SUPPLEMENTAL MATERIAL:

Comparison with the Weyl representation of Maxwell equations

To highlight the similarities and differences between the topological Klein-Gordon (KG) approach to acoustics and its electromagnetic counterpart [41], here we describe the Weyl representation and topological features of Maxwell equations in forms similar to the acoustic equations of the main text.

We start with Maxwell equations in a homogeneous isotropic lossless medium [1]:

$$\varepsilon \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H}, \qquad -\mu \frac{\partial \mathbf{H}}{\partial t} = \nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = 0. \tag{S1}$$

Introducing the "wavefunction" consisting of real physical fields, $\boldsymbol{\Psi} = (\mathbf{E}, \mathbf{H})^T$, the first two Eqs. (S1) can be written as [41]

$$i\hat{\sigma}^{(\mathrm{m})} \frac{\partial \boldsymbol{\Psi}}{\partial t} = \left(\hat{\mathbf{S}} \cdot \hat{\mathbf{p}}\right) \boldsymbol{\Psi}, \tag{S2}$$

where $\hat{\mathbf{p}} = -i\nabla$ is the canonical momentum operator, $\hat{\mathbf{S}}$ are the spin-1 matrices, which act on the three-vector degrees of freedom such that $\hat{\mathbf{S}} \cdot \hat{\mathbf{p}} = \nabla \times$, the speed of light in vacuum is assumed to be $c_0 = 1$, and

$$\hat{\sigma}^{(\mathrm{m})} = \begin{pmatrix} 0 & i\mu \\ -i\varepsilon & 0 \end{pmatrix}. \tag{S3}$$

is the matrix with the medium parameters which acts on the "electric-magnetic" degrees of freedom, i.e., intermixes the electric and magnetic fields.

Akin to Eqs. (5) in the main text, we introduce the scalar and "cross" products for this 6-component wavefunction $\boldsymbol{\Psi}$ producing scalars and three-vectors, respectively:

$$\boldsymbol{\Psi}_1 \bullet \boldsymbol{\Psi}_2 \equiv \varepsilon \mathbf{E}_1 \cdot \mathbf{E}_2 + \mu \mathbf{H}_1 \cdot \mathbf{H}_2,$$
$$\boldsymbol{\Psi}_1 \otimes \boldsymbol{\Psi}_2 \equiv \mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{H}_1 \times \mathbf{E}_2. \tag{S4}$$

Using these definitions, the electromagnetic energy density and energy flux density (the Poynting vector) take the forms of Eqs. (9):

$$W = \frac{1}{2} \boldsymbol{\Psi} \bullet \boldsymbol{\Psi}, \qquad \boldsymbol{\Pi} = \frac{1}{2} \boldsymbol{\Psi} \otimes \boldsymbol{\Psi}. \tag{S5}$$

The approach of Ref. [41] is based on the helicity operator, which for monochromatic fields ($\partial/\partial t \to -i\omega$) and using Eq. (S2) can be written as

$$\hat{\mathfrak{S}} = \frac{\hat{\mathbf{S}} \cdot \hat{\mathbf{p}}}{|\mathbf{p}|} = \frac{1}{|n|} \hat{\sigma}^{(\mathrm{m})}. \tag{S6}$$

Here, the dispersion relation $|\mathbf{p}| = |n|\omega$ was used, whereas $n = \sqrt{\varepsilon\mu}$ is the refractive index of the medium. To distinguish the four different types of media, corresponding to the four quadrants of the parameter (ε, μ) space, we adopt the natural convention that the phase of the $\sqrt{\varepsilon\mu}$ grows uniformly with the number of the quadrant, i.e., $n = |n|$ for $(\varepsilon > 0, \mu > 0)$, $n = i|n|$ for $(\varepsilon < 0, \mu > 0)$, $n = -|n|$ for $(\varepsilon < 0, \mu < 0)$, and $n = -i|n|$, for $(\varepsilon > 0, \mu < 0)$ [41]. The helicity operator (S6) is non-Hermitian and has the following paired eigenvalues:

$$\mathfrak{S} = \pm \frac{n}{|n|}. \tag{S7}$$

These eigenvalues rotate in the complex plane at the transitions between the above four different types of media, which provides the $\mathbb{Z}_4$ Möbius-strip-like topology described by a single $\mathbb{Z}_4$ topological number $w = \frac{2}{\pi}\mathrm{Arg}(n)$ or, equivalently, a pair of $\mathbb{Z}_2$ topological numbers [41]:

$$w = \frac{1}{2}\left[1 - \mathrm{sgn}(\varepsilon), 1 - \mathrm{sgn}(\mu)\right]. \tag{S8}$$

The bulk-boundary correspondence, similar to Eq. (11) yields the phase diagram Fig. 1(a) for electromagnetic surface modes in Maxwell equations.

