Supplemental Material for

"Quantum Squeezing Induced Optical Nonreciprocity"

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In this Supplemental Material, we first present more details about the Bogoliubov squeezing transformation for the forward-input Hamiltonian. Then, we derive the master equations of the system including the squeezing-induced noise for calculating the steady-state transmission of the classical light. We also show more details of the elimination of the squeezing-induced noise and derive the master equation of the quantum cascaded system without the noise, to attain the transmission matrix of single-photon pulses. After that, we present the quantum Langevin equations of the system, in the cases with and without the squeezing-induced noise, for deriving the steady-state analytical transmissions. Then, we discuss the optimal condition for achieving the maximal isolation ratio. Finally, we derive the relation between the pump strength Ω_p and the pump power P_p.

I. SQUEEZING PARAMETER IN THE FORWARD-INPUT CASE

For convenience, we recall the Hamiltonian given by Eq. (1) in the main text for the forward-input case

$$\begin{aligned}
\mathcal{H}_{\rm fw} &= \mathcal{H}_{\rm A} + \mathcal{H}_{\rm B} + \mathcal{H}_J \ , \\
\mathcal{H}_{\rm A}/\hbar &= \Delta_p^a a_{\circlearrowright}^\dagger a_{\circlearrowright} + i\sqrt{2\kappa_{\rm ex1}}(a_{\rm in} a_{\circlearrowright}^\dagger e^{-i\Delta_{\rm in} t} - a_{\rm in}^\dagger a_{\circlearrowright} e^{i\Delta_{\rm in} t}) \ , \\
\mathcal{H}_{\rm B}/\hbar &= \Delta_p^b b_{\circlearrowright}^\dagger b_{\circlearrowright} + \frac{\Omega_p}{2}(e^{-i\theta_p} b_{\circlearrowright}^{\dagger 2} + e^{i\theta_p} b_{\circlearrowright}^2) \ , \\
\mathcal{H}_J/\hbar &= J_0(a_{\circlearrowright}^\dagger b_{\circlearrowright} + a_{\circlearrowright} b_{\circlearrowright}^\dagger) \ ,
\end{aligned} \tag{S1}$$

where the detuning $\Delta_p^{a/b} = \omega_{a/b} - \omega_p/2$ and $\Delta_{\rm in} = \omega_{\rm in} - \omega_p/2$. By applying the Bogoliubov squeezing transformation $b_s = \cosh(r_p)b + e^{-i\theta_p}\sinh(r_p)b^\dagger$ [1–3], we obtain the Hamiltonian in the squeezing picture. Under the condition $\Delta_p^a + \Delta_p^b\sqrt{1-\beta^2} \gg \sinh(r_p)J_0$, with $\beta = \Omega_p/\Delta_p^b$, we can apply the rotating-wave approximation and neglect the counter-rotating terms. Then, the Hamiltonian in the squeezing picture is given by

$$\begin{aligned}
\mathcal{H}_{\rm fw}^s &= \mathcal{H}_{\rm A}^s + \mathcal{H}_{\rm B}^s + \mathcal{H}_J^s \ , \\
\mathcal{H}_{\rm A}^s/\hbar &= \Delta_p^a a_{\circlearrowright}^\dagger a_{\circlearrowright} + i\sqrt{2\kappa_{\rm ex1}}(a_{\rm in} a_{\circlearrowright}^\dagger e^{-i\Delta_{\rm in} t} - a_{\rm in}^\dagger a_{\circlearrowright} e^{i\Delta_{\rm in} t}) \ , \\
\mathcal{H}_{\rm B}^s/\hbar &= \Delta_p^{bs} b_{s\circlearrowright}^\dagger b_{s\circlearrowright} \ , \\
\mathcal{H}_J^s/\hbar &= J_s(a_{\circlearrowright}^\dagger b_{s\circlearrowright} + a_{\circlearrowright} b_{s\circlearrowright}^\dagger) \ .
\end{aligned} \tag{S2}$$

Here, after the Bogoliubov squeezing transformation, the bare mode $b_{\circlearrowright}$ is transformed to the squeezed mode $b_{s\circlearrowright}$ with detuning

$$\Delta_p^{bs} = \Delta_p^b\sqrt{1-\beta^2} \ , \tag{S3}$$

and squeezing parameter

$$r_p = \frac{1}{4}\ln\frac{1+\beta}{1-\beta} \ . \tag{S4}$$

Consequently, for $r_p > 0$, the coupling rate between the bare mode $a_{\circlearrowright}$ and the squeezed mode $b_{s_{\circlearrowright}}$ is exponentially enhanced [2, 3], and given by

$$J_s = \cosh(r_p) J_0 \, . \tag{S5}$$

In a frame rotating at frequency $\Delta_{\rm in}$, the Hamiltonian Eq. (S2) can be rewritten as

$$\mathcal{H}_{\rm fw}^s/\hbar = \Delta_a a_{\circlearrowright}^\dagger a_{\circlearrowright} + i\sqrt{2\kappa_{\rm ex1}}(a_{\rm in} a_{\circlearrowright}^\dagger - a_{\rm in}^\dagger a_{\circlearrowright}) + \Delta_b^s b_{s_{\circlearrowright}}^\dagger b_{s_{\circlearrowright}} + J_s(a_{\circlearrowright}^\dagger b_{s_{\circlearrowright}} + b_{s_{\circlearrowright}}^\dagger a_{\circlearrowright}) \, , \tag{S6}$$

where $\Delta_a = \Delta_p^a - \Delta_{\rm in} = \omega_a - \omega_{\rm in}$, $\Delta_b^s = \Delta_p^{bs} - \Delta_{\rm in}$.

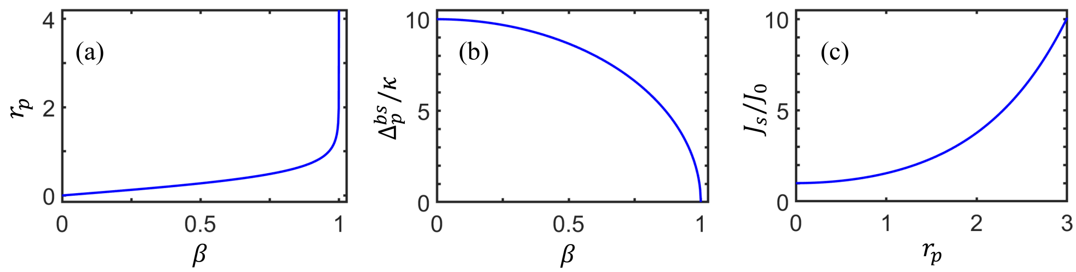

Fig. S1. (a) The squeezing parameter r_p versus $\beta \in [0, 1)$. (b) The detuning Δ_p^{bs} versus $\beta \in [0, 1)$, $\Delta_p^b/\kappa_a = 10$. (c) The enhanced coupling rate J_s varying with the squeezing parameter $r_p \in [0, 3]$.

