

# Supplemental Material to Multielectron Ground State Electroluminescence

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## Overview of the Supplemental Material

The Supplemental Material (SM), which comprises several sections, can be divided in three main parts:

The first part is given in Sec. **I, II** where, firstly, we perform the mathematical passages needed to properly *trace out the environmental degrees of freedom* in second quantization; secondly, we recast the fermionic Hamiltonian in terms of collective spins exploiting the symmetries of the light-matter interaction. We derive the Dicke model by reabsorbing the diamagnetic term in the photonic terms.

The second part, comprising Secs. **III-IV**, illustrates the details of the *bosonic model*, in which a Holstein-Primakoff approximation is performed to treat the matter-like excitations. In Sec. **III** we derive the polariton eigenstates using a perturbation expansion beyond the rotating-wave approximation (RWA) on the bosonic Jaynes-Cummings (JC) part of the Hamiltonian. In Sec. **IV** instead we provide an alternative route, explicitly deriving the polariton eigenstates beyond the perturbation theory used in the main text, by diagonalizing, in Sec. **IV A**, the full bosonic Hamiltonian,

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which contains the counter-rotating wave terms, and then deriving the transition matrix elements in terms of such states, Sec. IV B, showing qualitative and quantitative agreement with the perturbation theory. In Sec. IV C, we also estimate the effective photonic emission from the cavity after the polariton scattering rates.

The third part of the SM instead, comprising Sec. V, develops the *fermionic model* [Eq. (1) of the main text], retaining the full nonlinearities for the matter-like excitations. Following this route, we will arrive to an expression for the single-polariton processes involved in the multielectron ground state electroluminescence that can be compared to the one obtained for the bosonic models. Moreover, in this formalism, we can capture more confidently also the scattering processes involving double-polariton states. However, in this framework, we will need to develop a more refined second-quantization theory capable of grasping the microscopic processes involved in this effect, to leading order in perturbation theory, with light-matter states comprising both fermionic and bosonic degrees of freedom. In order to do so, we derive a model for electron addition and subtraction, which will lead us to map the electron current effects in terms of Dicke states with non-fixed particle number, beyond a RWA analysis.

In particular, in Sec. V, using the Dicke state formalism, we develop perturbative fermionic eigenstates in terms of the RWA Hamiltonian of the light-matter system. In Sec. V A, we highlight the properties of the Tavis-Cummings model eigenstates, used to derive the perturbed basis of the Dicke model. In Sec. V B, we introduce the difference between the microscopic and macroscopic state formalism and derive the corresponding master equations. In Sec. V C, we calculate the general transition matrix elements induced by the interaction Hamiltonian relative to the electronic environment. In Sec. V D, we unravel the compact but cumbersome formal expression derived in Sec. V C and specialize it to the case of ground state electroluminescence, calculating both the single-polariton and double-polariton transition rates. Finally, in Sec. V E, we compare the fermionic results with the bosonic calculations, finding excellent agreement.

## I. TRACING OUT THE ENVIRONMENT

We begin by considering Eq. (1) of the main text,

$$H = \omega_c a^\dagger a + \sum_n \left( \omega_1 c_{1,n}^\dagger c_{1,n} + \omega_2 c_{2,n}^\dagger c_{2,n} \right) + D(a + a^\dagger)^2 + \chi(a + a^\dagger) \sum_n (c_{2,n}^\dagger c_{1,n} + c_{1,n}^\dagger c_{2,n}), \quad (\text{S1})$$

which describes light-matter interaction in the model system under study. We are interested in studying the effects of three environments on this model: a left ( $L$ ) and right ( $R$ ) electronic reservoir, and the extra-cavity electromagnetic modes. The aim of this section is to compute the transition rates among eigenstates of the system induced by a generic Hamiltonian  $H^I = H_{\text{el}}^I + H_{\text{cav}}^I$  representing the physical interaction with the electronic and bosonic environmental degrees of freedom. By considering  $H^I$  as a perturbation over the coherent dynamics, the electron scattering rates can be computed by using the Fermi golden rule

$$\Gamma_{\text{el}}^{\alpha \rightarrow \beta} = 2\pi \sum_i \rho_i^{\text{FD}} \sum_f |\langle \beta, f | H_{\text{el}}^I | \alpha, i \rangle|^2 \delta(\tilde{\Delta}), \quad (\text{S2})$$

where  $\alpha$  ( $i$ ) and  $\beta$  ( $f$ ) are the initial and final states for the system (environment). The fermionic part of the environment follows a Fermi-Dirac distribution  $\rho^{\text{FD}}$ , diagonal in the basis  $|i\rangle$ . This distribution depends on macroscopic parameters characterizing the reservoir, such as the temperature and chemical potential. The energy conservation is imposed by the delta function with argument  $\tilde{\Delta} = \omega_f - \omega_i + \Delta_{\alpha\beta}$ , with  $\Delta_{\alpha\beta} = \omega_\beta - \omega_\alpha$ , where  $\omega_\beta$  and  $\omega_\alpha$  are the final and initial frequencies of the system. Specifically, we model the interaction with the electronic reservoirs as  $H_{\text{el}}^I = H_L^I + H_R^I$ , where

$$H_L^I = \lambda \sum_{n,\zeta} [(c_{1,n} + c_{2,n})c_{L;n,\zeta}^\dagger + (c_{1,n}^\dagger + c_{2,n}^\dagger)c_{L;n,\zeta}], \quad (\text{S3})$$

and similarly for  $H_R^I$ , thus assuming that the energy scale,  $\lambda$ , is equal for the two fermionic reservoirs. The operators  $c_{L(R);n,\zeta}$  label the annihilation operators for a fermion associated with a degree of freedom  $n$  and  $\zeta$  in the left (right) reservoir. The label  $\zeta$  represents a continuum of properties, so we replace  $\sum_\zeta \rightarrow \int d\zeta \nu(\zeta)$  with a density of states  $\nu(\zeta)$ .

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By introducing a generic dispersion relation of the kind  $E = f(\zeta)$  we can further write  $\sum_{\zeta} \rightarrow \int d\zeta \nu(\zeta) = \int dE \nu(E)$ , so that  $\nu(\zeta) = \frac{d\zeta}{dE} \nu(E)$  and we obtain

$$\Gamma_{\text{el}}^{\alpha \rightarrow \beta} = \sum_n \Gamma_{\text{el},n}^{\alpha \rightarrow \beta} \quad (\text{S4})$$

with

$$\begin{aligned} \Gamma_{\text{el},n}^{\alpha \rightarrow \beta} &= 2\pi\lambda^2 \sum_i \rho_i^{\text{FD}} \left[ \int dE \nu(E) [1 - n_i(E)] \delta(\tilde{\Delta}) |\langle \beta | (c_{1,n} + c_{2,n}) | \alpha \rangle|^2 \right. \\ &\quad \left. + \int dE \nu(E) n_i(E) \delta(\tilde{\Delta}) |\langle \beta | (c_{1,n}^\dagger + c_{2,n}^\dagger) | \alpha \rangle|^2 \right] = \Gamma_{\text{out},n}^{\alpha \rightarrow \beta} + \Gamma_{\text{in},n}^{\alpha \rightarrow \beta}, \end{aligned} \quad (\text{S5})$$

where  $n_i(E)$  counts the number of electrons with energy  $E$  in the state  $|i\rangle$  of the reservoir according to the Fermi-Dirac distribution  $\rho_i^{\text{FD}}$ . The rate  $\Gamma_{\text{out},n}^{\alpha \rightarrow \beta}$  ( $\Gamma_{\text{in},n}^{\alpha \rightarrow \beta}$ ) appearing in the r.h.s. of Eq. (S5) accounts for processes in which the environment has one more (less) electron associated with some quantum number  $\zeta$ , hence the definition of in/out rates:

$$\Gamma_{\text{out}}^{\alpha \rightarrow \beta} = \sum_n \Gamma_{\text{out},n}^{\alpha \rightarrow \beta} = \sum_n 2\pi\lambda^2 [1 - \bar{n}_{\text{FD}}(-\Delta_{\alpha\beta})] \nu(-\Delta_{\alpha\beta}) |\langle \beta | (c_{1,n} + c_{2,n}) | \alpha \rangle|^2, \quad (\text{S6a})$$

$$\Gamma_{\text{in}}^{\alpha \rightarrow \beta} = \sum_n \Gamma_{\text{in},n}^{\alpha \rightarrow \beta} = \sum_n 2\pi\lambda^2 \bar{n}_{\text{FD}}(\Delta_{\alpha\beta}) \nu(\Delta_{\alpha\beta}) |\langle \beta | (c_{1,n}^\dagger + c_{2,n}^\dagger) | \alpha \rangle|^2, \quad (\text{S6b})$$

where  $\bar{n}_{\text{FD}}(E) = \sum_i \rho_i^{\text{FD}} n_i(E)$  is the average electron number in the left or right electronic reservoir with energy  $E$ . In Eqs. (S4),(S5),(S6), the rates can be specified to the left and right reservoir, something we omitted above for simplicity, but which will hereafter be labelled for clarity in the *electron-current transition rates*

$$\Gamma_{L(R)}^{\alpha \rightarrow \beta} = \Gamma_{L(R),\text{out}}^{\alpha \rightarrow \beta} + \Gamma_{L(R),\text{in}}^{\alpha \rightarrow \beta} \quad (\text{S7})$$

