- Supplemental Material -
Quantum Phase Transitions in Optomechanical Systems

PHOTON OCCUPATIONS IN GROUND STATE

In this section, we interpret why the Hamiltonian,

$$\widetilde{H}_{\rm om} = a^\dagger a - \xi^2 a^\dagger a a^\dagger a \tag{S1}$$

with eigenvalues

$$\widetilde{E}_{\rm om} = n - \xi^2 n^2, \tag{S2}$$

does not have a well-defined ground state in the whole parameter space if the free term $a^\dagger a$ is not considered as a perturbation.

For a general Hamiltonian $H = H_0 + \lambda H_{\rm I}$, the interaction term $H_{\rm I}$ would be conventionally considered as a perturbation if the coupling strength λ is much smaller than the work frequency of the system. In this scenario, the ground state is mainly determined by the free term H_0. When increasing λ, $H_{\rm I}$ becomes dominant, leading to the ground state being determined by both the free and interaction terms.

The above description can be applied to the Dicke model and Rabi model. However, it is not true in Eq. (S1) even though the parameter ξ is extremely small. This is because we can always find an eigenstate $|n\rangle$ in $\widetilde{H}_{\rm om}$ where its corresponding eigenvalue is smaller than that of the vacuum state $|0\rangle$ (which is, in fact, the lowest energy state of $a^\dagger a$), regardless of the value of this parameter ξ.

To clearly illustrate this point, we can evaluate the extrema of the energy,

$$\frac{d\widetilde{E}_{\rm om}}{d\xi} = -2n^2\xi = 0 \qquad \text{and} \qquad \frac{d^2\widetilde{E}_{\rm om}}{d\xi^2} = -2n^2 < 0, \tag{S3}$$

and find that for each value of n, the energy spectrum Eq. (S2) has only a maximum value but no minimum value, as shown in Fig. S1. Additionally, Fig. S1 shows that, as n increases, the value of the intersection ξ_0 between $\widetilde{E}_{\rm om}(n)$ and $\widetilde{E}_{\rm om} = 0$ (vacuum energy) will tend to zero. More precisely, by evaluating the formula $n - \xi_0^2 n^2 = 0$, it is not hard to find that when $n \to \infty, \xi_0 \to 0$. These analyses illustrate that no well-defined ground state can be determined by the term $a^\dagger a$.

If the free term $a^\dagger a$ is considered as a perturbation, we can examine whether the ground state is determined by the quartic term $a^\dagger a a^\dagger a$. Under such a strategy, the negative half-axis of $\widetilde{E}_{\rm om}$ in Fig. S1 constitutes the component of the excitation spectrum of the system. Therefore, for $\xi \in [1, \infty]$, the lowest energy of system should be $\widetilde{E}_G = 0$, giving the lowest level $|0\rangle$.

When ξ decreases until $\xi = 1$, there occurs a level crossing between the states $|0\rangle$ and $|1\rangle$, indicating that the perturbation term $a^\dagger a$ gradually becomes dominant. For $\xi < 1$, the lowest-energy state will meet a series of level crossings between the states $|n\rangle$ and $|n+1\rangle$, showing the instability. Finally, we obtain a well-defined ground state.

In the main text, the parameter ξ becomes the dimensionless coupling strength $\kappa = \sqrt{\omega_m \omega_c}/g$. Due to the excitation spectrum on the negative half-axis of $\widetilde{E}_{\rm om}$ shown in Fig. S1, performing the transformation $\widetilde{E}_{\rm om} \to -\widetilde{E}_{\rm om}$, which only changes the reference frame, can conveniently capture the physics, as shown in Figs.1(a), 2(a-c) in the main text.

SQUEEZED VACUUM IN THE PHOTON AND PHONON MODES

In this section, we show the squeezed vacuum of the cavity and mechanical modes at critical points, both of which have infinitely squeezed vacuum when $\eta \to \infty$ and have inseparable relation between the two squeezed states when η is finite. First, by applying a displacement transformation with $U_1 = \exp[-(g/\omega_m)(b^\dagger - b)]$ to the system Hamiltonian,

$$H = \omega_c a^\dagger a + \omega_m b^\dagger b + g(a + a^\dagger)^2 (b + b^\dagger), \tag{S4}$$

the transformed Hamiltonian becomes

$$\bar{H} = \omega_c a^\dagger a + \omega_m b^\dagger b - \frac{2g^2}{\omega_m}(a + a^\dagger)^2 + \frac{g^2}{\omega_m} + g(a^2 + a^{\dagger 2})(b + b^\dagger) + 2g a^\dagger a (b + b^\dagger). \tag{S5}$$