As depicted in Fig. S1, when the ratio β approaches 1, the squeezing parameter increases greatly and the detuning Δ_p^{bs} decreases to 0. Furthermore, increasing the squeezing parameter r_p leads to an exponential enhancement [2, 3] of the coupling rate J_s.

II. MASTER EQUATION

A. Squeezing-induced noise and classical-light input

For our system in the forward-input case, the dynamics of the system coupling to a normal vacuum reservoir is described by a standard master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\rm fw} = -i[\mathcal{H}_{\rm fw}, \rho_{\rm fw}] + \mathcal{L}[L_a]\rho_{\rm fw} + \mathcal{L}[L_b]\rho_{\rm fw} \, , \tag{S7}$$

where $\rho_{\rm fw}$ is the density matrix of the system in the forward-input case, $\mathcal{H}_{\rm fw}$ is the Hamiltonian given by Eq. (S1), and the operators $L_a = \sqrt{\kappa_a} a_{\circlearrowright}$ and $L_b = \sqrt{\kappa_b} b_{\circlearrowright}$ describe the decay of the resonator A ($\mathrm{R_A}$) and B ($\mathrm{R_B}$) with rates κ_a and κ_b, respectively. Moreover, $\mathcal{L}[o]\rho = 2o\rho o^\dagger - o^\dagger o \rho - \rho o^\dagger o$.

Next, we apply the Bogoliubov squeezing transformation, $b = \cosh(r_p) b_s - e^{-i\theta_p}\sinh(r_p) b_s^\dagger$, to respectively replace the Hamiltonian $\mathcal{H}_{\rm fw}$ with $\mathcal{H}_{\rm fw}^s$ [Eq. (S6)] and L_b with $L_b^s = \sqrt{\kappa_b}[\cosh(r_p) b_{s_{\circlearrowright}} - e^{-i\theta_p}\sinh(r_p) b_{s_{\circlearrowright}}^\dagger]$. Therefore, we can rewrite Eq. (S7) in the squeezed picture as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\rm fw} &= -i[\mathcal{H}_{\rm fw}^s, \rho_{\rm fw}] + \mathcal{L}[L_a]\rho_{\rm fw} + \mathcal{L}[L_b^s]\rho_{\rm fw} \\ &= -i[\mathcal{H}_{\rm fw}^s, \rho_{\rm fw}] + \mathcal{L}[L_a]\rho_{\rm fw} + \mathcal{L}[L_{bs}]\rho_{\rm fw} + \pounds_{\rm n}[L_{bs}]\rho_{\rm fw} \, ,\end{aligned} \tag{S8}$$

Here, we use the operator $L_{bs} = \sqrt{\kappa_b} b_{s_{\circlearrowright}}$ and the noise-related Lindblad term

$$\pounds_{\rm n}[L_{bs}]\rho_{\rm fw} = N_p \mathcal{L}[L_{bs}]\rho_{\rm fw} + N_p \mathcal{L}[L_{bs}^\dagger]\rho_{\rm fw} - M_p \mathcal{L}'[L_{bs}]\rho_{\rm fw} - M_p^* \mathcal{L}'[L_{bs}^\dagger]\rho_{\rm fw} \, , \tag{S9}$$

where $N_p = \sinh^2(r_p)$, $M_p = e^{i\theta_p}\cosh(r_p)\sinh(r_p)$, and $\mathcal{L}'[o]\rho = 2o\rho o - oo\rho - \rho oo$.

In the backward-input case, the Hamiltonian reads

$$\mathcal{H}_{\rm bw}/\hbar = \Delta_a a_{\circlearrowright}^\dagger a_{\circlearrowright} + i\sqrt{2\kappa_{\rm ex1}}(a_{\rm in} a_{\circlearrowright}^\dagger - a_{\rm in}^\dagger a_{\circlearrowright}) \; + \Delta_b^0 b_{\circlearrowright}^\dagger b_{\circlearrowright} + J_0(a_{\circlearrowright}^\dagger b_{\circlearrowright} + b_{\circlearrowright}^\dagger a_{\circlearrowright}) \, , \tag{S10}$$

where $\Delta_b^0 = \omega_b - \omega_{\rm in}$. The corresponding master equation of the system is given by

$$\frac{\rm d}{{\rm d}t}\rho_{\rm bw} = -i[\mathcal{H}_{\rm bw}, \rho_{\rm bw}] + \mathcal{L}[L_a]\rho_{\rm bw} + \mathcal{L}[L_b]\rho_{\rm bw} \ . \tag{S11}$$

Note that the term $\pounds_{\rm n}[L_{bs}]\rho_{\rm fw}$ in Eq. (S8) is induced by quantum squeezing for the forward-input case. This noise-related term is absent in Eq. (S11) in the backward-input case.

According to the input-output relation [4], we have

$$a_{\rm out} = a_{\rm in} - \sqrt{2\kappa_{\rm ex1}}a \ , \quad a^\dagger_{\rm out}a_{\rm out} = a^\dagger_{\rm in}a_{\rm in} - \sqrt{2\kappa_{\rm ex1}}\left(a^\dagger_{\rm in}a + a^\dagger a_{\rm in}\right) + 2\kappa_{\rm ex1}a^\dagger a \ , \tag{S12a}$$

$$b_{\rm out} = \sqrt{2\kappa_{\rm ex2}}b \ , \quad b^\dagger_{\rm out}b_{\rm out} = 2\kappa_{\rm ex2}b^\dagger b \ , \tag{S12b}$$

where $\kappa_{\rm ex2}$ is the external decay rate of $\rm R_B$.

The transmissions are defined as

$$T_{12/21} = \frac{\langle a^\dagger_{\rm out}a_{\rm out}\rangle}{\langle a^\dagger_{\rm in}a_{\rm in}\rangle} \ , \quad T_{23} = \frac{\langle b^\dagger_{\rm out}b_{\rm out}\rangle}{\langle a^\dagger_{\rm in}a_{\rm in}\rangle} \ , \tag{S13}$$

where T_{ij} is the transmission from port i to port j, with $i,\ j = 1,\ 2,\ 3$. According to Eqs. (S12) and (S13), replacing the operators $a_{\rm in}$ and $a^\dagger_{\rm in}$ with their average value $\alpha_{\rm in}$ and $\alpha^*_{\rm in}$ and setting $\theta_p = 0$ (by adjusting the phase of the pump), we can solve the transmission for a coherent signal field by numerically solving Eqs. (S8) and (S11).

B. Elimination of squeezing-induced noise and single-photon input

In the forward-input case, to suppress the squeezing-induced noise, we input a broadband squeezed-vacuum field with squeezing parameter r_e and reference phase θ_e into $\rm R_B$ from port 3. The squeezed-vacuum field can be regarded as a squeezed-vacuum reservoir for CCW modes in $\rm R_B$. Therefore, the dynamics of the system is described by the master equation [1]

$$\frac{\rm d}{{\rm d}t}\rho^{\rm sv}_{\rm fw} = -\,i[\mathcal{H}_{\rm fw}, \rho^{\rm sv}_{\rm fw}] + \mathcal{L}[L_a]\rho^{\rm sv}_{\rm fw} + (N_e+1)\mathcal{L}[L_b]\rho^{\rm sv}_{\rm fw} + N_e\mathcal{L}[L^\dagger_b]\rho^{\rm sv}_{\rm fw} - M_e\mathcal{L}'[L_b]\rho^{\rm sv}_{\rm fw} - M^*_e\mathcal{L}'[L^\dagger_b]\rho^{\rm sv}_{\rm fw} \ , \tag{S14}$$

where $N_e = \sinh^2(r_e)$ and $M_e = e^{-i\theta_e}\cosh(r_e)\sinh(r_e)$.