If the electronic reservoirs are at zero temperature, the Fermi-Dirac distribution is only a function of the chemical potential,  $\mu_L$  and  $\mu_R$  respectively, such that we obtain a very compact expression relating the transition matrix elements to the “in” and “out” electron-current transition rates

$$\begin{aligned} \Gamma_{L(R),\text{out}}^{\alpha \rightarrow \beta} &= \Gamma_{\text{el}} \theta(-\mu_{L(R)} - \Delta_{\alpha\beta}) \sum_n |M_{\alpha\beta}^n|^2, \\ \Gamma_{L(R),\text{in}}^{\alpha \rightarrow \beta} &= \Gamma_{\text{el}} \theta(\mu_{L(R)} - \Delta_{\alpha\beta}) \sum_n |M_{\beta\alpha}^n|^2, \end{aligned} \quad (\text{S8})$$

where  $\Gamma_{\text{el}} = 2\pi\lambda^2\nu$  and  $\theta(x)$  is the Heaviside function. At zero temperature it is possible to define a frequency-independent density of states  $\nu$  for the two electronic reservoirs. We recall that the *electron-current transition matrix element*, introduced in Eq. (3) in the main text, reads

$$M_{\alpha\beta}^n = \langle \beta | (c_{1,n} + c_{2,n}) | \alpha \rangle, \quad (\text{S9})$$

where we consider that an electron in the upper or lower band can be lost, given the fact that, as we will show in the next Section, the light-matter ground state of the Hamiltonian in Eq. (1), also Eq. (S1), contains a superposition of such populations.

Similarly to Eq. (S3), the interaction of the cavity field with the extra-cavity electromagnetic modes is described by

$$H_{\text{cav}}^I = \lambda_{\text{cav}} \sum_{\zeta} (a + a^\dagger)(a_{\zeta} + a_{\zeta}^\dagger), \quad (\text{S10})$$

where  $\lambda_{\text{cav}}$  is the photon-photon coupling strength and  $a_{\zeta}$  is the annihilation operator for the  $\zeta$ th extra-cavity mode. By analyzing this formula, in analogy to what done for the electronic environment, we can derive the *cavity-photon emission rate*,

$$\Gamma_{\text{cav}}^{\alpha \rightarrow \beta} = 2\pi \sum_i \rho_i^{\text{BE}} \sum_f |\langle \beta, f | H_{\text{cav}}^I | \alpha, i \rangle|^2 \delta(\tilde{\Delta}) = 2\pi\lambda_{\text{cav}}^2 [\bar{n}_{\text{BE}}(\Delta_{\alpha\beta}) \nu_{\text{EM}}(\Delta_{\alpha\beta}) + [1 + \bar{n}_{\text{BE}}(-\Delta_{\alpha\beta})] \nu_{\text{EM}}(-\Delta_{\alpha\beta})] M_{\alpha\beta}^{\text{cav}2}, \quad (\text{S11})$$

where  $\rho_i^{\text{BE}}$  is the Bose-Einstein distribution for photons in the state  $|i\rangle$  and  $\bar{n}_{\text{BE}}$  counts the average number of photons in the electromagnetic reservoir. Defining the *cavity-photon transition matrix elements* as

$$M_{\alpha\beta}^{\text{cav}} = \langle\beta|(a + a^\dagger)|\alpha\rangle. \quad (\text{S12})$$

At zero temperature and for a frequency-independent photonic density of states,  $\nu_{\text{EM}}(\omega) \simeq \nu_{\text{EM}}$ , we obtain

$$\Gamma_{\text{cav}}^{\alpha\rightarrow\beta} = \Gamma_{\text{cav}} M_{\alpha\beta}^{\text{cav}2}, \quad (\text{S13})$$

where  $\Gamma_{\text{cav}} = 2\pi\lambda_{\text{cav}}^2\nu_{\text{EM}}$  is the characteristic emission rate of the photonic cavity.

## II. DICKE MODEL

We can rewrite Eq. (S1) in terms of spin angular momentum operators. We define  $S^\pm = \sum_n S_n^\pm$ , with  $S_n^+ = c_{2,n}^\dagger c_{1,n}$  and thus

$$2S^3 = 2\sum_n S_n^3 = \sum_n \sigma_{z,n} = \sum_n [S_n^+, S_n^-] = \sum_n (c_{2,n}^\dagger c_{2,n} - c_{1,n}^\dagger c_{1,n}). \quad (\text{S14})$$

We immediately find that the total angular momentum,  $\mathbf{S}^2$ , is a symmetry of the model. We obtain

$$H = \omega_c a^\dagger a + \omega_0 S^3 + \chi(a + a^\dagger)(S^- + S^+) + D(a + a^\dagger)^2 + \tilde{E}_0(N, N_2), \quad (\text{S15})$$

where we shifted the energy to absorb a term

$$E_0(N, N_2) = \omega_1 N + 2\omega_1 N_2 + \omega_0(j_N + N_2) \quad (\text{S16})$$

with  $j_N = N/2$ . We performed a fermion-to-spin transformation which, with respect to Eq. (S1), involves no approximations. In order to absorb the diamagnetic term, we rewrite the bosonic operator for the light field as a displaced operator,

$$\tilde{a} = \cosh(\lambda)a + \sinh(\lambda)a^\dagger, \quad (\text{S17})$$

which allows us to rewrite the Hamiltonian of Eq. (1) in the main text, repeated as

$$H_{N, N_2} = \tilde{\omega}_c \tilde{a}^\dagger \tilde{a} + \omega_0 S^3 + \tilde{\chi}(\tilde{a} + \tilde{a}^\dagger)(S^- + S^+) + \tilde{E}_0(N, N_2), \quad (\text{S18})$$

where

$$\tilde{\omega}_c = \omega_c e^{2\lambda}, \quad (\text{S19})$$

$$\tilde{\chi} = \chi e^{-\lambda}, \quad (\text{S20})$$

$$\tilde{E}_0 = \frac{\omega_c}{2} (e^{-2\lambda} - 1) + E_0, \quad (\text{S21})$$

$$\lambda = \frac{1}{2} \text{arctanh}[D/(\omega_c + 2D)]. \quad (\text{S22})$$

Renaming  $\tilde{a} \rightarrow a$ ,  $\tilde{\omega}_c \rightarrow \omega_c$ , and  $\tilde{\chi} \rightarrow \chi$  as implicitly assumed elsewhere in the text, and reabsorbing the constant term  $\tilde{E}_0$ , we obtain Eq. (2) of the main text, the Hamiltonian of the Dicke model,

$$H = \omega_c a^\dagger a + \omega_0 S^3 + \chi(a + a^\dagger)(S^- + S^+). \quad (\text{S23})$$

## III. BOSONIC MODEL: PERTURBATIVE THEORY OF THE BOSONIC MODEL

In the main text, the polariton modes of the JC model are defined in terms of the *rotating-wave part* of the bosonic Hamiltonian, Eq. (4). Here we illustrate explicitly the passages involved to derive the polariton eigenstates involved in the transition processes, developing them perturbatively onto the bosonic Jaynes-Cummings polariton eigenstates.

### A. Holstein-Primakoff transformation

The introduction of bosonic operators for the matter-excitations,  $S_+ = \sqrt{N}b^\dagger + O(|b^\dagger b/\sqrt{N}|)$ , and  $S_z = b^\dagger b - j_N$  allow us to rewrite Eq. (S23) as

$$H_{\text{bos}} = \omega_c a^\dagger a + \omega_0 b^\dagger b + g_N (a^\dagger + a)(b + b^\dagger), \quad (\text{S24})$$

by performing an approximation that is valid in the dilute regime,  $|b^\dagger b/\sqrt{N}| \ll 1$ . In the following subsections we provide the explicit derivation of the perturbed polariton eigenstates on the RWA part of this bosonic Hamiltonian.