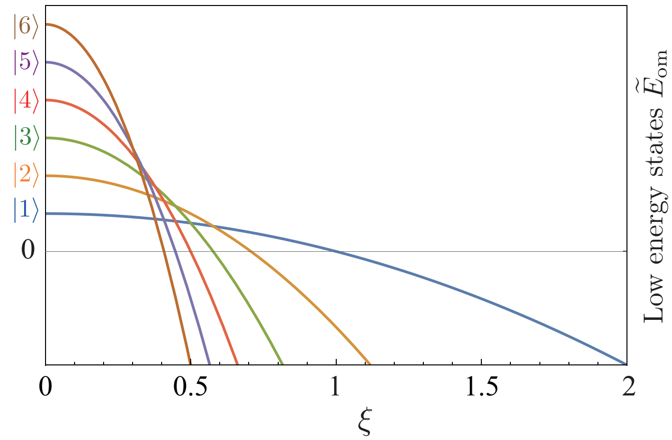

FIG. S1. Level crossings of the low-energy states $\widetilde{E}_{\rm om}$, according to Eq. (S2). As n increases, the value of the intersection between the states $|n\rangle$ and $|0\rangle$ tends to zero, meaning that there is no well-defined ground state if the term $a^\dagger a a^\dagger a$ is considered as a perturbation.

For convenience, this transformed Hamiltonian can also be rewritten as

$$\bar{H}_f = a^\dagger a + \eta^{-1} b^\dagger b - \frac{1}{4}\gamma^2 (a+a^\dagger)^2 + \frac{1}{8}\gamma^2 + \frac{1}{2\sqrt{2}}\gamma\eta^{-\frac{1}{2}}(a^2 + a^{\dagger 2})(b+b^\dagger) + \frac{1}{\sqrt{2}}\gamma\eta^{-\frac{1}{2}} a^\dagger a (b+b^\dagger), \tag{S6}$$

where $\gamma = 2\sqrt{2}g/\sqrt{\omega_c\omega_m}$ is the dimensionless coupling strength and $\eta = \omega_c/\omega_m$. Note that the last two terms in Eq. (S6) have a factor with a negative power of η. In the limit $\eta \to \infty$, the coefficients of these nonquadratic terms Eq. (S6) become zero, leading to

$$\bar{H}_f = a^\dagger a - \frac{1}{4}\gamma^2 (a+a^\dagger)^2 + \frac{1}{8}\gamma^2. \tag{S7}$$

Obviously, the Hamiltonian satisfies a Z_2 parity symmetry, which is obtained by eliminating the non-symmetry part with the classical limit. Further, equation (S7) can be diagonalized as

$$\bar{H}_f = 2\varepsilon_{\rm np} d^\dagger d + \frac{1}{8}\gamma^2 + \varepsilon_{\rm np} - \frac{1}{2}, \tag{S8}$$

with $\varepsilon_{\rm np} = \sqrt{(1/4)(1-\gamma^2)}$, which is vaild for $\gamma < 1$ and vanishes at $\gamma = 1$. The eigenstates of $\bar{H}_f$ for $\gamma < 1$ are $|\psi\rangle(\gamma) = S(\xi(\gamma))|n\rangle|k\rangle$, where $S(x) = \exp[(x/2)(a^{\dagger 2} - a^2)]$ and $\xi(\gamma) = -(1/4)\ln(1-\gamma^2)$.

From Eq. (S8), we can find an infinitely squeezed vacuum of cavity mode at $\gamma = 1$, which implies an infinitely squeezed photon condensate in the ground state. It is not hard to imagine that the squeezing phenomenon of the cavity mode will give feedback to the mechanical mode by the radiation pressure.