Next, we respectively transform the Hamiltonian $\mathcal{H}_{\rm fw}$ to $\mathcal{H}^s_{\rm fw}$, and the operator L_b to L^s_b according to the Bogoliubov squeezing transformation $b = \cosh(r_p)b_s - e^{-i\theta_p}\sinh(r_p)b^\dagger_s$. After the replacement, the master equation for the forward input can accordingly be written as

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\rho^{\rm sv}_{\rm fw} =& -\,i[\mathcal{H}^s_{\rm fw}, \rho^{\rm sv}_{\rm fw}] + \mathcal{L}[L_a]\rho^{\rm sv}_{\rm fw} + (N_e+1)\mathcal{L}[L^s_b]\rho^{\rm sv}_{\rm fw} + N_e\mathcal{L}[L^{s\dagger}_b]\rho^{\rm sv}_{\rm fw} - M_e\mathcal{L}'[L^s_b]\rho^{\rm sv}_{\rm fw} - M^*_e\mathcal{L}'[L^{s\dagger}_b]\rho^{\rm sv}_{\rm fw}\\ =& -\,i[\mathcal{H}^s_{\rm fw}, \rho^{\rm sv}_{\rm fw}] + \mathcal{L}[L_a]\rho^{\rm sv}_{\rm fw} + \mathcal{L}[L_{bs}]\rho^{\rm sv}_{\rm fw} + N^s_e\mathcal{L}[L_{bs}]\rho^{\rm sv}_{\rm fw} + N^s_e\mathcal{L}[L^\dagger_{bs}]\rho^{\rm sv}_{\rm fw} - M^s_e\mathcal{L}'[L_{bs}]\rho^{\rm sv}_{\rm fw} - M^{s*}_e\mathcal{L}'[L^\dagger_{bs}]\rho^{\rm sv}_{\rm fw} \ ,\end{aligned} \tag{S15}$$

where

$$N^s_e = \cosh^2(r_p)\sinh^2(r_e) + \sinh^2(r_p)\cosh^2(r_e) + \frac{1}{2}\sinh(2r_p)\sinh(2r_e)\cos(\theta_p+\theta_e) \ , \tag{S16a}$$

$$\begin{aligned}M^s_e =& \exp(i\theta_p)\left[\sinh(r_p)\cosh(r_e) + \exp[-i(\theta_p+\theta_e)]\cosh(r_p)\sinh(r_e)\right]\\ &\times\left[\cosh(r_p)\cosh(r_e) + \exp[i(\theta_p+\theta_e)]\sinh(r_p)\sinh(r_e)\right] \ .\end{aligned} \tag{S16b}$$

When $r_e = r_p$ and $\theta_e + \theta_p = \pm n\pi$ $(n = 1, 3, 5, ...)$, we have $N^s_e = 0$ and $M^s_e = 0$. Thus, Eq. (S15) becomes

$$\frac{\rm d}{{\rm d}t}\rho^{\rm sv}_{\rm fw} = -i[\mathcal{H}^s_{\rm fw}, \rho^{\rm sv}_{\rm fw}] + \mathcal{L}[L_a]\rho^{\rm sv}_{\rm fw} + \mathcal{L}[L_{bs}]\rho^{\rm sv}_{\rm fw} \ . \tag{S17}$$

Here, the squezzing-noise-induced term $\pounds_{\rm n}[L_{bs}]\rho_{\rm fw}$ in Eq. (S8) is cancelled by the squeezed-vacuum field in the forward-input case. Therefore, the squeezed mode equivalently couples to a normal vacuum bath. As a result, the decay rate of the squeezed mode equals that of the original bare mode. The term $\mathcal{L}[L_{bs}]\rho_{\rm fw}$ with operator $L_{bs} = \sqrt{\kappa_b}b_{s_\circlearrowleft}$ describes the decay of the mode $b_{s_\circlearrowleft}$ with a rate κ_b.

In the backward-input case, the squeezed-vacuum field from port 3 has no influence on the system dynamics. So the motion of the system coupling to a normal vacuum bath is governed by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\mathrm{bw}} = -i[\mathcal{H}_{\mathrm{bw}}, \rho_{\mathrm{bw}}] + \mathcal{L}[L_a]\rho_{\mathrm{bw}} + \mathcal{L}[L_b]\rho_{\mathrm{bw}}\ . \tag{S18}$$

Then, we use a quantum cascaded system to simulate the propagation of single-photon pulses incident to ports 1 and 2 simultaneously [5–7]. In this quantum cascaded system, single-photon pulses emitted from the source resonator are input into our optical nonreciprocal device. Therefore, when the squeezing-vacuum field is applied, the master equation describing the quantum cascaded system for the forward-input case is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho^{\mathrm{sv}}_{\mathrm{qcs,fw}} = -i[\mathcal{H}_d, \rho^{\mathrm{sv}}_{\mathrm{qcs,fw}}] - i[\mathcal{H}^s_{\mathrm{qcs,fw}}, \rho^{\mathrm{sv}}_{\mathrm{qcs,fw}}] + \mathcal{L}[L_d]\rho^{\mathrm{sv}}_{\mathrm{qcs,fw}} + \mathcal{L}[L_a]\rho^{\mathrm{sv}}_{\mathrm{qcs,fw}} + \mathcal{L}[L_{bs}]\rho^{\mathrm{sv}}_{\mathrm{qcs,fw}} + \pounds_{\mathrm{qcs,fw}}\rho^{\mathrm{sv}}_{\mathrm{qcs,fw}}\ , \tag{S19}$$

