SUPPLEMENTAL MATERIAL:
Acoustic radiation force and torque on small particles as measures of the canonical momentum and spin densities

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1. MIE SCATTERING OF AN ACOUSTIC PLANE WAVE BY A SPHERE

Here we describe the exact Mie-type solution for the acoustic plane wave scattering by a spherical particle [1, 2]. Note that throughout this work we consider only monochromatic fields, which are described by the complex coordinate-dependent fields $p(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$. The real time-dependent fields are obtained by applying the $\mathrm{Re}[\,...\exp(-i\omega t)]$ operator. Accordingly, all quadratic forms (such as energy density, momentum density, etc.) are considerfed as cycle-averaged quantities. This means that the form $f_1 f_2$ of real time-dependent fields becomes $\frac{1}{2}\mathrm{Re}(f_1^* f_2)$ in terms of complex time-independent fields. Note also that it is often sufficient to write the explicit form of the $p(\mathbf{r})$ field, and the velocity field can be obtained as $\mathbf{v} = (i\omega\rho)^{-1}\nabla p$.

As in the main text, we consider a spherical particle with the parameters ρ_1, β_1, and the radius a, located at $\mathbf{r} = \mathbf{0}$ in a homogeneous medium with parameters ρ and β. Using spherical coordinates (r, θ, ϕ), the incident z-propagating plane-wave field can be written as [1]

$$p^{(\mathrm{in})} = A\, e^{ikr\cos\theta} = \sum_{n=0}^{\infty} A_n j_n(kr) P_n(\cos\theta), \tag{1}$$

where $A_n = A\, i^n (2n+1)$, j_n are the spherical Bessel functions of the first kind, and P_n are the Legendre polynomials. Taking into account the azimuthal symmetry of the problem, the field inside the spherical particles and scattered field outside the particles can be written as:

$$p^{(\mathrm{part})} = \sum_{n=0}^{\infty} A_n c_n j_n(k_1 r) P_n(\cos\theta), \quad p^{(\mathrm{sc})} = \sum_{n=0}^{\infty} A_n a_n h_n^{(1)}(kr) P_n(\cos\theta). \tag{2}$$

where $k_1 = k\sqrt{\bar\rho\bar\beta}$ is the wave number inside the particle, and $h_n^{(1)}$ are the spherical Hankel functions of the first kind.

The coefficients a_n and c_n in Eqs. (2) should be determined from the boundary conditions, i.e., the continuity of the pressure and normal velocity component at the interface $r = a$. Using $\mathbf{v} = (i\omega\rho)^{-1}\nabla p$, we have:

$$p^{(\mathrm{in})} + p^{(\mathrm{sc})} = p^{(\mathrm{part})}, \quad \frac{1}{\rho}\left(\frac{\partial p^{(\mathrm{in})}}{\partial r} + \frac{\partial p^{(\mathrm{sc})}}{\partial r}\right) = \frac{1}{\rho_1}\frac{\partial p^{(\mathrm{part})}}{\partial r}. \tag{3}$$

Substituting the fields (1) and (2) into the boundary conditions (3), we derive [2, 3]:

$$c_n = \frac{i/(ka)^2}{j_n(k_1 a)\, h_n^{(1)\prime}(ka) - \delta\, j_n'(k_1 a)\, h_n^{(1)}(ka)}, \quad a_n = \frac{\delta\, j_n'(k_1 a)\, j_n(ka) - j_n(k_1 a)\, j_n'(ka)}{j_n(k_1 a)\, h_n^{(1)\prime}(ka) - \delta\, j_n'(k_1 a)\, h_n^{(1)}(ka)}, \tag{4}$$

where $\delta = (k_1\rho_0)/(k\rho_1) = \sqrt{\bar\beta/\bar\rho}$, $\bar\rho = \rho_1/\rho$, $\bar\beta = \beta_1/\beta$, and the prime stands for the derivative with respect to the argument of the functions.

The terms with the coefficients a_0, a_1, a_2, ... in the decomposition (2) of the scattered field can be associated with the corresponding multipole radiations: the monopole, dipole, quadrupole, ... ones. In the case of a small subwavelength particle, $ka \ll 1$, the higher-n terms have higher leading orders in ka (but each term a_n has all orders higher than the leading ones), see Table I. In this work, we restrict our consideration by the leading monopole and dipole terms, which generally have the same order of smallness. Their coefficients (4) can be expanded in the Taylor series as:

$$\begin{aligned} a_0 &= \frac{i}{3}(\bar\beta - 1)(ka)^3 + \frac{i}{45}\left[\bar\beta^2(\bar\rho + 5) - 15\bar\beta + 9\right](ka)^5 - \frac{1}{9}(\bar\beta - 1)^2 (ka)^6 + \ldots, \\ a_1 &= \frac{i}{3}\frac{\bar\rho - 1}{2\bar\rho + 1}(ka)^3 + \frac{i}{5}\frac{\bar\rho^2(\bar\beta - 1) - \bar\rho + 1}{(2\bar\rho + 1)^2}(ka)^5 - \frac{1}{9}\left(\frac{\bar\rho - 1}{2\bar\rho + 1}\right)^2 (ka)^6 + \ldots. \end{aligned} \tag{5}$$

TABLE I. The leading orders of different multipole terms in the decomposition (2) and (4) of the scattered field for the case of a lossless particle $[\mathrm{Im}(\bar{\rho}) = \mathrm{Im}(\bar{\beta}) = 0]$ [4].