To further explore the underlying physics, one can employ a transformation with the unitary operator, $U_2 = \exp[-(g/2\omega_c)(b+b^\dagger)(a^{\dagger 2} - a^2)]$, to the Hamiltonian $\bar{H}$ in Eq. (S5). The transformed Hamiltonian can be expressed as $\widetilde{H} = \widetilde{H}_e + \widetilde{H}_o$, where

$$\begin{aligned}\widetilde{H}_e = {} & \omega_c a^\dagger a + \omega_m b^\dagger b - \frac{1}{2!}\frac{g^2\omega_m}{2\omega_c^2}(a^{\dagger 2} - a^2)^2 + \sum_{n=1}^{\infty}\left[\frac{1}{(2n)!} - \frac{1}{(2n-1)!}\right] 2^{2n-3}\frac{g^{2n}}{\omega_c^{2n-1}}(b+b^\dagger)^{2n}(8a^\dagger a + 4) \\ & - \sum_{n=0}^{\infty}\frac{1}{(2n)!}2^{2n+1}\frac{g^{2n+2}}{\omega_m\omega_c^{2n}}(b+b^\dagger)^{2n}(a^\dagger + a)^2 - \sum_{n=1}^{\infty}\frac{1}{(2n-1)!}2^{2n-1}\frac{g^{2n}}{\omega_c^{2n-1}}(b+b^\dagger)^{2n}(a^{\dagger 2} + a^2) + \frac{g^2}{\omega_m},\end{aligned} \tag{S9}$$

which consists of even-order operators, while $\widetilde{H}_o$ consists of odd-order operators.

This Hamiltonian $\widetilde{H}$ still cannot be diagonalized, but a variational method can be used for analysis. We propose a trial wave function $|\psi(r,s)\rangle = \mathcal{S}_a(r)\mathcal{S}_b(s)|0_a, 0_b\rangle$ for $\widetilde{H}$, where $\mathcal{S}_y(x) = \exp[(x^2/2)(y^{\dagger 2} - y^2)]$ is the squeezing operator with a bosonic operator $y \in [a,b]$ and a variational parameter $x \in [r,s]$. The energy function can be obtained as follows

$$\begin{aligned}\widetilde{E}(r,s) =\ & \sinh^2(r) + \eta^{-1}\sinh^2(s) + \sum_{n=1}^{\infty}\left[\frac{1}{(2n)!} - \frac{1}{(2n-1)!}\right]2^{2n-3}\frac{1}{8^n}\gamma^{2n}\eta^{-n}\frac{(2n)!}{2^n n!}e^{2ns}[8\sinh^2(r)+4] + \frac{1}{8}\gamma^2 \\ & - \sum_{n=0}^{\infty}\frac{1}{(2n)!}2^{2n+1}\frac{1}{8^{n+1}}\gamma^{2n+2}\eta^{-n}\frac{(2n)!}{2^n n!}e^{2ns}e^{2r} - \sum_{n=1}^{\infty}\frac{1}{(2n-1)!}2^{2n-1}\frac{1}{8^n}\gamma^{2n}\eta^{-n}\frac{(2n)!}{2^n n!}e^{2ns}\sinh(2r),\end{aligned} \tag{S10}$$

where $\widetilde{E}(r,s)$ has been renormalized by ω_c. By minimizing the energy function with respect to the r and s, we can obtain

$$\exp(4r) = \frac{1+A_1+C_1}{1+A_1-B_1-C_1}, \tag{S11}$$

where

$$\begin{aligned}A_1 &= \sum_{n=1}^{\infty}\left[\frac{1}{(2n)!} - \frac{1}{(2n-1)!}\right]2^{2n-3}\frac{1}{8^{n-1}}\gamma^{2n}\eta^{-n}\frac{(2n)!}{2^n n!}e^{2ns}, \\ B_1 &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}2^{2n+3}\frac{1}{8^{n+1}}\gamma^{2n+2}\eta^{-n}\frac{(2n)!}{2^n n!}e^{2ns}, \\ C_1 &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)!}2^{2n}\frac{1}{8^n}\gamma^{2n}\eta^{-n}\frac{(2n)!}{2^n n!}e^{2ns},\end{aligned} \tag{S12}$$

and

$$\exp(4s) = \frac{1+A_2}{1-\gamma^2\left[\sinh^2(r)+\frac{1}{2}+\sinh(2r)+\frac{1}{4}\gamma^2 e^{2r}\right]}, \tag{S13}$$

where

$$\begin{aligned}A_2 =\ & -\left[8\sinh^2(r)+4\right]\sum_{n=2}^{\infty}\left[\frac{1}{(2n)!} - \frac{1}{(2n-1)!}\right]2^{2n-2}\frac{1}{8^n}\gamma^{2n}\eta^{1-n}\frac{(2n)!}{2^n n!}e^{2ns+2s} \\ & + e^{2r}\sum_{n=2}^{\infty}\frac{1}{(2n)!}2^{2n+2}\frac{1}{8^{n+1}}\gamma^{2n+2}\eta^{1-n}\frac{(2n)!}{2^n n!}e^{2ns+2s}2n \\ & + \sinh(2r)\sum_{n=2}^{\infty}\frac{1}{(2n-1)!}2^{2n}\frac{1}{8^n}\gamma^{2n}\eta^{1-n}\frac{(2n)!}{2^n n!}e^{2ns+2s}2n.\end{aligned} \tag{S14}$$