### B. Perturbative eigenstates of the bosonic model

In the main text, we restricted ourselves to diagonalize the Jaynes-Cummings part of the bosonic Hamiltonian in Eq. (4), rewritten above as Eq. (S24), i.e.,

$$H_{\text{JC}} = \omega_c a^\dagger a + \omega_0 b^\dagger b + g_N (a^\dagger b + b^\dagger a), \quad (\text{S25})$$

whose kernel reads

$$\bar{H} = \begin{pmatrix} \omega_c & 0 & g_N & 0 \\ 0 & -\omega_c & 0 & -g_N \\ g_N & 0 & \omega_0 & 0 \\ 0 & -g_N & 0 & -\omega_0 \end{pmatrix}. \quad (\text{S26})$$

and its polariton modes are given as

$$p_\pm^\dagger = \alpha_a^\pm a^\dagger + \alpha_b^\pm b^\dagger, \quad (\text{S27})$$

with

$$\begin{aligned} \alpha_a^\pm &= \pm \left( \frac{\sqrt{4+x^2} \mp x}{2\sqrt{4+x^2}} \right)^{\frac{1}{2}}, \\ \alpha_b^\pm &= \left( \frac{2}{\sqrt{4+x^2}(\sqrt{4+x^2} \mp x)} \right)^{\frac{1}{2}}, \end{aligned} \quad (\text{S28})$$

where

$$x = \Delta/g_N \quad (\text{S29})$$

with

$$\Delta = \omega_0 - \omega_c. \quad (\text{S30})$$

Note that through Eq. (S28), it becomes explicit the dependence on  $N$  of the polariton modes,  $p_\pm = p_\pm^N$ , which is implicit in Eq. (S27). Hereafter, for simplicity, we will omit this explicit dependence besides for the expressions in which it will be clearer to highlight it.

In terms of these modes the bosonic Jaynes-Cummings Hamiltonian reads

$$H_{\text{JC}} = \omega^+ p_+^\dagger p_+ + \omega^- p_-^\dagger p_-, \quad (\text{S31})$$

with

$$\omega^\pm = \frac{1}{2} \left( \omega_0 + \omega_c \pm \sqrt{4g_N^2 + \Delta^2} \right). \quad (\text{S32})$$

As a consequence, the full Hamiltonian now reads

$$H = H_{\text{JC}} + V, \quad (\text{S33})$$

with

$$V = g_N \left[ -\alpha_a^+ \alpha_b^+ p_-^\dagger p_-^\dagger - \alpha_a^- \alpha_b^- p_+^\dagger p_+^\dagger + (\alpha_b^+ \alpha_a^- + \alpha_b^- \alpha_a^+) p_-^\dagger p_+^\dagger \right] + \text{H.c.} \quad (\text{S34})$$

where we used the identity  $\alpha_a^- \alpha_b^+ - \alpha_b^- \alpha_a^+ = -1$  to write the operators  $a$  and  $b$  in terms of the Jaynes-Cummings polaritons of Eq. (S27). We can then define the eigenstates of the bosonic Jaynes-Cummings Hamiltonian as

$$|G_N^{(0)}\rangle = \bigotimes_{n \in N} c_{1,n}^\dagger |0_{\text{el}}\rangle |0_{\text{ph}}\rangle, \quad (\text{S35a})$$

$$|\pm_N^{(0)}\rangle = p_\pm^\dagger |G_N^{(0)}\rangle, \quad (\text{S35b})$$

$$|\pm \pm_N^{(0)}\rangle = \frac{p_\pm^\dagger p_\pm^\dagger}{\sqrt{2}} |G_N^{(0)}\rangle, \quad (\text{S35c})$$

$$|+ -_N^{(0)}\rangle = p_+^\dagger p_-^\dagger |G_N^{(0)}\rangle, \quad (\text{S35d})$$

where  $|0_{\text{el}}\rangle$  and  $|0_{\text{ph}}\rangle$  are the electron and photonic vacuum, respectively. By using first-order perturbation theory in the potential  $V$  we then find that

$$|G_N\rangle = |G_N^{(0)}\rangle - \beta_{++} |+_+^{(0)}\rangle - \beta_{--} |-_-^{(0)}\rangle - \beta_{+-} |+_+^{(0)}\rangle - \beta_{-+} |-_-^{(0)}\rangle, \quad (\text{S36a})$$

$$|\pm_N\rangle = |\pm_N^{(0)}\rangle + \dots, \quad (\text{S36b})$$

$$|\pm \pm_N\rangle = |\pm \pm_N^{(0)}\rangle + \beta_{\pm\pm} |G_N^{(0)}\rangle + \dots, \quad (\text{S36c})$$

$$|+ -_N\rangle = |+ -_N^{(0)}\rangle + \beta_{+-} |G_N^{(0)}\rangle + \dots, \quad (\text{S36d})$$

with

$$\beta_{++} = \langle +_+^{(0)} | V | G_N^{(0)} \rangle = -\sqrt{2} \frac{g_N}{2\omega^+} \alpha_a^- \alpha_b^-, \quad (\text{S37a})$$

$$\beta_{--} = \langle -_-^{(0)} | V | G_N^{(0)} \rangle = -\sqrt{2} \frac{g_N}{2\omega^-} \alpha_a^+ \alpha_b^+, \quad (\text{S37b})$$

$$\beta_{+-} = \langle +_+^{(0)} | V | G_N^{(0)} \rangle = \frac{g_N}{\omega^+ + \omega^-} (\alpha_b^+ \alpha_a^- + \alpha_b^- \alpha_a^+), \quad (\text{S37c})$$

which are the results used in the main text. Note that also the coefficients  $\beta_{xy}$  have a dependence on  $N$ .

### C. Commutation relations between Jaynes-Cummings polaritons and electrons

We wish to calculate the commutators

$$C_{1,n} = [b^\dagger, c_{1,n}], \quad C_{2,n} = [b^\dagger, c_{2,n}], \quad (\text{S38})$$

in order to calculate the electron-current transition matrix elements within the RWA bosonic model. Using the approximate Holstein-Primakoff transformation

$$b^\dagger = \frac{S_+}{\sqrt{N}} = \sum_{\bar{n}} \frac{c_{2,\bar{n}}^\dagger c_{1,\bar{n}}}{\sqrt{N}}, \quad (\text{S39})$$

we obtain

$$\begin{aligned} \sqrt{N} C_{1,n} &= \sum_{\bar{n}} c_{2,\bar{n}}^\dagger c_{1,\bar{n}} c_{1,n} - c_{1,n} c_{2,\bar{n}}^\dagger c_{1,\bar{n}}, \\ \sqrt{N} C_{2,n} &= \sum_{\bar{n}} c_{2,\bar{n}}^\dagger c_{1,\bar{n}} c_{2,n} - c_{2,n} c_{2,\bar{n}}^\dagger c_{1,\bar{n}}. \end{aligned} \quad (\text{S40})$$

We now note that, if  $\bar{n} \neq n$  the fermionic anticommutation rules give zero. As a consequence, the sum only contributes for  $\bar{n} = n$  giving

$$\begin{aligned} \sqrt{N} C_{1,n} &= c_{2,n}^\dagger c_{1,n} c_{1,n} - c_{1,n} c_{2,n}^\dagger c_{1,n} = 0 \\ \sqrt{N} C_{2,n} &= c_{2,n}^\dagger c_{1,n} c_{2,n} - c_{2,n} c_{2,n}^\dagger c_{1,n} = -c_{1,n}, \end{aligned} \quad (\text{S41})$$

where we used again the fermionic anticommutation rules.

Similarly, it is possible to calculate the analogous commutators involving  $b$ , resulting in the following commutators

$$\begin{aligned} [b, c_{1,n}] &= -\frac{1}{\sqrt{N}}c_{2,n}, \\ [b, c_{2,n}] &= 0, \\ [b^\dagger, c_{1,n}] &= 0, \\ [b^\dagger, c_{2,n}] &= -\frac{1}{\sqrt{N}}c_{1,n}, \end{aligned} \quad (\text{S42})$$

which provides a prescription for the actual calculation of the electron-current transition matrix elements, performed in the following subsection.