where $\mathcal{H}_d = \Delta_d d^\dagger d$ is the Hamiltonian of the source resonator, $\Delta_d = \omega_d - \omega_{\mathrm{in}}$, ω_d is the resonance frequency of the source resonator, and $\rho^{\mathrm{sv}}_{\mathrm{qcs,fw}}$ is the joint density matrix of the source resonator and our device in the forward-input case. Here we set $\Delta_d = 0$. The Hamiltonian of the quantum cascaded system is $\mathcal{H}^s_{\mathrm{qcs,fw}} = \Delta_a a^\dagger_{\circlearrowleft} a_{\circlearrowleft} + \Delta^s_b b^\dagger_{s\circlearrowleft} b_{s\circlearrowleft} + J_s(a^\dagger_{\circlearrowleft} b_{s\circlearrowleft} + b^\dagger_{s\circlearrowleft} a_{\circlearrowleft})$. The Lindblad operator $L_d = \sqrt{\kappa_{\mathrm{ex0}}}d$ describes the external decay of the source resonator, where κ_{ex0} is the decay rate from the source resonator to the device. To apply a Gaussian-like single-photon pulse, we set $\kappa_{\mathrm{ex0}}(t) = \kappa_a \exp(-(t-\tau_{\mathrm{d}})^2/2\tau_{\mathrm{p}}^2)$, where τ_p is the pulse duration and τ_d is the pulse delay [8, 9]. The relevant Lindblad terms are $\mathcal{L}[L_d]\rho = \kappa_{\mathrm{ex0}}(2d\rho d^\dagger - d^\dagger d\rho - \rho d^\dagger d)$ and $\pounds_{\mathrm{qcs,fw}}\rho = \sqrt{4\kappa_{\mathrm{ex0}}\kappa_{\mathrm{ex1}}}\left([a^\dagger_{\circlearrowleft}, d\rho] + [\rho d^\dagger, a_{\circlearrowleft}]\right)$.

In the backward-input case, the master equation of the quantum cascaded system is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\mathrm{qcs,bw}} = -i[\mathcal{H}_d, \rho_{\mathrm{qcs,bw}}] - i[\mathcal{H}_{\mathrm{qcs,bw}}, \rho_{\mathrm{qcs,bw}}] + \mathcal{L}[L_d]\rho_{\mathrm{qcs,bw}} + \mathcal{L}[L_a]\rho_{\mathrm{qcs,bw}} + \mathcal{L}[L_b]\rho_{\mathrm{qcs,bw}} + \pounds_{\mathrm{qcs,bw}}\rho_{\mathrm{qcs,bw}}\ , \tag{S20}$$

where $\mathcal{H}_{\mathrm{qcs,bw}} = \Delta_a a^\dagger_{\circlearrowleft} a_{\circlearrowleft} + \Delta^0_b b^\dagger_{\circlearrowleft} b_{\circlearrowleft} + J_0(a^\dagger_{\circlearrowleft} b_{\circlearrowleft} + b^\dagger_{\circlearrowleft} a_{\circlearrowleft})$ and $\pounds_{\mathrm{qcs,bw}}\rho = \sqrt{4\kappa_{\mathrm{ex0}}\kappa_{\mathrm{ex1}}}\left([a^\dagger_{\circlearrowleft}, d\rho] + [\rho d^\dagger, a_{\circlearrowleft}]\right)$, and $\rho_{\mathrm{qcs,bw}}$ is the joint density matrix of the source resonator and our device in the backward-input case.

According to Eqs. (S12) and (S13), we can attain the propagation (left panel in Fig. S2) and transmissions (right panel in Fig. S2) of the single-photon pulses by solving Eqs. (S19) and (S20) numerically. As show in Fig. S2, the system can function as a three-port quasi-circulator, allowing single-photon pulses propagating along port $1 \to 2 \to 3$ [10].

III. QUANTUM LANGEVIN EQUATION

According to Eq. (S8), the quantum Langevin equation for an arbitrary system operator Q in the forward-input case is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}Q = i[\mathcal{H}^s_{\mathrm{fw}}, Q] + \mathcal{L}[L_a]Q + \mathcal{L}[L_{bs}]Q + \pounds_{\mathrm{n}}[L_{bs}]Q\ , \tag{S21}$$

where $L_a = \sqrt{\kappa_a}a_{\circlearrowleft}$, $L_b = \sqrt{\kappa_b}b_{s\circlearrowleft}$, $\pounds_{\mathrm{n}}[L_{bs}]Q = N_p\mathcal{L}[L_{bs}]Q + N_p\mathcal{L}[L^\dagger_{bs}]Q - M_p\mathcal{L}'[L_{bs}]Q - M^*_p\mathcal{L}'[L^\dagger_{bs}]Q$, $\mathcal{L}[o]Q = 2o^\dagger Qo - Qo^\dagger o - o^\dagger oQ$, and $\mathcal{L}'[o]Q = 2oQo - Qoo - ooQ$. Hence, we have the equations of motion for the specific operators $Q = \{a_{\circlearrowleft},\ b_{s\circlearrowleft},\ a^\dagger_{\circlearrowleft} b_{s\circlearrowleft},\ b^\dagger_{s\circlearrowleft} b_{s\circlearrowleft},\ a^\dagger_{\circlearrowleft} a_{\circlearrowleft}\}$ reading as

$$\frac{\mathrm{d}}{\mathrm{d}t}a_{\circlearrowleft} = -(i\Delta_a + \kappa_a)a_{\circlearrowleft} + \sqrt{2\kappa_{\mathrm{ex1}}}a_{\mathrm{in}} - iJ_s b_{s\circlearrowleft}\ , \tag{S22a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b_{s\circlearrowleft} = -(i\Delta^s_b + \kappa_b)b_{s\circlearrowleft} - iJ_s a_{\circlearrowleft}\ , \tag{S22b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a^\dagger_{\circlearrowleft} b_{s\circlearrowleft} = (i\Delta^s_{ab} - \kappa_{ab})a^\dagger_{\circlearrowleft} b_{s\circlearrowleft} + \sqrt{2\kappa_{\mathrm{ex1}}}a^\dagger_{\mathrm{in}} b_{s\circlearrowleft} - iJ_s\Xi\ , \tag{S22c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b^\dagger_{s\circlearrowleft} b_{s\circlearrowleft} = iJ_s(a^\dagger_{\circlearrowleft} b_{s\circlearrowleft} - a_{\circlearrowleft} b^\dagger_{s\circlearrowleft}) - 2\kappa_b b^\dagger_{s\circlearrowleft} b_{s\circlearrowleft} + \Psi_{\mathrm{noise}}\ , \tag{S22d}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a^\dagger_{\circlearrowleft} a_{\circlearrowleft} = -iJ_s(a^\dagger_{\circlearrowleft} b_{s\circlearrowleft} - a_{\circlearrowleft} b^\dagger_{s\circlearrowleft}) + \sqrt{2\kappa_{\mathrm{ex1}}}(a_{\mathrm{in}} a^\dagger_{\circlearrowleft} + a^\dagger_{\mathrm{in}} a_{\circlearrowleft}) - 2\kappa_a a^\dagger_{\circlearrowleft} a_{\circlearrowleft}\ . \tag{S22e}$$