Now we have derived the results of the squeezed vacuum of photon and phonon mode, as shown in Eqs. (S11) and (S13).

When $\eta \to \infty$, Eqs. (S11) and (S13) can be rewritten as

$$\exp(4r) = \frac{1}{1-\gamma^2} \tag{S15}$$

and

$$\exp(4s) = \frac{1}{1-\gamma^2\left[\sinh^2(r)+\frac{1}{2}+\sinh(2r)+\frac{1}{4}\gamma^2 e^{2r}\right]}, \tag{S16}$$

which are given in the main text. Note that after implementing the unitary transformation $U_2^\dagger \bar{H} U_2$, we obtain not only the result in Eq. (S15), which is consistent with that in Eq. (S8), but also the squeezing information of the mechanical mode shown in Eq. (S16), which is induced by the squeezed field of the cavity mode.

It is clear that upon displacing the mechanical mode by g/ω_m, the squeezed vacuum of cavity mode could be induced and subsequently gives feedback to the mechanical mode via radiation pressure. Notably, such feedback would not once again transfer to the cavity mode in the classical limit $\eta \to \infty$, Eq. (S15), except for finite η, Eq. (S11).

In the main text, without loss of generality, we have chosen the energy function $\widetilde{E}(r,s)$ up to the fourth order for analyzing the case of $\eta \to \infty$ and finite η, whose conclusions are the same as that of Eqs. (S11) and (S13).

MODIFYING A QPT IN OPTOMECHANICAL SYSTEM

In the previous section, Eq. (S8) showed that the normal phase occurs for $\gamma < 1$. In this section, we will show that when the system is driven by a squeezed field of the cavity mode with a suitable squeezing parameter, the occurrence of the normal phase can be allowed to occur in the region $\gamma > 1$, and the coupling strength required to reach the critical point can be significantly reduced.

By employing a transformation with the unitary operator, $S_\zeta = \exp\left[(1/2)(\zeta^* a^2 - \zeta a^{\dagger 2})\right]$, where $\zeta = \xi e^{i\theta}$ represents the squeezed parameter, to the system Hamiltonian H in Eq. (S4), with the squeezing driven term $\xi(a^{\dagger 2}e^{-i\theta} + a^2 e^{i\theta})$, we can obtain a transformed Hamiltonian

$$\begin{aligned} H_{(\theta,\xi)} = & \frac{\omega_c}{2}\cosh(2\xi)(2a^\dagger a+1) - \frac{\omega_c}{2}\sinh(2\xi)(e^{i\theta}a^{\dagger 2} + e^{-i\theta}a^2) - \frac{\omega_c}{2} \\ & + \omega_m b^\dagger b + g(b+b^\dagger)\left[\cosh(\xi)(a+a^\dagger) - \sinh(\xi)(e^{-i\theta}a + e^{i\theta}a^\dagger)\right]^2 \\ & + S_\zeta^\dagger\left[\xi(a^{\dagger 2}e^{-i\theta} + a^2 e^{i\theta})\right]S_\zeta, \end{aligned} \tag{S17}$$

which depends on the squeezing direction θ and the squeezing amplitude ξ.

When the squeezing direction is $\theta = 0$, the transformed Hamiltonian can be written as

$$\begin{aligned} H_{(\theta=0,\xi)} = & \frac{\omega_c}{2}\cosh(2\xi)(2a^\dagger a+1) - \frac{\omega_c}{2}\sinh(2\xi)(a^{\dagger 2} + a^2) - \frac{\omega_c}{2} \\ & + \omega_m b^\dagger b + g e^{-2\xi}(a+a^\dagger)^2(b+b^\dagger) + S_{(\theta=0,\xi)}^\dagger\left[\xi(a^{\dagger 2} + a^2)\right]S_{(\theta=0,\xi)}. \end{aligned} \tag{S18}$$