#### D. Transition matrix elements in the bosonic model using Jaynes-Cummings polaritons

We now calculate the transition matrix elements between the ground state and the first excited states. The transition to single-polariton states,  $M_{G\pm}^n$ , using the results of the perturbation theory, Eqs. (S35),(S36),(S37), gives, to first order in the perturbative parameters  $\beta_{ij}$  given in Eq. (S36)

$$\begin{aligned} M_{G\pm}^n &= \langle \pm_{N-1} | (c_{1,n} + c_{2,n}) | G_N \rangle = \langle G_{N-1}^{(0)} | p_{\pm}(c_{1,n} + c_{2,n}) | G_N \rangle \\ &= \langle G_{N-1}^{(0)} | p_{\pm}(c_{1,n} + c_{2,n}) | G_N^{(0)} \rangle - \frac{\beta_{++}}{\sqrt{2}} \langle G_{N-1}^{(0)} | p_{\pm}(c_{1,n} + c_{2,n}) p_+^\dagger p_+^\dagger | G_N^{(0)} \rangle \\ &\quad - \frac{\beta_{--}}{\sqrt{2}} \langle G_{N-1}^{(0)} | p_{\pm}(c_{1,n} + c_{2,n}) p_-^\dagger p_-^\dagger | G_N^{(0)} \rangle - \beta_{+-} \langle G_{N-1}^{(0)} | p_{\pm}(c_{1,n} + c_{2,n}) p_+^\dagger p_-^\dagger | G_N^{(0)} \rangle. \end{aligned} \quad (\text{S43})$$

The first term on the r.h.s. of Eq. (S43) is 0, as  $c_{2,n}$  is annihilated on the Fermi sea at zero temperature and  $c_{1,n}$  describes the Fermi sea without one electron, i.e.,

$$c_{2,n} | G_N^{(0)} \rangle = 0 \quad (\text{S44a})$$

$$c_{1,n} | G_N^{(0)} \rangle = | G_{N-1}^{(0)} \rangle. \quad (\text{S44b})$$

In Eq. (S43), the average of a polariton creation operator on the ground state is zero. To unravel the other terms in Eq. (S43) it is useful to calculate

$$[c_{1,n} + c_{2,n}, p_{\pm}^\dagger] = \alpha_b^\pm [c_{1,n} + c_{2,n}, b^\dagger] = \alpha_b^\pm \frac{c_{1,n}}{\sqrt{N}}. \quad (\text{S45})$$

Equation (S45) implies that the second term on the r.h.s. of Eq. (S43), proportional to  $\beta_{++}$ , can be written as

$$\begin{aligned} \langle G_{N-1}^{(0)} | p_{\pm}(c_{1,n} + c_{2,n}) p_+^\dagger p_+^\dagger | G_N^{(0)} \rangle &= \alpha_b^\pm \frac{1}{\sqrt{N}} \langle G_{N-1}^{(0)} | p_{\pm} c_{1,n} p_+^\dagger | G_N^{(0)} \rangle + \langle G_{N-1}^{(0)} | p_{\pm} p_+^\dagger (c_{1,n} + c_{2,n}) p_+^\dagger | G_N^{(0)} \rangle \\ &= 2\alpha_b^\pm \frac{1}{\sqrt{N}} \langle G_{N-1}^{(0)} | p_{\pm} p_+^\dagger c_{1,n} | G_N^{(0)} \rangle + \langle G_{N-1}^{(0)} | p_{\pm} p_+^\dagger p_+^\dagger (c_{1,n} + c_{2,n}) | G_N^{(0)} \rangle \\ &= 2\alpha_b^\pm \frac{1}{\sqrt{N}} \langle G_{N-1}^{(0)} | p_{\pm} p_+^\dagger | G_{N-1}^{(0)} \rangle + \langle G_{N-1}^{(0)} | p_{\pm} p_+^\dagger p_+^\dagger | G_{N-1}^{(0)} \rangle = \delta_{\pm,+} \frac{2\alpha_b^\pm}{\sqrt{N}}, \end{aligned} \quad (\text{S46})$$

where  $\delta_{\pm,+}$  is the Krönecker delta giving 1 for  $M_{G_+}^n$  and 0 for  $M_{G_-}^n$ . As shown by the commutator in Eq. (S45), this matrix element depends on the presence of  $c_{2,n}$  in the interaction Hamiltonian, which means that this process relies on a  $c_{2,n}$  electron to tunnel out of the system.

Similarly to Eq. (S46), one can calculate the third and fourth term in Eq. (S43), proportional to  $\beta_{--}$  and  $\beta_{+-}$ , respectively. The result is then

$$M_{G\pm}^n = \frac{\sqrt{2}\beta_{\pm\pm}\alpha_b^\pm + \beta_{+-}\alpha_b^\mp}{\sqrt{N}}. \quad (\text{S47})$$

We now consider the transition to double-polariton states, characterized by the matrix elements  $M_{G_{\pm\pm}}^n$ ,

$$M_{G_{\pm\pm}}^n = \langle \pm \pm_{N-1} | (c_{1,n} + c_{2,n}) | G_N \rangle = \beta_{\pm\pm}^N \langle G_{N-1}^{(0)} | (c_{1,n} + c_{2,n}) | G_N^{(0)} \rangle - \frac{\beta_{\pm\pm}^{N-1}}{2} \langle G_{N-1}^{(0)} | p_+ p_+ (c_{1,n} + c_{2,n}) p_{\pm}^{\dagger} p_{\pm}^{\dagger} | G_N^{(0)} \rangle. \quad (\text{S48})$$

We already see a major difference with respect to the previous case. In fact, the first element is non-zero because of the  $c_{1,n}$  term. We also immediately see that, when calculating a commutator, the second term ends up with an odd number of electrons and vanishes. This means that the only non-zero result will be when no commutators are computed. At that point only  $c_{1,n}$  will contribute to transform the Fermi sea with  $N$  electrons to a Fermi sea with  $(N-1)$  electrons. Therefore,

$$M_{G_{\pm\pm}}^n = \beta_{\pm\pm}^N - \beta_{\pm\pm}^{N-1} \simeq \partial_N \beta_{\pm\pm}^N, \quad (\text{S49})$$

which needs to be compared with the transition matrix element calculated in the full bosonic model, Eq. (4), and the same applies for the single-polariton rates, Eq. (S47).

#### IV. BOSONIC MODEL: FULL BOSONIC MODEL DIAGONALIZATION BEYOND PERTURBATION THEORY

Here we derive the *full* polariton spectrum of the Hamiltonian in Eq. (4), as well as its eigenstates, and use them to calculate the transition rates between the ground state and the relevant excited states.

##### A. Polariton spectrum of the full bosonic model

We reconsider Eq. (4) by writing it as

$$H = \omega_c a^{\dagger} a + \omega_0 b^{\dagger} b + g_N (a + a^{\dagger})(b + b^{\dagger}), \quad (\text{S50})$$

whose normal modes  $\hat{v}$  can be found by solving the Hopfield equation

$$[H, \hat{v}] = E_v \hat{v}. \quad (\text{S51})$$

The commutator structure of this equation implies a built-in ‘‘time reversal’’ symmetry which simply reads

$$[H, \hat{v}^{\dagger}] = -E_v \hat{v}^{\dagger}. \quad (\text{S52})$$

By writing a generic operator  $\hat{v}$  as

$$\hat{v} = c_1 a^{\dagger} + c_2 a + c_3 b^{\dagger} + c_4 b = \sum_i v_i \cdot \hat{O}_i, \quad (\text{S53})$$

where  $\vec{v} \in \mathbb{R}^4$  and  $\hat{O} = \{a^{\dagger}, a, b^{\dagger}, b\}$ , we obtain

$$\sum_i [H, \hat{O}_i] v_i = E_v \sum_i v_i \hat{O}_i. \quad (\text{S54})$$

Since the set of operators in  $\hat{O}$  is closed under commutations with the Hamiltonian we can define a Hamiltonian kernel  $\bar{H}$ , such that

$$[H, \hat{O}_i] = \sum_j \bar{H}_{ji} \hat{O}_j. \quad (\text{S55})$$

This leads to  $\sum_j \sum_i \bar{H}_{ji} v_i \hat{O}_j = E_v \sum_i v_i \hat{O}_i$ , or, equivalently

$$\sum_j \sum_i \bar{H}_{ji} v_i \hat{O}_j = E_v \sum_j v_j \hat{O}_j, \quad (\text{S56})$$



whose solution can be found by solving

$$\sum_j \bar{H}_{ij} v_j = E_v v_j, \quad (\text{S57})$$

or, equivalently,

$$\bar{H} \vec{v} = E_v \vec{v}. \quad (\text{S58})$$

Explicitly, the matrix  $\bar{H}$  can be constructed column by column, taking the commutator of  $H$  with  $a^\dagger$  (first column),  $a$  (second column),  $b^\dagger$  (third column), and  $b$  (fourth column). The entry of the first (second) row corresponds to taking the component proportional to  $a^\dagger$  ( $a$ ) of the column by column procedure. The entry of the third (fourth) row corresponds to taking the component proportional to  $b^\dagger$  ( $b$ ) of the column by column procedure. Explicitly we have

$$\bar{H} = \begin{pmatrix} \omega_c & 0 & g_N & -g_N \\ 0 & -\omega_c & g_N & -g_N \\ g_N & -g_N & \omega_0 & 0 \\ g_N & -g_N & 0 & -\omega_0 \end{pmatrix}. \quad (\text{S59})$$

There are two eigenvectors  $\vec{v}_\pm$  of the previous equation with positive energy and they correspond to the *polariton modes* of the *full bosonic model*

$$\tilde{p}_\pm^\dagger = \sum_i v_i^\pm \hat{O}_i. \quad (\text{S60})$$

The normalization follows from the imposition  $[\tilde{p}_\pm, \tilde{p}_\pm^\dagger] = 1$  and requires

$$v_1^{\pm 2} + v_3^{\pm 2} - v_2^{\pm 2} - v_4^{\pm 2} = 1. \quad (\text{S61})$$

Notice that, if we used  $\tilde{p}_\pm$  in the definition of Eq. (S60), a similar analysis for the operators  $\tilde{p}_\pm$  we would have imposed  $v_2^2 + v_4^2 - v_1^2 - v_3^2 = 1$  instead.