To reveal the similarity between this approach and the acoustic KG formalism, we represent Maxwell equations (S2) in the relativistic Weyl form. This representation is based on the Riemann-Silberstein "wavefunction" $\boldsymbol{\psi} = \sqrt{\varepsilon}\,\mathbf{E} + i\sqrt{\mu}\,\mathbf{H}$ [58], for which Eq. (S2) acquires the form

$$in\frac{\partial \boldsymbol{\psi}}{\partial t} = \left(\hat{\mathbf{S}} \cdot \hat{\mathbf{p}}\right)\boldsymbol{\psi}, \qquad -in\frac{\partial \boldsymbol{\psi}^*}{\partial t} = \left(\hat{\mathbf{S}} \cdot \hat{\mathbf{p}}\right)\boldsymbol{\psi}^*. \tag{S9}$$

Thus, this representation diagonalizes the helicity operator, which becomes, using the 6-component wavefunction $\left(\boldsymbol{\psi}, \boldsymbol{\psi}^*\right)^T$ [58], $\hat{\mathfrak{S}}' = \frac{n}{|n|}\mathrm{diag}(1, -1)$. Remarkably, Eqs. (S9) can be written in the relativistic covariant form of the Weyl equation [48]:

$$\left(\hat{S}^{\mu}\hat{p}_{\mu}\right)\boldsymbol{\psi} = 0, \quad \text{where} \quad \hat{p}_{\mu} = \left(in\partial_t, i\nabla\right) \tag{S10}$$

is the *four-momentum* operator, and $\hat{S}^{\mu} = \left(I_3, \hat{\mathbf{S}}\right)$. Comparing these equations with Eqs. (6) and (7) of the main text, we see that *the electromagnetic four-momentum operator contains* $n = \sqrt{\varepsilon\mu}$ *and differs in the four quadrants of the* (ε, μ) *space, while the acoustic four-momentum depends only on* $\sqrt{\rho}$ *and is independent of* β. This explains the main topological difference between the acoustic and Maxwell equations. Note that the Weyl four-momentum (S10) is clearly connected with the helicity operator (S6) because the coefficient at ∂_t exactly determines the behavior of the helicity $\propto \hat{\mathbf{S}} \cdot \hat{\mathbf{p}}$. We finally note that the connection between the real-field and Weyl (Riemann-Silberstein) wavefunctions can be written as

$$\begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}^* \end{pmatrix} = \hat{M}\,\boldsymbol{\Psi}, \qquad \hat{M} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \sqrt{\varepsilon} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix}. \tag{S11}$$

This connection differs from the acoustic Eq. (7) but still involves the same square roots of the two parameters (ε, μ).

To summarize, both electromagnetic and acoustic equations can be expressed via real physical fields (entering the boundary conditions): $(\mathbf{E},\mathbf{H})$ and $(P,\mathbf{v})$, as well as via "relativistic wavefunctions": the KG and Weyl ones. In this manner, the connections between these wavefunctions, as well as the four-momentum operators in the corresponding relativistic wave equations, involve square roots of the medium parameters: (ε,μ) and ρ. These determine the separation of the electromagnetic and acoustic parameter spaces into topologically-different sectors, labeled by the bulk topological indices (10) and (S8), and the appearance of surface modes at interfaces between topologically-different media, as shown in Fig. 1. The comparison of the main electromagnetic and acoustic quantities used in this work is shown in the Table SI.

	Acoustics	**Electromagnetism**
Real fields	$\boldsymbol{\Psi}^{\mu}=(P,\mathbf{v})$	$\boldsymbol{\Psi}=(\mathbf{E},\mathbf{H})$
Energy density and flux	$W=\frac{1}{2}\boldsymbol{\Psi}^{\mu}\bullet\boldsymbol{\Psi}^{\mu}=\frac{1}{2}(\beta P^2+\rho\mathbf{v}^2)$ $\boldsymbol{\Pi}=\frac{1}{2}\boldsymbol{\Psi}^{\mu}\otimes\boldsymbol{\Psi}^{\mu}=P\mathbf{v}$	$W=\frac{1}{2}\boldsymbol{\Psi}\bullet\boldsymbol{\Psi}=\frac{1}{2}(\varepsilon\mathbf{E}^2+\mu\mathbf{H}^2)$ $\boldsymbol{\Pi}=\frac{1}{2}\boldsymbol{\Psi}\otimes\boldsymbol{\Psi}=\mathbf{E}\times\mathbf{H}$
Connection with "relativistic wavefunctions"	$\boldsymbol{\Psi}^{\mu}=i\hat{p}^{\mu}\psi$ $=\left(-\sqrt{\rho}\,\partial_t\psi,\frac{\nabla\psi}{\sqrt{\rho}}\right)$	$\boldsymbol{\Psi}=\left(\frac{\operatorname{Re}\boldsymbol{\psi}}{\sqrt{\varepsilon}},\frac{\operatorname{Im}\boldsymbol{\psi}}{\sqrt{\mu}}\right)$
Relativistic wave equations	$(\hat{p}^{\mu}\bullet\hat{p}_{\mu})\psi=0$	$(\hat{S}^{\mu}\hat{p}_{\mu})\boldsymbol{\psi}=0$
Four-momentum operator	$\hat{p}^{\mu}=\left(i\sqrt{\rho}\,\partial_t,\frac{-i\nabla}{\sqrt{\rho}}\right)$	$\hat{p}^{\mu}=\left(i\sqrt{\varepsilon\mu}\,\partial_t,-i\nabla\right)$
Topological indices	$w(\rho)=\frac{1}{2}[1-\operatorname{sgn}(\rho)]$	$w(\varepsilon,\mu)=\frac{1}{2}[1-\operatorname{sgn}(\varepsilon),1-\operatorname{sgn}(\mu)]$

Table I. Comparison of acoustic and electromagnetic quantities.