In order to find the anti-squeezing term of the cavity mode, which can induce a singularity of the intrinsic squeezed vacuum, one can perform a displacement transformation with $\widetilde{U}_{\theta\to 0} = \exp[-(ge^{-2\xi}/\omega_m)(b^\dagger - b)]$, to the Hamiltonian $H_{(\theta=0,\xi)}$. The transformed Hamiltonian becomes

$$\begin{aligned} \widetilde{H}_{(\theta=0,\xi)} = & \frac{1}{2}\cosh(2\xi)(2a^\dagger a+1) - \frac{1}{2}\sinh(2\xi)(a^{\dagger 2} + a^2) - \frac{1}{2} + \eta^{-1}b^\dagger b - \frac{1}{4}\gamma^2 e^{-4\xi}(a+a^\dagger)^2 \\ & + \frac{1}{2\sqrt{2}}\gamma\eta^{-\frac{1}{2}}e^{-2\xi}(a^2 + a^{\dagger 2})(b+b^\dagger) + \frac{1}{\sqrt{2}}\gamma\eta^{-\frac{1}{2}}e^{-2\xi}a^\dagger a(b+b^\dagger) + \frac{1}{8}\gamma^2 e^{-4\xi} \\ & + \eta^{-1}\frac{1}{\omega_m}S_{(\theta=0,\xi)}^\dagger\left[\xi(a^{\dagger 2} + a^2)\right]S_{(\theta=0,\xi)}, \end{aligned} \tag{S19}$$

where $\widetilde{H}_{(\theta=0,\xi)}$ has been renormalized by the cavity frequency ω_c. In the classical limit, $\eta \to \infty$, Eq. (S19) can be written as

$$\widetilde{H}_{(\theta=0,\xi)} = \frac{1}{2}\cosh(2\xi)(2a^\dagger a+1) - \frac{1}{2}\sinh(2\xi)(a^{\dagger 2} + a^2) - \frac{1}{4}\gamma^2 e^{-4\xi}(a+a^\dagger)^2 + \frac{1}{8}\gamma^2 e^{-4\xi} - \frac{1}{2}. \tag{S20}$$

Equation (S20) can be diagonalized giving

$$\widetilde{H}_{(\theta=0,\xi)} = 2\varepsilon_\xi d^\dagger d + \frac{1}{8}\gamma^2 e^{-4\xi} + \varepsilon_\xi - \frac{1}{2}, \tag{S21}$$

with

$$\varepsilon_\xi = \frac{1}{2}\sqrt{1 - \gamma^2\exp(-2\xi)}. \tag{S22}$$

When $\xi = 0$, the result of Eq. (S22) is the same as that of Eq. (S8). However, for $\xi = 2\ln(\gamma)$, Eq. (S22) becomes

$$\varepsilon_{\xi\to 2\ln(\gamma)} = \frac{1}{2}\sqrt{1-\gamma^{-2}}, \tag{S23}$$

which is real only for $\gamma \geq 1$ and vanishes at $\gamma = 1$. This result shows that exploiting a squeezed field of the cavity mode with an appropriate squeezing parameter ζ to drive the system can alter the region where the normal phase occurs. Moreover, by employing the strategy used in the section "superradiant phase"of the main text to Eq. (S18), we can also determine that the corresponding superradiant phase occurs in the region $\gamma < 1$ for the same squeezing amplitude $\xi = 2\ln(\gamma)$.

Now we discuss the case when the squeezing direction is chosen as $\theta=\pi$. The transformed Hamiltonian becomes

$$\begin{aligned}H_{\theta=\pi,\xi}=&\frac{\omega_c}{2}\cosh(2\xi)(2a^\dagger a+1)+\frac{\omega_c}{2}\sinh(2\xi)(a^{\dagger 2}+a^2)-\frac{\omega_c}{2}\\&+\omega_m b^\dagger b+ge^{2\xi}(a+a^\dagger)^2(b+b^\dagger)+S^\dagger_{(\theta=\pi,\xi)}\left[\xi(a^{\dagger 2}+a^2)\right]S_{(\theta=\pi,\xi)}.\end{aligned}\tag{S24}$$

After implementing the above similarity transformation, we can obtain

$$\widetilde{H}_{(\theta=\pi,\xi)}=2\varepsilon_{(\theta=\pi,\xi)}d^\dagger d+\frac{1}{8}\gamma^2e^{4\xi}+\varepsilon_{(\theta=\pi,\xi)}-\frac{1}{2},\tag{S25}$$

with

$$\varepsilon_{(\theta=\pi,\xi)}=\frac{1}{2}\sqrt{1-\gamma^2\exp(2\xi)}.\tag{S26}$$

This result shows that increasing the squeezing amplitude ξ can exponentially reduce the coupling strength g required to reach the critical point.