To summarize, we now write explicitly the solution of

$$\bar{H} \vec{v}_\pm = \lambda_\pm \vec{v}_\pm, \quad (\text{S62})$$

such that  $\lambda_\pm > 0$ . We have that the eigenenergies of the upper and lower polariton branches are then given by

$$\lambda_\pm = \left[ \frac{\omega_0^2 + \omega_c^2 \pm (\omega_0^4 + 16g_N^2 \omega_0 \omega_c - 2\omega_0^2 \omega_c^2 + \omega_c^4)^{\frac{1}{2}}}{2} \right]^{\frac{1}{2}}, \quad (\text{S63})$$

and the eigenvectors are defined by

$$\begin{aligned} v_1^\pm &= g_N(\lambda_\pm + \omega_0)(\lambda_\pm + \omega_c)/Z_\pm, \\ v_2^\pm &= -g_N(\lambda_\pm + \omega_0)(\lambda_\pm - \omega_c)/Z_\pm, \\ v_3^\pm &= [2g_N^2 \omega_c + (\lambda_\pm + \omega_0)(\lambda_\pm^2 - \omega_c^2)]/Z_\pm, \\ v_4^\pm &= -2g_N^2 \omega_c/Z_\pm, \end{aligned} \quad (\text{S64})$$

where  $Z_\pm$  is set to ensure Eq. (S61). The final expression for the polaritonic modes for the full bosonic Hamiltonian of Eq. (S50) is then

$$\tilde{p}_\pm^\dagger = \sum_i v_i^\pm \hat{O}_i = v_1^\pm a^\dagger + v_2^\pm a + v_3^\pm b^\dagger + v_4^\pm b. \quad (\text{S65})$$

Note that we can also relate  $\hat{\tilde{p}} = \{\tilde{p}_+^\dagger, \tilde{p}_+, \tilde{p}_-^\dagger, \tilde{p}_-\}^T$  to  $\hat{\tilde{O}}$ . By defining

$$P = \begin{pmatrix} v_+^1 & v_+^2 & v_+^3 & v_+^4 \\ v_+^2 & v_+^1 & v_+^4 & v_+^3 \\ v_-^1 & v_-^2 & v_-^3 & v_-^4 \\ v_-^2 & v_-^1 & v_-^4 & v_-^3 \end{pmatrix}, \quad (\text{S66})$$

we can finally write the compact expression

$$\hat{\tilde{p}} = P \cdot \hat{\tilde{O}}, \quad (\text{S67})$$

which greatly simplifies the calculation of the emission rates in terms of the  $P_{ij}$  matrix elements. We recall that, as in the previous sections, with regard to the Jaynes-Cummings polaritons, also the polariton modes of Eq. (S65) do depend on  $N$ , a feature that will appear in the next section in the calculation of matrix elements between states with different particle number, and it will be exploited to give intuitive estimates on the order of each of such effects.

### B. Transition rates in terms of the polariton eigenstates of the full bosonic model

While in the main text the rates were derived in perturbation theory for clarity of exposition, it is also straightforward to compute them after exact diagonalization. Essential for this calculation are the commutation relations in Eq. (S42) and the definition of the polaritonic excitations in Eq. (S65) and their matrix form in Eq. (S67). We start calculating the single-polariton states matrix elements,  $M_{G_{\pm}}^n$ . We have

$$\begin{aligned} M_{G_+}^n &= \langle +_{N-1} | (c_{1,n} + c_{2,n}) | G_N \rangle = \langle G_{N-1} | \tilde{p}_{\pm}^{N-1} (c_{1,n} + c_{2,n}) | G_N \rangle \simeq -P_{23} \langle G_{N-1} | [b^\dagger, c_{2,n}] | G_N \rangle = -\frac{P_{23}}{\sqrt{N}}, \\ M_{G_-}^n &= \langle -_{N-1} | (c_{1,n} + c_{2,n}) | G_N \rangle \simeq -\frac{P_{43}}{\sqrt{N}}, \end{aligned} \quad (\text{S68})$$

where the symbol  $\simeq$  implies keeping only the lowest non-trivial order in the asymptotic expansion in  $1/N$ , and where we made explicit the dependence of the polariton mode operator on the number of single-excited fermionic sites,  $N$ , writing  $\tilde{p}_{\pm} = \tilde{p}_{\pm}^{N-1}$  for the mode operator coming from the definition of the bra. The coefficients  $M_{G_{\pm\pm}}^n$  can be computed similarly. However, the lowest non-trivial order in an asymptotic expansion in  $1/N$  requires a little bit more work as

$$\begin{aligned} \sqrt{2}M_{G_{++}}^n &= \sqrt{2} \langle +_{N-1} | (c_{1,n} + c_{2,n}) | G_N \rangle = \langle G^{N-1} | \tilde{p}_+^{N-1} \tilde{p}_+^{N-1} (c_{1,n} + c_{2,n}) | G_N \rangle \\ &= \langle G_{N-1} | (\tilde{p}_+^N - \partial_N \tilde{p}_+^N) (\tilde{p}_+^N - \partial_N \tilde{p}_+^N) (c_{1,n} + c_{2,n}) | G_N \rangle \\ &\simeq \langle G_{N-1} | \tilde{p}_+^N \tilde{p}_+^N (c_{1,n} + c_{2,n}) | G_N \rangle - \langle G_{N-1} | \tilde{p}_+^N \partial_N \tilde{p}_+^N (c_{1,n} + c_{2,n}) | G_N \rangle - \langle G_{N-1} | \partial_N \tilde{p}_+^N \tilde{p}_+^N (c_{1,n} + c_{2,n}) | G_N \rangle \\ &\simeq -\frac{1}{\sqrt{N}} \langle G_{N-1} | \tilde{p}_+^N (P_{24}c_{2,n} + P_{23}c_{1,n}) | G_N \rangle - \langle G_{N-1} | \tilde{p}_+^N \left( \sum_j \partial_N P_{1j} \right) P_{jk}^{-1} \hat{p}_k (c_{1,n} + c_{2,n}) | G_{N-1} \rangle \\ &\quad - \langle G_{N-1} | \partial_N \tilde{p}_+^N \tilde{p}_+^N (c_{1,n} + c_{2,n}) | G_N \rangle \simeq \frac{1}{N} P_{24} P_{23} - \sum_j \partial_N P_{2j} P_{j1}^{-1} \end{aligned} \quad (\text{S69})$$

where, in the second line of the equation above, we transformed

$$\tilde{p}_+^{N-1} = (\tilde{p}_+^N - \partial_N \tilde{p}_+^N), \quad (\text{S70})$$

and, in the third and fourth line of the equation above, we took advantage of the matrix notation of Eq. (S67) to evaluate the operator derivative only on the matrix of coefficients  $P$ , approximating to leading order in  $1/N$  the dependence on  $N$  only in  $g_N$ .