Coefficient	a_n	$O(\mathrm{Re}\, a_n)$	$O(\mathrm{Im}\, a_n)$
Monopole	a_0	$\sim (ka)^6$	$\sim (ka)^3$
Dipole	a_1	$\sim (ka)^6$	$\sim (ka)^3$
Quadrupole	a_2	$\sim (ka)^{10}$	$\sim (ka)^5$
Octupole	a_3	$\sim (ka)^{14}$	$\sim (ka)^7$
⋮			
n-th multipole	a_n	$\sim (ka)^{2(2n+1)}$	$\sim (ka)^{2n+1}$

2. MONOPOLE AND DIPOLE POLARIZABILITIES OF THE PARTICLE

Using the monopole and dipole coefficients in the Mie plane-wave scattering series, one can determine the generic monopole and dipole responses of the particle to an *arbitrary* incident monochromatic field. A similar approach is well known in optics [5–11].

To do this, note that the monochromatic radiation of the oscillating acoustic point monopole and z-oriented dipole can be written as [1, 2]:

$$p_m = i\, Q\, \frac{\rho\, \omega}{4\pi r}\, e^{ikr}, \qquad p_d = k\, D\, \frac{\rho\, \omega}{4\pi r} \cos\theta \left(1 + \frac{i}{kr}\right) e^{ikr}, \tag{6}$$

where Q and D are the *monopole and dipole strengths*, respectively, which we define with the signs opposite to those in [1, 2]. These expressions have the same form as the first two terms in the Mie series (2) for the scattered field: $A_0 a_0 h_0^{(1)}(kr) P_0(\cos\theta)$ and $A_1 a_1 h_1^{(1)}(kr) P_1(\cos\theta)$. Writing these terms in the form of Eqs. (6) with the monopole and dipole strength presented as

$$Q = -i\omega\beta\, \alpha_m\, p^{(\mathrm{in})}(\mathbf{0}), \qquad D = \alpha_d\, p^{(\mathrm{in})}(\mathbf{0}), \tag{7}$$

where $p^{(\mathrm{in})}(\mathbf{0}) = A$ is the incident field at the particle's position, we obtain the *monopole and dipole polarizabilities* of the particle:

$$\alpha_m = -\frac{4\pi i}{k^3}\, a_0\,, \qquad \alpha_d = -\frac{4\pi i}{k^3}\, 3a_1\,. \tag{8}$$

Using the leading terms in the Taylor series (5) yields Eqs. (6) of the main text. For lossless particles, the leading terms in the Taylor series (5) yield purely real polarizabilities, while the *third* terms in Eqs. (5) provide small imaginary corrections, responsible for the "radiation friction" effect [5, 12, 13]. Due to this effect, even a lossless particle experiences a non-zero scattering (radiation pressure) force, while the radiation torque vanishes identically (see the main text).

3. EXACT CALCULATIONS OF THE ACOUSTIC FORCE AND TORQUE

The radiation force and torque acting on a scattering particle can be calculated using the momentum and angular momentum fluxes through a closed surface Σ enclosing the particle [14–18]:

$$\mathbf{F} = -\oint_{\Sigma} \hat{\mathcal{P}}\mathbf{n}\, d\Sigma, \qquad \mathbf{T} = -\oint_{\Sigma} \hat{\mathcal{M}}\mathbf{n}\, d\Sigma, \tag{9}$$

where $\mathbf{n}$ is the outer normal unit vector to the surface, $\mathcal{P}_{ij} = \frac{1}{4}\left(\beta|p|^2 - \rho|\mathbf{v}|^2\right)\delta_{ij} + \frac{1}{2}\rho\,\mathrm{Re}(v_i^* v_j)$ is the cycle-averaged kinetic momentum flux density tensor (the acostic analogue of the Maxwell stress tensor), $\mathcal{M}_{ij} = \varepsilon_{jk\ell} r_k \mathcal{P}_{\ell i}$ is the corresponding angular momentum flux density, whereas δ_{ji} and $\varepsilon_{ik\ell}$ are the Kronecker and Levi-Civita symbols. Here, the acoustic wave field is the sum of the incident and scattered fields outside the particle: $p = p^{(\mathrm{in})} + p^{(\mathrm{sc})}$, $\mathbf{v} = \mathbf{v}^{(\mathrm{in})} + \mathbf{v}^{(\mathrm{sc})}$.

For the exact numerical calculations of the acoustic forces and torque shown in Fig. 3 in the main text, we used the Mie-scattering fields (1)–(4) (modified for the evanescent incident wave, as described below) and a spherical surface $\Sigma = \{r = R > a\}$. In this case, the expression for the torque (9) can be simplified to [19, 20]:

$$\mathbf{T} = -\frac{\rho}{2} R^3 \,\mathrm{Re} \oint (\mathbf{n}\cdot\mathbf{v}^*)\,[\mathbf{n}\times\mathbf{v}]\; d\Omega\,, \tag{10}$$

where $d\Omega = \sin\theta\, d\phi\, d\theta$ is the element of the spherical solid angle.