In addition, these results do not apply to the Rabi or Dicke models, as the cavity fields of these two models have no deterministic symmetry. Therefore, if one controllable squeezed field of the cavity mode is exploited to drive the hybrid quantum system described in the main text, it is possible to find two superradiant phases, which are induced by the optomechanical system and light-atom system, respectively, and separated by the hybrid critical point. The two ordered phases could be characterized by two types of thermodynamic limits. Therefore, they belong to distinct symmetry-broken phases. In this scenario, it naturally arises the exciting topic of whether the hybrid system can undergo a direct second-order QPT between the two ordered phases beyond the Landau-Ginzburg-Wilson paradigm.

QUANTUM PHASE TRANSITIONS IN HYBRID QUANTUM SYSTEMS

In this section, we will examine the characteristics of the superradiant phase in the hybrid quantum system. To determine the superradiant phase, we must find the macroscopic coherence of the cavity mode in the ground state. Generally, when the ground-state energy has been weighted by one thermodynamic limit, the macroscopic coherence of the cavity mode can be evaluated using the mean-field approach. In the case of a hybrid quantum system, the macroscopic coherence would be determined by the two types of thermodynamic limits, where one of them is described by $\omega_c/\omega_m\to\infty$ and the other is described by $N\to\infty$.

Due to the possibility that these two limits may affect the nontrivial phase of the system, either competitively or independently, finding the solutions of macroscopic coherence in the ground-state energy using the mean-field approach with the two limits is very complicated. However, suppose we need to roughly capture the potential characteristics of the superradiant phase in the hybrid quantum system. In this case, we may only consider one of the two limits in the mean-field energy, while the other could become a large constant. Here, we will only consider the limit, $N\to\infty$, for the mean-field approach.

The hybrid quantum system can be described by

$$H_{\mathrm{h}}=\omega_c a^\dagger a+\omega_m b^\dagger b+g(a+a^\dagger)^2(b+b^\dagger)+\omega_a J_z+\frac{\lambda}{\sqrt{N_a}}(a+a^\dagger)(J_++J_-)+\frac{\alpha\lambda^2}{\omega_a}(a+a^\dagger)^2,\tag{S27}$$

where the cavity frequency and the atomic transition frequency are in resonance, namely, $\omega_c=\omega_a$. Note that using the Hamiltonian H_{op} in Eq.(10) in the main text, instead of $H=\omega_c a^\dagger a+\omega_m b^\dagger b+g(a+a^\dagger)^2(b+b^\dagger)$ to describe the optomechanical system, we can also reach the same conclusions. By displacing the mechanical mode with the single-photon coupling strength g/ω_m, the system Hamiltonian can be transformed to

$$H_{\mathrm{h}}=\omega_c a^\dagger a+\omega_m b^\dagger b+g(a^2+a^{\dagger 2}+2a^{\dagger 2}a)(b+b^\dagger)+\left(\frac{\alpha\lambda^2}{\omega_a}-\frac{2g^2}{\omega_m}\right)(a+a^\dagger)^2+\frac{g^2}{\omega_m}+\omega_a J_z+\frac{\lambda}{\sqrt{N_a}}(a+a^\dagger)(J_++J_-).\tag{S28}$$

After the displacement, the cavity field can reach a singularity induced by the two terms, $-2g^2/\omega_m(a+a^\dagger)^2$ and $(\lambda/\sqrt{N_a})(a+a^\dagger)(J_++J_-)$, together. However, the latter will be suppressed if we consider the $\boldsymbol{A}^2$ term. Now, we first consider the limit, $\omega_c/\omega_m\to\infty$, Eq. (S28) can be written as

$$\widetilde{H}_{\mathrm{h}}=a^\dagger a+\left(\frac{\alpha\lambda^2}{\omega_a\omega_c}-\frac{2g^2}{\omega_m\omega_c}\right)(a+a^\dagger)^2+\frac{g^2}{\omega_m\omega_c}+\frac{\omega_a}{\omega_c}J_z+\frac{\lambda}{\omega_c\sqrt{N_a}}(a+a^\dagger)(J_++J_-),\tag{S29}$$

where $\widetilde{H}_{\rm h}$ is renormalized by ω_c.