Finally, in order to compute the term

$$\langle G_{N-1} | \tilde{p}_+^N \left( \sum_j \partial_N P_{1j} \right) P_{jk}^{-1} \hat{p}_k (c_{1,n} + c_{2,n}) | G_{N-1} \rangle \quad (\text{S71})$$

in the last step of the equation above, we considered that, since we want to calculate the transition matrix elements up to order  $O(1/N)$ , we could at that point commute the polaritonic operators with the fermionic ones and annihilate the ground state on the right. Hence, the only non-zero contribution in the sum is given by  $j = 1$  leading to that result. We also used

$$(c_{1,n} + c_{2,n}) | G_N \rangle = | G_{N-1} \rangle = O\left(\frac{1}{\sqrt{N}}\right) \quad (\text{S72})$$

and the fact that since the only dependence on  $N$  is through  $g_N$ , the derivative in  $N$  can be written as

$$\partial_N = g_N / (2N) \partial_{g_N}. \quad (\text{S73})$$

Following the same steps, we can calculate the rates towards the other two-polariton states. We obtain

$$\begin{aligned} M_{G_{++}}^n &= \frac{P_{24}P_{23}}{\sqrt{2}N} - \frac{g_N \sum_j \partial_{g_N} (P_{2j}) P_{j1}^{-1}}{2\sqrt{2}N}, \\ M_{G_{--}}^n &= \frac{P_{44}P_{43}}{\sqrt{2}N} - \frac{g_N \sum_j \partial_{g_N} (P_{4j}) P_{j3}^{-1}}{2\sqrt{2}N}, \\ M_{G_{+-}}^n &= \frac{P_{24}P_{43}}{N} - \frac{g_N \sum_j \partial_{g_N} (P_{2j}) P_{j3}^{-1}}{2N}, \\ &= \frac{P_{44}P_{23}}{N} - \frac{g_N \sum_j \partial_{g_N} (P_{4j}) P_{j1}^{-1}}{2N}, \end{aligned} \quad (\text{S74})$$

which in this form shows clearly that, even in the full-bosonic model, the double-polariton transition rates have a  $1/N$  attenuation with respect to the single-polariton channels, Eq. (S68). Moreover, the form of Eq. (S74) clearly hints at the interpretation of *the electron-subtraction processes as an effective non-adiabatic modulation of the light-matter coupling*, hence allowing to draw a connection with previous experiments in which such process was investigated [1–3].

### C. Photon-scattering transition matrix elements

In a real quantum device, the photon emission rate cannot have a perfect efficiency conversion from cavity polariton to emitted light. This quantity is the one measured in photo-detection in a spectroscopic experiment. The photonic emission rate arising from GSE is the product of two processes: First, there is the polariton scattering due to the extraction of an electron, calculated using  $\Gamma_{\text{GSE}}$ , with

$$\Gamma_{\text{GSE}} = \sum_{E=\{\pm, \pm\pm, \pm\mp, \dots\}} \Gamma_{\text{el}}^{G \rightarrow E}, \quad (\text{S75})$$

and which we have shown to be dependent on the  $|G_N\rangle \rightarrow |\pm_{N-1}\rangle$  channel. Then there is a second relaxation process that involves the emission of a photon,  $|\pm_{N-1}\rangle \rightarrow |G_{N-1}\rangle$ , occurring with a probability  $|\alpha_{\text{ph}}^\pm|^2$ , proportional to the Hopfield coefficients associated with light.

Note that in  $\Gamma_{\text{GSE}}$  and elsewhere in rates, in the subscripts of matrix elements and superscripts of rate emissions we omit defining the number of electrons in the initial and final states, where the transition is always  $|G_N\rangle \rightarrow |E_{N-1}\rangle$ , and in the sum we omit the transition  $|G_N\rangle \rightarrow |G_{N-1}\rangle$ , which leads only to a *dark electron current* with no photon emission.

Here we calculate explicitly  $|\alpha_{\text{ph}}^\pm|^2$ , the probability of photon emission associated to the decay of the system from the polariton branches,  $|\pm_{N-1}\rangle$ , to the ground state ( $|G_{N-1}\rangle$  for the RWA case,  $|\tilde{G}_{N-1}\rangle$  in the non-RWA bosonic case). This is a fast process occurring *after* the electron scattering process quantified in the sections above. We can calculate  $|\alpha_{\text{ph}}^\pm|^2$  in the RWA boson model from

$$|\alpha_{\text{ph}}^\pm|^2 = |\langle +_{N-1} | (a + a^\dagger) | G_{N-1} \rangle|^2 = |(\alpha_a^\pm)^* + \alpha_a^\pm \frac{\beta_{\pm\pm}}{\sqrt{2}}|^2. \quad (\text{S76})$$

In the polariton ground state of the non-RWA Hamiltonian we also have a photonic component for that state, which complicates the calculation. Yet, for any fixed  $N$ , it is always valid the quantum harmonic oscillator relation  $\tilde{p}_\pm |\tilde{G}\rangle = 0$ , so all we need to compute is the commutator

$$[\tilde{p}_{\pm, N}, (a + a^\dagger)] = v_2^\pm - v_1^\pm \quad (\text{S77})$$

and obtain, for the full boson case (non-RWA)

$$|\alpha_{\text{ph}}^\pm|^2 = |\langle \tilde{G}_{N-1} | \tilde{p}_{\pm, N-1} (a + a^\dagger) | \tilde{G}_{N-1} \rangle|^2 = |v_1^\pm - v_2^\pm|^2. \quad (\text{S78})$$

As mentioned in the main text, this calculation allows us to derive the effective total photon emission rate,  $\Gamma_{\text{tot}} = \Gamma_{\text{tot}}^+ + \Gamma_{\text{tot}}^-$ .

The efficiency of this conversion is determined by the cavity characteristic rate,  $\Gamma_{\text{cav}}$ , and by the rate of conversion of the bright polaritons into dark polaritons,  $\Gamma_{\text{dark}}^\pm$ . If we assume  $\Gamma_{\text{cav}} \gg \Gamma_{\text{dark}}^\pm$ , we obtain

$$\Gamma_{\text{tot}}^\pm = |\alpha_{\text{ph}}^\pm|^2 \frac{\Gamma_{\text{em}}^\pm \Gamma_{\text{cav}}}{\Gamma_{\text{dark}}^\pm + \Gamma_{\text{cav}}} \simeq |\alpha_{\text{ph}}^\pm|^2 \Gamma_{\text{em}}^\pm. \quad (\text{S79})$$

Note that  $|\alpha_{\text{ph}}^\pm|^2$  is included in the plots in Figure 2 of the main text, here reproduced for clarity as Figure S1, to estimate the effective light emission of the GSE process in both the RWA and non-RWA bosonic models. In Figure S2, we resolve the total signals of Figure 3, into the contributions of the upper and lower polariton branches, shown in the upper and lower panels, respectively. For different coupling strengths, the lower (upper) polariton emission rates and fluxes are peaked at negative (positive) frequency detuning. As shown by the darker shading in the contour plots of Figure S2, the lower polariton signal is stronger than the upper polariton one, especially at stronger light-matter couplings.

In Figure S3 and Figure S4, we report the contour plots for the total extra-cavity emission, showing that the extraction of photons from the GSE is efficient. In particular in Figure S3 it is visible that this second decay process makes the intensity of the total extra-cavity flux symmetric with respect to detuning of the cavity-matter frequency.

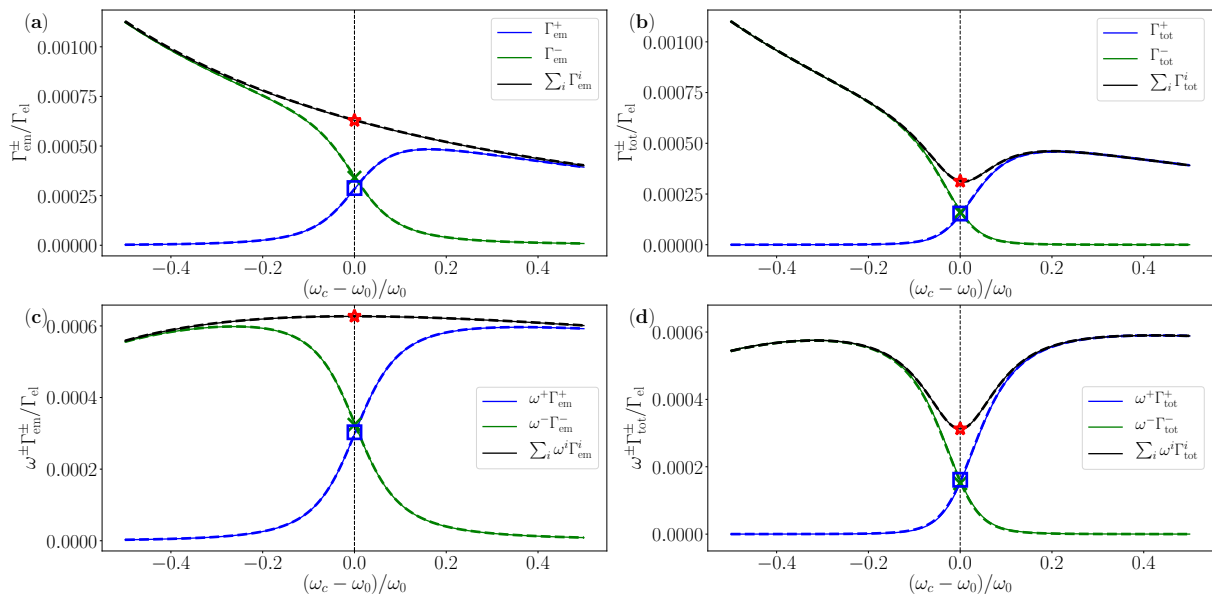


FIG. S1: (a,c) Polariton scattering rates  $\Gamma_{em}^{B'}$  and fluxes,  $\omega^{B'} \Gamma_{em}^{B'}$ , in units of the total electron transport rate  $\Gamma_{el}$  for the upper polariton ( $B' = +$ , blue curves) and lower polariton ( $B' = -$ , green curves), and sum of the two signals (black curves), as a function of the normalized detuning for  $g/\omega_0 = 0.05$ . (b,d) Total photon emission rates and fluxes. The resonance condition is marked by a dashed black vertical line and markers for the different quantities: blue open square for the upper polariton, green cross for the lower polariton, red star for the total signal. Solid curves correspond to the bosonic RWA quantities, dashed curves to the full boson model developed in the SM.