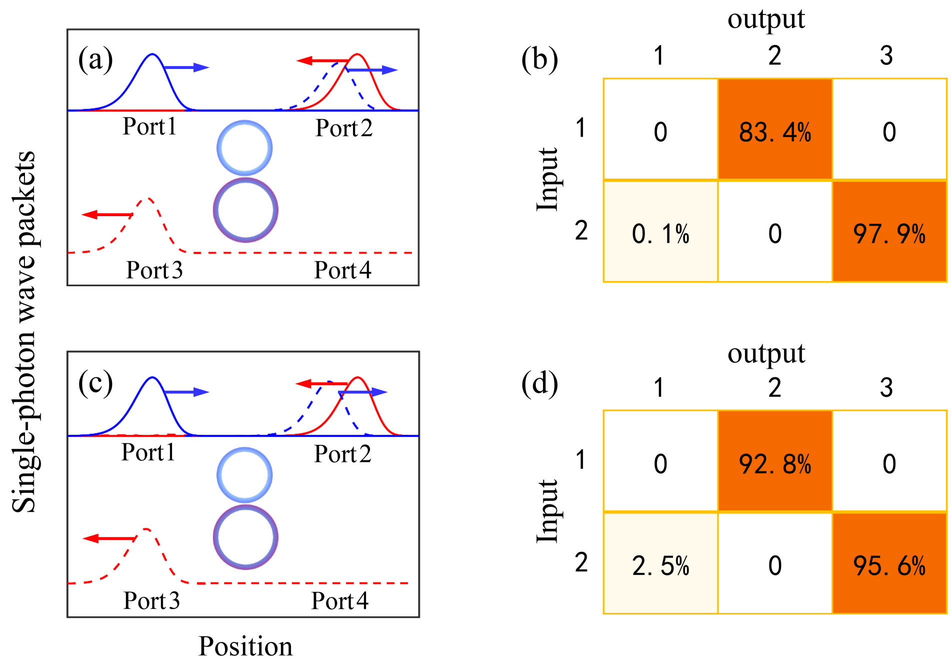

Fig. S2. (a) and (c) Propagation of single-photon pulses with a duration of $2\pi \times 6\kappa_a^{-1}$. The arrows indicate the propagating directions of the pulses. Blue (red) curves are for the incident (solid curves) and transmitted (dashed curves) pulses for the forward (backward) input. (b) and (d) Transmission matrix of the three-port quasi-circulator. (a) and (b) for the normal mode splitting (NMS) scenario with $\kappa_a = \kappa_b = \kappa$, $\kappa_{\mathrm{ex1,2}}/\kappa = 0.99$, $J_0/\kappa = 0.99$, $\Delta_p^b/\kappa = 10.3$, $\Omega_p/\kappa = 10$, and $\Delta_a = \Delta_b^0 = 0$. (c) and (d) for the mode resonance shift (MRS) scenario with $\kappa_a = \kappa_b = \kappa$, $\kappa_{\mathrm{ex1,2}}/\kappa = 0.99$, $J_0/\kappa = 2.8$, $\Delta_p^b/\kappa = 15$, $\Omega_p/\kappa = 13$, and $\Delta_a = \Delta_b^0 = 2.62\kappa$.

Here, $\Psi_{\mathrm{noise}} = 2\sinh^2(r_p)\kappa_b$, $\Delta_{ab}^s = \Delta_a - \Delta_b^s$, $\kappa_{ab} = \kappa_a + \kappa_b$ and $\Xi = a_{\circlearrowleft}^\dagger a_{\circlearrowleft} b_{s_{\circlearrowleft}} b_{s_{\circlearrowleft}}^\dagger - a_{\circlearrowleft} a_{\circlearrowleft}^\dagger b_{s_{\circlearrowleft}}^\dagger b_{s_{\circlearrowleft}}$. To a good approximation, using the commutation relations $[a, a^\dagger] = 1$ and $[b_s, b_s^\dagger] = 1$, we can derive $\Xi = a_{\circlearrowleft}^\dagger a_{\circlearrowleft} - b_{s_{\circlearrowleft}}^\dagger b_{s_{\circlearrowleft}}$. By setting $\frac{\mathrm{d}}{\mathrm{d}t}Q = 0$, we can obtain the steady-state mean values of the operators

$$\langle a_{\circlearrowleft} \rangle_{\mathrm{ss}} = \frac{(i\Delta_b^s + \kappa_b)\sqrt{2\kappa_{\mathrm{ex1}}}\alpha_{\mathrm{in}}}{(i\Delta_a + \kappa_a)(i\Delta_b^s + \kappa_b) + J_s^2}, \tag{S23a}$$

$$\langle b_{s_{\circlearrowleft}} \rangle_{\mathrm{ss}} = \frac{-iJ_s\sqrt{2\kappa_{\mathrm{ex1}}}\alpha_{\mathrm{in}}}{(i\Delta_a + \kappa_a)(i\Delta_b^s + \kappa_b) + J_s^2}, \tag{S23b}$$

$$\left\langle a_{\circlearrowleft}^\dagger a_{\circlearrowleft} \right\rangle_{\mathrm{ss}} = \frac{2\kappa_{\mathrm{ex1}}|\alpha_{\mathrm{in}}|^2(\kappa_b^2 + \Delta_b^{s\,2})}{\mathcal{G}_s} + \mathcal{N}_{\mathrm{noise}}, \tag{S23c}$$

$$\left\langle b_{s_{\circlearrowleft}}^\dagger b_{s_{\circlearrowleft}} \right\rangle_{\mathrm{ss}} = \frac{2\kappa_{\mathrm{ex1}}|\alpha_{\mathrm{in}}|^2 J_s^2}{\mathcal{G}_s} + \varrho\mathcal{N}_{\mathrm{noise}}. \tag{S23d}$$

Here, $\mathcal{N}_{\mathrm{noise}} = \kappa_b(\kappa_a + \kappa_b)\sinh^2(r_p)J_s^2/\mathcal{Q}_s$. The mean value $\alpha_{\mathrm{in}} = \langle a_{\mathrm{in}} \rangle$ is the coherent amplitude of the input signal field. We also have $\mathcal{G}_s = J_s^4 + 2J_s^2(\kappa_a\kappa_b - \Delta_a\Delta_b^s) + (\kappa_a^2 + \Delta_a^2)(\kappa_b^2 + \Delta_b^{s\,2})$, $\mathcal{Q}_s = J_s^2(\kappa_a + \kappa_b)^2 + \kappa_a\kappa_b[(\kappa_a + \kappa_b)^2 + \Delta_{ab}^{s\,2}]$, and $\varrho = J_s^2(\kappa_a + \kappa_b) + \kappa_a[(\kappa_a + \kappa_b)^2 + \Delta_{ab}^{s\,2}]/[(\kappa_a + \kappa_b)J_s^2]$.

According to Eqs. (S13) and (S23), the steady-state transmission for port $1 \to 2$ is

$$T_{12} = \frac{J_s^4 + 2\zeta_s J_s^2 + \Lambda_s}{\mathcal{G}_s} + \frac{2\kappa_{\mathrm{ex1}}\mathcal{N}_{\mathrm{noise}}}{|\alpha_{\mathrm{in}}|^2}, \tag{S24}$$

where $\zeta_s = \kappa_a\kappa_b - 2\kappa_b\kappa_{\mathrm{ex1}} - \Delta_a\Delta_b^s$ and $\Lambda_s = [(\kappa_a - 2\kappa_{\mathrm{ex1}})^2 + \Delta_a^2](\kappa_b^2 + \Delta_b^{s\,2})$.