By applying a squeezing transformation, $a = \cosh(r)d + \sinh(r)d^\dagger$ with a squeezing parameter $r = (-1/4)\ln(1 + \alpha\mu^2 - \gamma^2)$ ($\mu = 2\lambda/\sqrt{\omega_a\omega_c}$ is a dimensionless coupling strength of light-atoms interacting system), the Hamiltonian $\widetilde{H}_{\rm h}$ can be written in a more compact form,

$$\widetilde{H}_{\rm h} = \widetilde{\omega}_c d^\dagger d + \omega_a J_z + \frac{\widetilde{\lambda}}{\sqrt{N_a}}(d + d^\dagger)(J_+ + J_-) + \widetilde{Q}, \tag{S30}$$

where $\widetilde{\omega}_c = \omega_c e^{-2r}$, $\widetilde{\lambda} = \lambda e^r$, and $\widetilde{Q} = (\omega_c/2)(e^{-2r} - 1) + g^2/\omega_m$. By using the Holstein-Primakoff approach with the transformation $J_+ = d^\dagger\sqrt{N_a - d^\dagger d}$, $J_- = \sqrt{N_a - d^\dagger d}d$, and $J_z = d^\dagger d - N_a/2$, Eq. (S30) becomes

$$\widetilde{H}_{\rm h(np)} = \widetilde{\omega}_c d^\dagger d + \omega_a(c^\dagger c - \frac{N_a}{2}) + \frac{\widetilde{\lambda}}{\sqrt{N_a}}\left(d + d^\dagger\right)\left(c^\dagger\sqrt{N_a - c^\dagger c} + \sqrt{N_a - c^\dagger c}\ c\right) + \tilde{Q}, \tag{S31}$$

whose energy reads

$$\epsilon_\pm^{\rm np} = \sqrt{\frac{1}{2}\left(\widetilde{\omega}_c^2 + \omega_a^2 \pm \sqrt{(\widetilde{\omega}_c^2 - \omega_a^2)^2 + 16\widetilde{\lambda}^2\widetilde{\omega}_c\omega_a}\right)}. \tag{S32}$$

The excitation energy of the lowest branch ε_- vanishes at $\mu^2(1-\alpha) + \gamma^2 = 1$, locating the quantum critical point for the hybrid quantum system, as shown in Eq.(15) in the main text.

In the following, we can determine the superradiant phase of the system by only considering the thermodynamic limit, $N \to \infty$, in the mean-field approach. Through displacing the bosonic operator with respect to their mean value, i.e., $d \to d + \sqrt{N_a}\zeta$ and $c \to c + \sqrt{N_a}\beta$, for the Hamiltonian $\widetilde{H}_{\rm h(np)}$, we can derive the Hamiltonian $\widetilde{H}_{\rm h(sp)}$, whose ground state energy reads

$$E_G = N_a\omega_a|\beta|^2 + N_a\widetilde{\omega}_c|\zeta|^2 + 4\widetilde{\lambda}N_a\zeta\beta\sqrt{1-\beta^2} - \frac{N_a}{2}\omega_a. \tag{S33}$$

By minimizing the ground state energy with respect to α and β, we can obtain

$$\begin{aligned} \beta &= \pm\sqrt{\frac{1}{2}\left(1 - \widetilde{\delta}^{-2}\right)}, \\ \zeta &= \mp\sqrt{\frac{\omega_a}{4\widetilde{\omega}_c}}\sqrt{\widetilde{\delta}^2 - \widetilde{\delta}^{-2}}, \end{aligned} \tag{S34}$$

where $\widetilde{\delta} = 2\widetilde{\lambda}/\sqrt{\omega_a\widetilde{\omega}_c}$ is a dimensionless coupling strength. The nonzero value of ζ can be found for $\widetilde{\delta} > 1$ and indicates a nonzero coherence of the boson field in the ground state, which is an order parameter of the superradiant phase transition. According to those nonzero coherences in Eq. (S34), the Hamiltonian of the superradiant phase can be written as