## V. FERMIONIC MODEL: PERTURBATIVE EIGENSTATES

We now consider the full model in Eq. (2). By neglecting the counter-rotating terms which do not conserve the number of bare excitations, we can write the Dicke model in the rotating wave approximation (RWA), i.e. the Tavis-Cummings (TC) model,

$$H^{\text{RWA}} = \omega_c a^\dagger a + \omega_0 S^3 + \chi(aS^+ + a^\dagger S^-) + E_0(N, N_2). \quad (\text{S80})$$

The eigenstates of Eq. (S80) can be written in terms of the bare-basis states

$$|\psi\rangle = |j, m\rangle \otimes |N_2\rangle \otimes |\gamma\rangle, \quad (\text{S81})$$

where  $|j, m\rangle = |j, m\rangle_N$  is the Dicke state of  $N$  two-level systems with total angular momentum  $j$  and third component  $m$ , defined onto the partition of electronic sites containing single excitations,  $|N_2\rangle$  counts the double-occupied electron states, which cannot be accounted for in the spin basis, and  $|\gamma\rangle$  counts  $\gamma$  photons in the photonic subspace.

While Eq. (S81) is in an intuitive form and can be used to find the states diagonalizing Eq. (S80), it will be more convenient to use a different notation, which will be more transparent when the electron-scattering terms of the interaction Hamiltonian will be considered, explicitly showing the dependence on  $N$  and  $N_2$ ,

$$|\psi\rangle = |j, m; N, N_2, \gamma\rangle, \quad (\text{S82})$$

where the dependence on  $\psi$  of the quantum numbers  $j, m, N, N_2$ , and  $\gamma$  will be highlighted when necessary. We thus express the eigenstates of the TC Hamiltonian, Eq. (S80), as

$$|\beta^{(0)}\rangle = \sum_{m\gamma} U_{m\gamma}^{(0)\beta} |j^\beta, m; N^\beta, N_2^\beta, \gamma\rangle, \quad (\text{S83})$$

where the factor  $U_{m\gamma}^{(0)\beta}$  can be written as

$$U_{m\gamma}^{(0)\beta} = u_\gamma^{(0)\beta} \delta_{m, m_\gamma^\beta}, \quad (\text{S84})$$

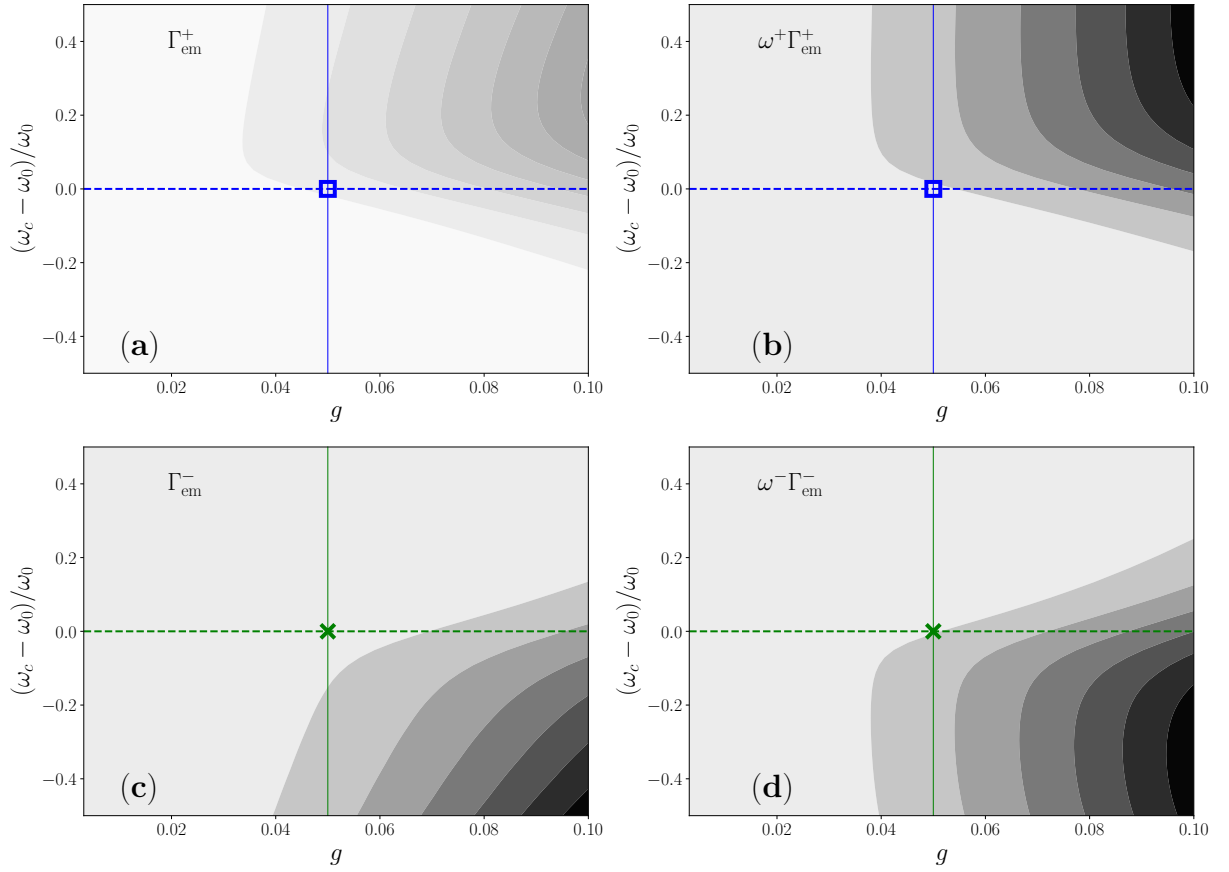


FIG. S2: Polariton emission rate  $\Gamma_{\text{em}}^{B'}$  [panels (a,c)] and flux [panels (b,d)] as a function of the frequency detuning and coupling strength, setting  $\chi = 3 \cdot 10^{-3} \omega_0$  fixed and varying  $N$ , on a logarithmic scale, and thus varying  $g = \sqrt{N} \chi$ , up to  $g = 0.1 \omega_0$ . The vertical solid blue and green lines correspond to cut shown Figure 2 in the main text. The blue open square and green cross correspond to the resonance point at  $g = 0.05 \omega_0$  for the upper and lower polariton, respectively. The contour plots of panels (a,c) are normalized by a common quantity, as well as those of panels (b,d), showing that the lower polariton rate reaches higher values at fixed  $g$ , limited in the case of fluxes,  $\omega \Gamma_{\text{em}}$ .

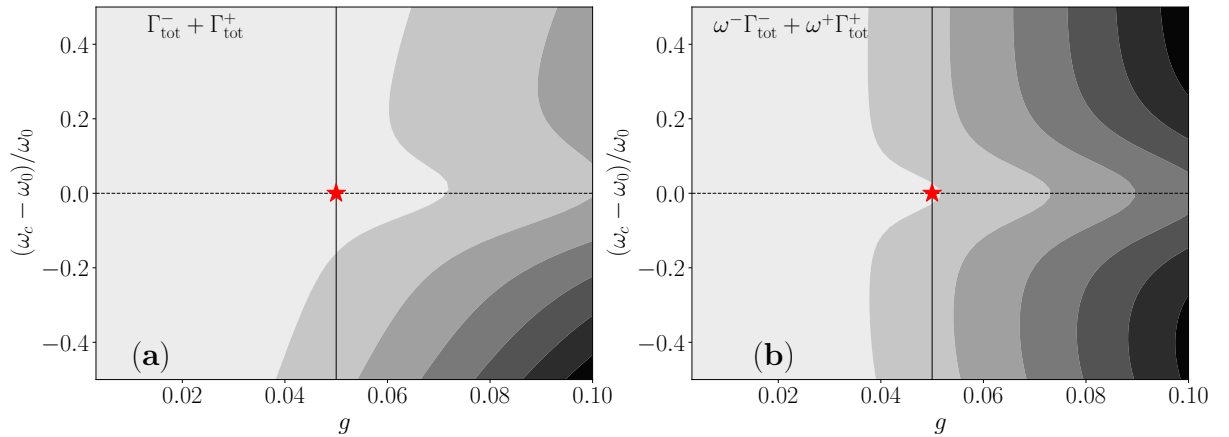


FIG. S3: Total extra-cavity photon emission rate  $\Gamma_{\text{tot}} = \Gamma_{\text{tot}}^- + \Gamma_{\text{tot}}^+$  [panel (a)] and flux,  $\sum_{B'} \omega^{B'} \Gamma_{\text{tot}}^{B'}$  [panel (b)], in units of the total electron scattering rate  $\Gamma_{\text{el}}$ .