Similarly, according to Eq. (S11), we obtain the quantum Langevin equation for an arbitrary system operator Q in the backward-input case

$$\frac{\mathrm{d}}{\mathrm{d}t}Q = i[\mathcal{H}_{\mathrm{bw}}, Q] + \mathcal{L}[L_a]Q + \mathcal{L}[L_b]Q \; . \tag{S25}$$

Thus, we have the equations of motion for the specific operators $Q = \{a_{\circlearrowleft},\ b_{\circlearrowleft},\ a^\dagger_{\circlearrowleft} b_{\circlearrowleft},\ b^\dagger_{\circlearrowleft} b_{\circlearrowleft},\ a^\dagger_{\circlearrowleft} a_{\circlearrowleft}\}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}a_{\circlearrowleft} = -(i\Delta_a + \kappa_a)a_{\circlearrowleft} + \sqrt{2\kappa_{\mathrm{ex1}}}a_{\mathrm{in}} - iJ_0 b_{\circlearrowleft} \; , \tag{S26a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b_{\circlearrowleft} = -(i\Delta_b^0 + \kappa_b)b_{\circlearrowleft} - iJ_0 a_{\circlearrowleft} \; , \tag{S26b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a^\dagger_{\circlearrowleft} b_{\circlearrowleft} = (i\Delta_{ab}^0 - \kappa_{ab})a^\dagger_{\circlearrowleft} b_{\circlearrowleft} + \sqrt{2\kappa_{\mathrm{ex1}}}a^\dagger_{\mathrm{in}} b_{\circlearrowleft} - iJ_0 \Xi \; , \tag{S26c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b^\dagger_{\circlearrowleft} b_{\circlearrowleft} = iJ_0(a^\dagger_{\circlearrowleft} b_{\circlearrowleft} - a_{\circlearrowleft} b^\dagger_{\circlearrowleft}) - 2\kappa_b b^\dagger_{\circlearrowleft} b_{\circlearrowleft} \; , \tag{S26d}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a^\dagger_{\circlearrowleft} a_{\circlearrowleft} = -iJ_0(a^\dagger_{\circlearrowleft} b_{\circlearrowleft} - a_{\circlearrowleft} b^\dagger_{\circlearrowleft}) + \sqrt{2\kappa_{\mathrm{ex1}}}(a_{\mathrm{in}} a^\dagger_{\circlearrowleft} + a^\dagger_{\mathrm{in}} a_{\circlearrowleft}) - 2\kappa_a a^\dagger_{\circlearrowleft} a_{\circlearrowleft} \; . \tag{S26e}$$

Here, we use the notation $\Delta_{ab}^0 = \Delta_a - \Delta_b^0$, $\Xi = a^\dagger_{\circlearrowleft} a_{\circlearrowleft} - b^\dagger_{\circlearrowleft} b_{\circlearrowleft}$. Setting $\frac{\mathrm{d}}{\mathrm{d}t}Q = 0$, we obtain the steady-state solutions

$$\langle a_{\circlearrowleft} \rangle_{\mathrm{ss}} = \frac{(i\Delta_b^0 + \kappa_b)\sqrt{2\kappa_{\mathrm{ex1}}}\alpha_{\mathrm{in}}}{(i\Delta_a + \kappa_a)(i\Delta_b^0 + \kappa_b) + J_0^2} \; , \tag{S27a}$$

$$\langle b_{\circlearrowleft} \rangle_{\mathrm{ss}} = \frac{-iJ_0\sqrt{2\kappa_{\mathrm{ex1}}}\alpha_{\mathrm{in}}}{(i\Delta_a + \kappa_a)(i\Delta_b^0 + \kappa_b) + J_0^2} \; , \tag{S27b}$$

$$\left\langle a^\dagger_{\circlearrowleft} a_{\circlearrowleft} \right\rangle_{\mathrm{ss}} = \frac{2\kappa_{\mathrm{ex1}}|\alpha_{\mathrm{in}}|^2(\kappa_b^2 + \Delta_b^{0\,2})}{\mathcal{G}_0} \; , \tag{S27c}$$

$$\left\langle b^\dagger_{\circlearrowleft} b_{\circlearrowleft} \right\rangle_{\mathrm{ss}} = \frac{2\kappa_{\mathrm{ex1}}|\alpha_{\mathrm{in}}|^2 J_0^2}{\mathcal{G}_0} \; . \tag{S27d}$$

where $\mathcal{G}_0 = J_0^4 + 2J_0^2(\kappa_a\kappa_b - \Delta_a\Delta_b^0) + (\kappa_a^2 + \Delta_a^2)(\kappa_b^2 + \Delta_b^{0\,2})$. According to Eqs. (S13) and (S27), the steady-state transmissions in the backward-input case are given by

$$T_{21} = \frac{J_0^4 + 2\zeta_0 J_0^2 + \Lambda_0}{\mathcal{G}_0} \; , \tag{S28a}$$

$$T_{23} = \frac{4\kappa_{\mathrm{ex1}}\kappa_{\mathrm{ex2}}J_0^2}{\mathcal{G}_0} \; , \tag{S28b}$$

where $\zeta_0 = \kappa_a\kappa_b - 2\kappa_b\kappa_{\mathrm{ex1}} - \Delta_a\Delta_b^0$ and $\Lambda_0 = [(\kappa_a - 2\kappa_{\mathrm{ex1}})^2 + \Delta_a^2](\kappa_b^2 + \Delta_b^{0\,2})$.

After applying a phase-matched squeezed-vacuum field to drive $\mathrm{R_B}$, as discussed in Sec. II B, the squeezing-induced noise can be completely eliminated. In this case, we have $\kappa_{bs} = \kappa_b$. Thus, the term $\mathcal{L}[L_{bs}]b^\dagger_{s_{\circlearrowleft}} b_{s_{\circlearrowleft}}$ with the operator $L_{bs} = \sqrt{\kappa_b}b_{s_{\circlearrowleft}}$ describes the decay of the mode $b_{s_{\circlearrowleft}}$ with a rate κ_b. Therefore, when the squeezed-vacuum field is applied, we can rewrite Eq. (S22d) as

$$\frac{\mathrm{d}}{\mathrm{d}t}b^\dagger_{s_{\circlearrowleft}} b_{s_{\circlearrowleft}} = iJ_s\left(a^\dagger_{\circlearrowleft} b_{s_{\circlearrowleft}} - a_{\circlearrowleft} b^\dagger_{s_{\circlearrowleft}}\right) - 2\kappa_b b^\dagger_{s_{\circlearrowleft}} b_{s_{\circlearrowleft}} \; . \tag{S29}$$

Note that the noise term Ψ_{noise} in Eq. (S22d) is eliminated, but Eqs. (S22a)-(S22c) and (S22e) have no change in this case. So, we can derive the steady-state mean value of the mode $a_{\circlearrowleft}$ which is given by

$$\left\langle a^\dagger_{\circlearrowleft} a_{\circlearrowleft} \right\rangle_{\mathrm{ss}} = \frac{2\kappa_{\mathrm{ex1}}|\alpha_{\mathrm{in}}|^2(\kappa_b^2 + \Delta_b^{s\,2})}{\mathcal{G}_s} \; . \tag{S30}$$

Here, the noise term $\mathcal{N}_{\mathrm{noise}}$ in Eq. (S23c) is also eliminated in this case.