$$\widetilde{H}_{\rm h(sp)} = \widetilde{\omega}_c d^\dagger d + \left(\omega_a - \frac{2\widetilde{\lambda}\zeta\beta}{\sqrt{1-\beta^2}}\right)c^\dagger c + \left(\widetilde{\lambda}\sqrt{1-\beta^2} - \frac{\widetilde{\lambda}\beta^2}{\sqrt{1-\beta^2}}\right)(c+c^\dagger)(d+d^\dagger) - \left(\frac{\zeta\beta\widetilde{\lambda}}{\sqrt{1-\beta^2}} + \frac{\widetilde{\lambda}\zeta\beta^3}{2\left(1-\beta^2\right)^{\frac{3}{2}}}\right)(c+c^\dagger)^2, \tag{S35}$$

whose energy spectrum reads

$$\epsilon_\pm^{\rm sp} = \sqrt{\frac{1}{2}\left(\widetilde{\omega}_c^2 + \widetilde{\delta}^4\omega_a^2 \pm \sqrt{(\widetilde{\omega}_c^2 - \widetilde{\delta}^4\omega_a^2)^2 + 4\widetilde{\omega}_c^2\omega_a^2}\right)}. \tag{S36}$$

From the results of the mean-field in Eq. (S34), we can also determine that $\widetilde{\delta} = 1$ is the critical point of the system, which can be expanded as

$$\mu^2(1-\alpha) + \gamma^2 = 1. \tag{S37}$$

When $\alpha = 0$ (meaning the absence of the $\boldsymbol{A}^2$ term), Eq. (S37) becomes $\mu^2 + \gamma^2 = 1$, indicating a *hybrid* critical point. Such a point features the boundary between the normal and superradiant phases. Therefore, based on the energy spectrum [Eq. (S32) and Eq. (S36)] or the nonzero coherence ζ, we can plot the phase diagram shown in Fig. S2(a), where the boundary has been characterized by the hybrid critical point $\mu^2 + \gamma^2 = 1$. When $\alpha = 1$ (meaning including the $\boldsymbol{A}^2$ term), the critical point of the system is $\gamma^2 = 1$ and the corresponding phase diagram is displayed in Fig. S2(b). In this case, we find that the region of the superradiant phase can still be affected by the light-atom interaction (μ), even though the critical point is only dominated by the optomechanical system (γ).

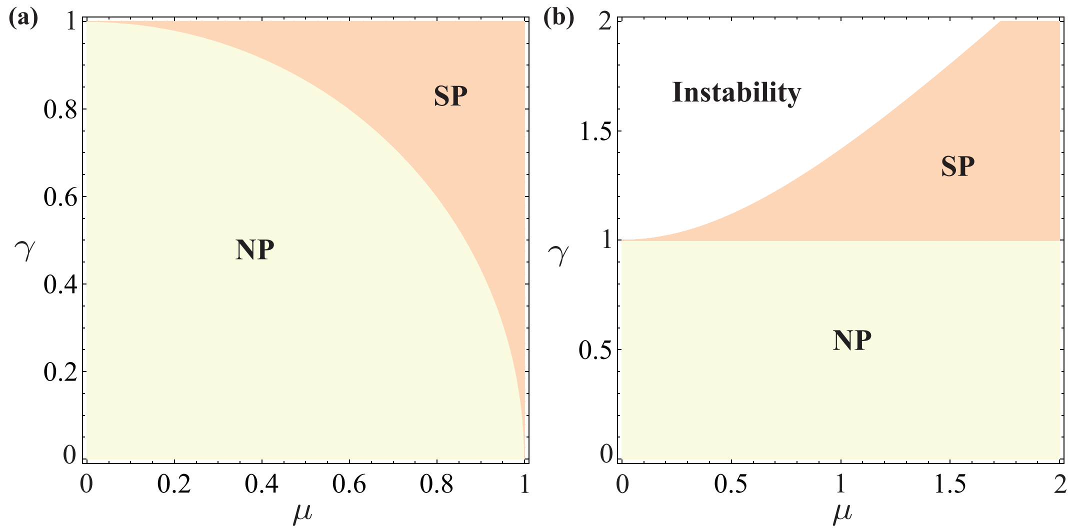

FIG. S2. Phase diagram of the hybrid quantum system in the (μ,γ) plane, described by Eq. (S32) and Eq. (S36). (a) $\alpha = 0$: the boundary (critical line) between the superradiant phase (SP) and the normal phase (NP) is characterized by $\mu^2 + \gamma^2 = 1$. (b) $\alpha = 1$: the boundary is given by $\gamma^2 = 1$.