,\ell_j}\}_n^l\rangle = \sum_{n,l} \beta_{n,l} |\Psi_{n,l}\rangle . \tag{S13}$$

The amplitude $\beta_{n,l}$ satisfies $\beta_{n,l} = 1/\mathcal{N}_{n,l} 2^{L/2}$, where $\mathcal{N}_{n,l}$ is the normalization factor defined in Eq. (6) of the main text. In addition, $\mathcal{N}_{n,l}^{-2}$ is the number of string bases for the scar state $|\Psi_{n,l}\rangle$, and it can be obtained as

$$\mathcal{N}_{n,l}^{-2} = \binom{l-1}{n-1} \left[\binom{L-l-1}{n} + 2\binom{L-l-1}{n-1} \right] + \binom{L-l-1}{n-1}\binom{l-1}{n}, \tag{S14}$$

where $\binom{\cdot}{\cdot}$ is the combinatorial number. In Fig. S1, we show the result of $\mathcal{N}_{n,l}^{-2}$ versus l for $L = 32$ and $n = 8$ (half filling). We can find that $\mathcal{N}_{n,l}^{-2}$ nearly satisfies a Gaussian distribution with the symmetric point at $l = L/2$. Therefore, for the initial state $|\psi_2\rangle$ the nonmesonic scar states dominate.

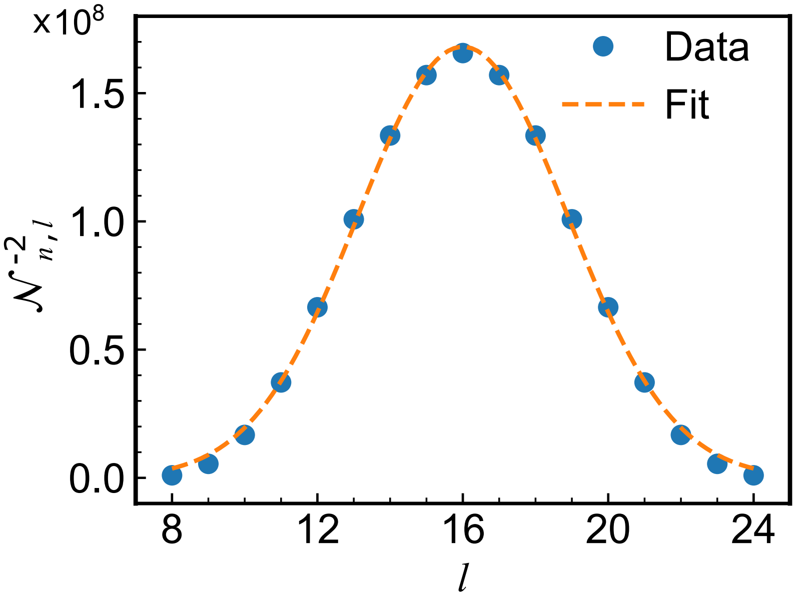

FIG. S1. Distribution of $\mathcal{N}_{n,l}^{-2}$ for $L = 32$ and $n = 8$ (half filling). The orange dashed curve is a Gaussian fit.