Therefore, the steady-state noise-free transmissions are obtain as

$$T_{12}^{\mathrm{sv}} = (J_s^4 + 2\zeta_s J_s^2 + \Lambda_s)/\mathcal{G}_s \; , \quad T_{21}^{\mathrm{sv}} = T_{21} \; , \quad T_{23}^{\mathrm{sv}} = T_{23} \; . \tag{S31}$$

From Eq. (S24), we can see that the noise-related term $2\kappa_{\rm ex1}\mathcal{N}_{\rm noise}/|\alpha_{\rm in}|^2$ in the transmission T_{12} can be completely eliminated by applying the squeezed-vacuum field.

The isolation ratio of transmissions between ports 1 and 2 in the case without the squeezed-vacuum field is defined as

$$\eta = 10\log_{10}(T_{12}/T_{21}) \, . \tag{S32}$$

The isolation ratio in the case with the squeezed-vacuum field is defined as

$$\eta^{\rm sv} = 10\log_{10}(T_{12}^{\rm sv}/T_{21}^{\rm sv}) \, . \tag{S33}$$

IV. THE MAXIMAL ISOLATION RATIO

To achieve the maximal available isolation ratio $\eta_{\rm max}$, we need to find the condition for the maximal forward transmission and the minimal backward transmission. In our cases, the forward transmission of interest is close to unity. As a result, the isolation ratio is dominantly determined by the near-zero backward transmission. Thus, we pay more attention to find an optimal condition allowing a vanishingly small backward transmission $T_{21} = T_{21}^{\rm sv}$ because $\eta_{\rm max}$ is crucially dependent on the near-zero T_{21}. In our backward-input case, our system can be modeled as the standard cavity system consisting of two coupled optical microring resonators. For simplicity in our analysis below, we take $\Delta_a = \Delta_b^0 = \Delta$ and $\kappa_a = \kappa_b = \kappa$.

From Eqs. (S28a) and (S32), we obtain the minimal transmission $T_{21}^{\rm min}$ and the corresponding maximal isolation ratio $\eta_{\rm max}$ under the optimal condition $[(J_0^2+\kappa_i^2)-(\kappa_{\rm ex1}^2+\Delta^2)]^2+(2\kappa_i\Delta)^2\approx 0$, where κ_i is the intrinsic decay rate of $\mathrm{R_A}$ and $\kappa_i+\kappa_{\rm ex1}=\kappa$. Below we discuss the isolation ratio $\eta_{\rm max}$ in the NMS and MRS scenarios, respectively.

A. $\eta_{\rm max}$ in the normal mode splitting scenario

In the NMS scenario, we apply $J_0\sim\kappa$ and $\kappa_i \ll \kappa$. In this case, the optimal coupling gives to an optimal detuning $\Delta = 0$. Then, we obtain the minimal transmission

$$T_{21}^{\rm min} = \frac{(J_0^2-\kappa_{\rm ex1}^2+\kappa_i^2)^2}{(J_0^2+\kappa^2)^2} \, . \tag{S34}$$

We consider the practical implementations $\kappa_{\rm ex1}\gg\kappa_i$ and apply the approximation $\kappa\approx\kappa_{\rm ex1}$ and $J_0^2\approx\kappa_{\rm ex1}^2$ for the exact optimal coupling $J_0^2+\kappa_i^2=\kappa_{\rm ex1}^2$ because κ_i^2 is small. In this case, we have

$$T_{21}^{\rm min} \approx \frac{\kappa_i^4}{4J_0^4} \, , \tag{S35}$$

yielding the maximal isolation ratio

$$\eta_{\rm max} \approx 10\log_{10}\left[\left(1-\sigma+\frac{2\kappa\mathcal{N}_{\rm noise}}{|\alpha_{\rm in}|^2}\right)\frac{4J_0^4}{\kappa_i^4}\right] \, , \tag{S36}$$

where $\sigma\approx 4J_s^2\kappa^2\big/\left[(J_s^2+\kappa^2)^2+\kappa^2\Delta_b^{s\,2}\right]$.

B. $\eta_{\rm max}$ in the mode resonance shift scenario

In the MRS scenario, the coupling rate J_0 and the detuning Δ are larger than κ. Thus, we need a small κ_i to meet the optimal condition that $J_0^2+\kappa_i^2=\kappa_{\rm ex1}^2+\Delta^2$ and $\kappa_i\Delta\sim 0$.

$$T_{21}^{\rm min} = \frac{(J_0^2-\kappa_{\rm ex1}^2+\kappa_i^2)\kappa_i^2}{(J_0^2+\kappa_i^2)\kappa^2} \approx \frac{(J_0^2-\kappa^2)\kappa_i^2}{J_0^2\kappa^2} \, . \tag{S37}$$

Here, we use the approximation $\kappa \approx \kappa_{\rm ex1}$. Thus, the maximal isolation ratio is

$$\eta_{\max} \approx 10\log_{10}\left[\left(1+\frac{2\kappa\mathcal{N}_{\rm noise}}{|\alpha_{\rm in}|^2}\right)\frac{J_0^2\kappa^2}{(J_0^2-\kappa^2)\kappa_i^2}\right] . \tag{S38}$$

Here, we apply a good approximation $\Delta\Delta_b^s \pm \kappa^2 \sim \Delta\Delta_b^s$ with $|\Delta\Delta_b^s| \gg \kappa^2$.

V. SECOND-ORDER NONLINEAR PARAMETRIC PROCESS

We now consider the full quantum description of the degenerate nonlinear parametric process in $\mathrm{R_B}$. Then, the Hamiltonian for the forward-input case is given by (for simplicity, we replace $a_{\circlearrowleft}$ with a, $b_{s_{\circlearrowleft}}$ with b, and $c_{\circlearrowleft}$ with c)

$$\begin{aligned}\mathcal{H}/\hbar =&\omega_a a^\dagger a+\omega_b b^\dagger b+\omega_c c^\dagger c+J_0(a^\dagger b+b^\dagger a)+g\left({b^\dagger}^2 c+b^2c^\dagger\right)\\ &+i\sqrt{2\kappa_{\rm ex1}}\alpha_{\rm in}\left(a^\dagger e^{-i\omega_{\rm in}t}-ae^{i\omega_{\rm in}t}\right)+i\sqrt{2\kappa_{\rm ex2}^p}\alpha_p\left(c^\dagger e^{-i\omega_p t}-ce^{i\omega_p t}\right) ,\end{aligned} \tag{S39}$$