Note that the integral (9) for the radiation force on a spherical particle from the incident plane wave (1) can be evaluated analytically [21–23]:

$$F_z = -\frac{2\pi}{\rho\,\omega^2}\,|A|^2 \sum_{n=0}^{\infty} \left[(2n+1)\,\mathrm{Re}\,(a_n) + 2(n+1)\,\mathrm{Re}\,(a_n^* a_{n+1})\right]. \tag{11}$$

For an absorbing small particle, the leading-order approximation $\sim (ka)^3$ yields:

$$F_z \simeq -\frac{2\pi}{\rho\,\omega^2}\,|A|^2 \left[\mathrm{Re}\,(\bar{a}_0) + 3\,\mathrm{Re}\,(\bar{a}_1)\right], \tag{12}$$

where we denoted the first terms in the Taylor series (5) as $\bar{a}_0$ and $\bar{a}_1$. Using Eqs. (3) and (6) of the main text, we find that the canonical momentum of the plane-wave field (1) is $P_z^{(p)} = P_z^{(\mathbf{v})} = \beta k/4\omega$, and Eq. (12) coincides with the scattering (radiation-pressure) force expression (10) of the main text.

For a lossless particle, $\mathrm{Re}\,\bar{a}_0 = \mathrm{Re}\,\bar{a}_1 = 0$, the approximate expression (12) vanishes, and one has to involve higher-order terms from the exact Eq. (11). Using the Taylor series (5), where the third terms equal $\bar{a}_0^2$ and $\bar{a}_1^2$, the first non-vanishing approximation for the radiation-pressure force can be written as

$$F_z \simeq -\frac{2\pi}{\rho\,\omega^2}\,|A|^2 \Big[-(\mathrm{Im}\,\bar{a}_0)^2 - 3\,(\mathrm{Im}\,\bar{a}_1)^2 + 2\,\mathrm{Im}(\bar{a}_0)\,\mathrm{Im}(\bar{a}_1)\Big]. \tag{13}$$

Here the first two terms can be associated with the "radiation-friction" corrections to the monopole and dipole polarizabilities (6) in the main text [12, 13, 24], while the third term originates from the interference of the monopole and dipole responses (the $\mathrm{Re}\,(a_n^* a_{n+1})$ term in Eq. (11)). An analogous higher-order force from the interference of electric and magnetic dipoles plays an important role in optics [12, 25–28]. All the three terms in Eq. (13) are generally $\sim (ka)^6$.

4. COMPLEX-ANGLE APPROACH FOR THE EVANESCENT INCIDENT WAVE

To apply the Mie scattering solutions (1)–(4) to the case of the evanescent incident wave, we use the approach described in [29]. Namely, we note that the incident plane wave (1), $p^{(\mathrm{in})}(\mathbf{r}) = Ae^{ikz}$, can be transformed to the evanescent wave, Eq. (12) in the main text, by the rotation of its argument on the *imaginary* angle:

$$p^{(\mathrm{in\ evan})}(\mathbf{r}) = p^{(\mathrm{in})}\left(\hat{R}(i\gamma)\mathbf{r}\right) = Ae^{ikz\cosh\gamma - kx\sinh\gamma}\,. \tag{14}$$

Here $k\cosh\gamma = k_z$, $k\sinh\gamma = \kappa$, i.e., $\gamma = \tanh^{-1}(\kappa/k_z)$, and

$$\hat{R}(i\gamma) = \begin{pmatrix} \cosh\gamma & 0 & -i\sinh\gamma \\ 0 & 1 & 0 \\ i\sinh\gamma & 0 & \cosh\gamma \end{pmatrix} \tag{15}$$

is the rotational operator of the imaginary argument.

Since the Mie scattering problem is linear, the field scattered from the evanescent wave can be obtained by the same transformation (14) of the plane-wave scattered field (2) and (4) [29]:

$$p^{(\mathrm{sc\ evan})}(\mathbf{r}) = p^{(\mathrm{sc})}\left(\hat{R}(i\gamma)\mathbf{r}\right). \tag{16}$$

Notice that the transformation (15) is written for the Cartesian coordinates: $(x, y, z) \to (x', y', z') = (x\cosh\gamma - iz\sinh\gamma, y, ix\sinh\gamma + z\cosh\gamma)$. The corresponding transformation of the spherical coordinates is: $(r, \theta, \phi) \to \left(r, \cos^{-1}(z'/r), \tan^{-1}(y/x')\right)$.

After applying the transformations (14)–(16), we have analytical expressions for all the fields in the evanescent Mie scattering problem. Then, these fields can be directly used in the calculations of the acoustic force and torque, Eqs. (9) and (10). This significantly decreases the numerical computational efforts compared to the general Lorenz-Mie-like theories [30–32].

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