where $\omega_{a/b}$ is the resonance frequency of the fundamental signal mode in $\mathrm{R_A}$ or $\mathrm{R_B}$, ω_c is the frequency of the second-harmonic modes in $\mathrm{R_B}$, $\alpha_{\rm in}=\sqrt{2\pi P_{\rm in}/\hbar\omega_{\rm in}}$ is the coherent amplitude of the incident signal light with the power $P_{\rm in}$, $\alpha_p=\sqrt{2\pi P_p/\hbar\omega_p}$ corresponds to the pump light with the power P_p and the angular frequency ω_p, $\kappa_{\rm ex2}^p$ is the external decay rate for the pump field mode in $\mathrm{R_B}$, and g is the nonlinear single-photon coupling strength in the parametric nonlinear process. Note that the factor 2π in $\alpha_{\rm in}$ and α_p is needed to keep the dimension consistent in the angular frequency. In the rotating frame defined by $U=\exp[(-i\frac{\omega_p}{2}a^\dagger a-i\frac{\omega_p}{2}b^\dagger b-i\omega_p c^\dagger c)t]$, the Hamiltonian becomes

$$\begin{aligned}\mathcal{H}/\hbar =&\Delta_p^a a^\dagger a+\Delta_p^b b^\dagger b+\Delta_p^c c^\dagger c+J_0(a^\dagger b+b^\dagger a)+g\left({b^\dagger}^2 c+b^2c^\dagger\right)\\ &+i\sqrt{2\kappa_{\rm ex1}}\alpha_{\rm in}\left(a^\dagger e^{-i\Delta_{\rm in}t}-ae^{i\Delta_{\rm in}t}\right)+i\sqrt{2\kappa_{\rm ex2}^p}\alpha_p\left(c^\dagger-c\right) ,\end{aligned} \tag{S40}$$

where $\Delta_p^{a/b}=\omega_{a/b}-\omega_p/2$, $\Delta_p^c=\omega_c-\omega_p$, and $\Delta_{\rm in}=\omega_{\rm in}-\omega_p/2$. The dynamical equation of c can be solved by the Heisenberg equation

$$\dot{c}=i[\mathcal{H},c]-\kappa_p c=-(i\Delta_p^c+\kappa_p)c+\sqrt{2\kappa_{\rm ex2}^p}\alpha_p-igb^2 . \tag{S41}$$

Here, we consider a strong continuous pump field to excite the mode c in $\mathrm{R_B}$ with amplitude $\langle c\rangle \gg \langle b\rangle$. In this strong pump case, we can omit the terms related to g in Eqs. (S40) and (S41) for the purpose of calculating the steady state of mode c. In doing so, we obtain the reduced Hamiltonian $H_p=\Delta_p^c c^\dagger c+i\sqrt{2\kappa_{\rm ex2}^p}\alpha_p\left(c^\dagger-c\right)$ and the steady-state solution

$$\langle c\rangle_{\rm ss}=\frac{\sqrt{2\kappa_{\rm ex2}^p}\alpha_p}{i\Delta_p^c+\kappa_p} . \tag{S42}$$

For a slowly varying mode c, we can replace c with its steady-state mean value $\langle c\rangle_{\rm ss}$ in the Hamiltonian Eq. (S40). Then, the Hamiltonian Eq. (S40) can be rewritten as

$$\mathcal{H}/\hbar=\Delta_p^a a^\dagger a+\Delta_p^b b^\dagger b+J_0(a^\dagger b+b^\dagger a)+g\left({b^\dagger}^2\langle c\rangle_{\rm ss}+b^2\langle c\rangle_{\rm ss}^*\right)+i\sqrt{2\kappa_{\rm ex1}}\alpha_{\rm in}\left(a^\dagger e^{-i\Delta_{\rm in}t}-ae^{i\Delta_{\rm in}t}\right) . \tag{S43}$$

Comparing Eq. (S43) with Eq. (S1), we can estimate the amplitude and phase of the pump as

$$\Omega_p=2g|\langle c\rangle_{\rm ss}|=4g\sqrt{\frac{\pi\kappa_{\rm ex2}^p P_p}{({\Delta_p^c}^2+\kappa_p^2)\hbar\omega_p}} , \quad \theta_p=-\mathrm{Arg}[\langle c\rangle_{\rm ss}] . \tag{S44}$$

The resonance pump field at frequency $\omega_p=\omega_c$ leads to $\Delta_p^c=0$. The pump power is given by

$$P_p=\frac{\hbar\omega_p\kappa_p^2\Omega_p^2}{16\pi g^2\kappa_{\rm ex2}^p} . \tag{S45}$$

To evaluate the optical transistor, we define the gain of the transistor as

$$G = \frac{P_{\rm in}}{P_p}\Delta T = \frac{8\omega_{\rm in} g^2 \kappa_{\rm ex2}^p \alpha_{\rm in}^2}{\omega_p \kappa_p^2 \Omega_p^2}\Delta T = \frac{2\kappa_{\rm ex2} g^2 \alpha_{\rm in}^2}{\kappa_a^2 \Omega_p^2}\Delta T\ . \tag{S46}$$

Here, owing to $\Delta_p^b \ll \{\omega_a,\ \omega_b,\ \omega_{\rm in},\ \omega_p\}$, we have applied the following approximations, $\omega_p = 2\omega_{\rm in}$, $\kappa_p = 2\kappa_a$, and $\kappa_{\rm ex2}^p = 2\kappa_{\rm ex2}$, to obtain the final form of the gain.

[1] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, England, 1997).
[2] X.-Y. Lü, Y. Wu, J. R. Johansson, H. Jing, J. Zhang, and F. Nori, Squeezed optomechanics with phase-matched amplification and dissipation, Phys. Rev. Lett. **114**, 093602 (2015).
[3] W. Qin, A. Miranowicz, P.-B. Li, X.-Y. Lü, J. Q. You, and F. Nori, Exponentially enhanced light-matter interaction, cooperativities, and steady-state entanglement using parametric amplification, Phys. Rev. Lett. **120**, 093601 (2018).
[4] C. W. Gardiner and M. J. Collett, Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation, Phys. Rev. A **31**, 3761 (1985).
[5] C. W. Gardiner, Driving a quantum system with the output field from another driven quantum system, Phys. Rev. Lett. **70**, 2269 (1993).
[6] H. J. Carmichael, Quantum trajectory theory for cascaded open systems, Phys. Rev. Lett. **70**, 2273 (1993).
[7] K. Stannigel, P. Rabl, and P. Zoller, Driven-dissipative preparation of entangled states in cascaded quantum-optical networks, New J. Phys. **14**, 063014 (2012).
[8] K. Xia, F. Jelezko, and J. Twamley, Quantum routing of single optical photons with a superconducting flux qubit, Phys. Rev. A **97**, 052315 (2018).
[9] A. H. Kiilerich and K. Mølmer, Input-output theory with quantum pulses, Phys. Rev. Lett. **123**, 123604 (2019).
[10] S. V. Kutsaev, A. Krasnok, S. N. Romanenko, A. Y. Smirnov, K. Taletski, and V. P. Yakovlev, Up-and-coming advances in optical and microwave nonreciprocity: From classical to quantum realm, Adv. Photonics Res. **2**, 2000104 (2021).