in terms of a  $(n^\beta + 1)$ -dimensional eigenvector  $w^\beta$  of the RWA Hamiltonian in the subspace with  $n^\beta = j^\beta + m + \gamma$  bare excitations (derived explicitly in Sec. V A) and where we defined  $m_\gamma^\beta = -j^\beta + n^\beta - \gamma$ . We will further denote

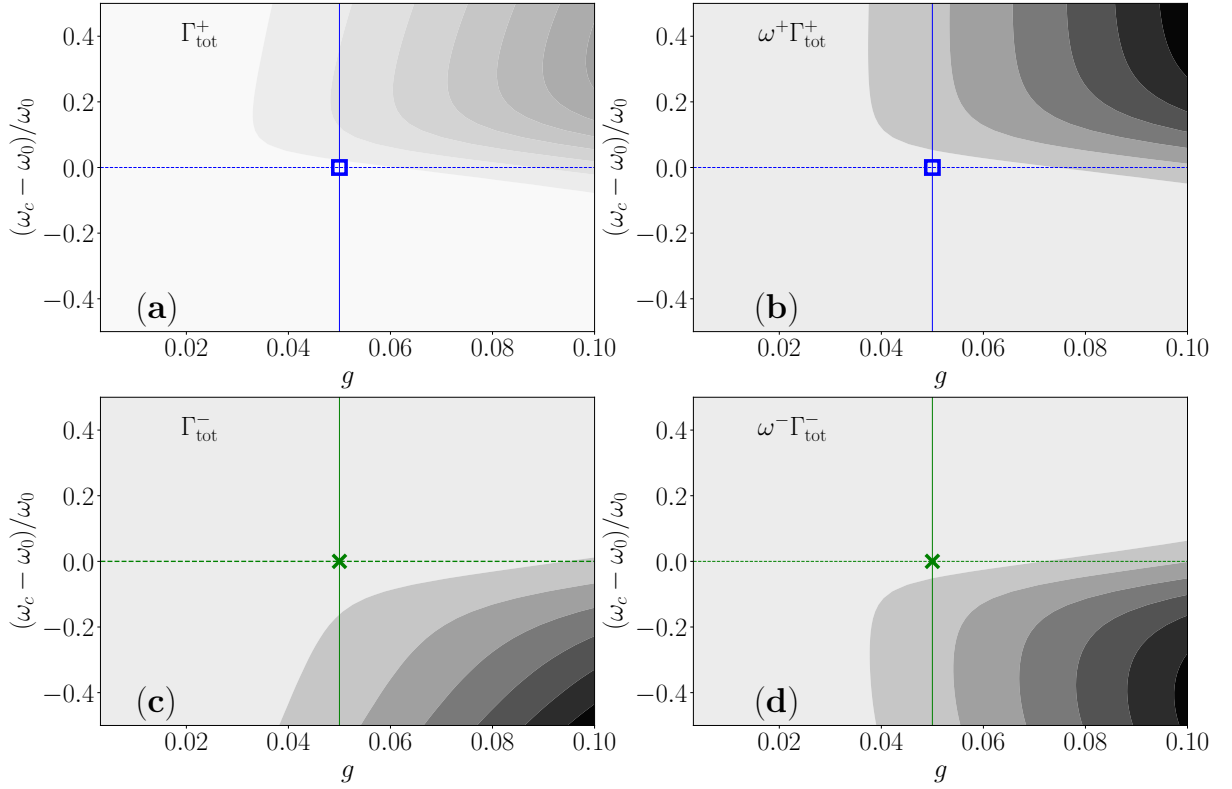


FIG. S4: Total extra-cavity photon emission rate  $\Gamma_{\text{tot}}^{B'}$  [panels (a,c)] and flux [panels (b,d)], for the upper and lower polariton, as a function of the frequency detuning and coupling strength, setting  $\chi = 3 \cdot 10^{-3}\omega_0$  fixed and varying  $N$  and thus  $g = \sqrt{N}\chi$ . The vertical solid blue and green lines correspond to the parameter ranges of the plots of Figure 2 in the main text. The blue open square and green cross correspond to the resonance point at  $g = 0.05\omega_0$  for the upper and lower polariton, respectively.

the energies of the associated states by  $E_{\beta}^{(0)}$ .

Using perturbation theory in the potential  $V = \chi(S^+a^+ + S^-a)$ , it is now possible to compute the coefficients  $U_{m\gamma}^{\beta}$  for the full Hamiltonian of the Dicke model. To first order,

$$|\beta\rangle = |\beta^{(0)}\rangle - \sum_{\bar{\beta}} \frac{\langle \bar{\beta}^{(0)} | V | \beta^{(0)} \rangle}{\Delta E_{\beta\bar{\beta}}} |\bar{\beta}^{(0)}\rangle = \sum_{m\gamma} U_{m\gamma}^{\beta} |j^{\beta}, m; N^{\beta}, N_2^{\beta}, \gamma\rangle, \quad (\text{S85})$$

where  $\Delta E_{\beta\bar{\beta}} = E_{\bar{\beta}}^{(0)} - E_{\beta}^{(0)}$ , and we introduced the matrix elements  $U_{m\gamma}^{\beta}$ . Equivalently, we can write

$$U_{m\gamma}^{\beta} = u_{\gamma}^{(0)\beta} \delta_{m, m_{\gamma}^{\beta}} - \sum_{\bar{\beta}} (c_{+}^{\beta\bar{\beta}} + c_{-}^{\beta\bar{\beta}}) u_{\gamma}^{(0)\bar{\beta}} \delta_{m, m_{\gamma}^{\bar{\beta}}}, \quad (\text{S86})$$

where

$$c_{\pm}^{\beta\bar{\beta}} = \frac{\chi}{\Delta E_{\beta\bar{\beta}}} \sum_{\bar{\gamma}} \bar{u}_{\bar{\gamma}\pm 1}^{(0)\bar{\beta}} u_{\bar{\gamma}}^{(0)\beta} A_{\pm, \bar{\gamma}}^{\beta} \delta_{n^{\bar{\beta}}, n^{\beta} \pm 2} \delta_{j^{\bar{\beta}}, j^{\beta}} \delta_{N^{\bar{\beta}}, N^{\beta}} \delta_{N_2^{\bar{\beta}}, N_2^{\beta}}, \quad (\text{S87})$$

and

$$A_{+, \gamma}^{\beta} = \sqrt{(\gamma + 1)(2j^{\beta} - n^{\beta} + \gamma)(n^{\beta} - \gamma + 1)}, \quad (\text{S88})$$

$$A_{-, \gamma}^{\beta} = \sqrt{\gamma(n^{\beta} - \gamma)(2j^{\beta} - n^{\beta} + \gamma + 1)}. \quad (\text{S89})$$

The result of this section is thus the formal definition of the eigenstates of the light-matter system, retaining the full fermionic nonlinearity of Eq. (1), recasting the problem in terms of collective pseudo-spin states. The calculation of the eigenstates has been reduced to finding the matrix elements  $u_{\gamma}^{(0)\beta}$ , a task that is performed in the next subsection.

### A. Single-excitation states of the Tavis-Cummings Hamiltonian

In this subsection, we derive an explicit expression for the eigenstates of the rotating wave Hamiltonian in Eq. (S80). To do this, we consider sets of basis states with fixed total angular momentum  $j$  and relative number of bare excitations  $n = j + \langle (a^\dagger a + S^3) \rangle_\beta = j + \gamma + m$ . In this subspace the matrix elements for the Hamiltonian kernel,  $\bar{H}^{j,N,N_2,n}$ , read

$$\bar{H}_{ik}^{j,N,N_2,n} = \langle j, m_i; N, N_2, \gamma_i | H^{\text{RWA}} | j, m_k; N, N_2, \gamma_k \rangle \quad (\text{S90})$$

$$= \langle j, m_i; N, N_2, \gamma_i | [\omega_c a^\dagger a + \omega_0 S^3 + \chi(aS^+ + a^\dagger S^-) + E_0(N, N_2)] | j, m_k; N, N_2, \gamma_k \rangle \quad (\text{S91})$$

$$= [\omega_c \gamma_k + \omega_0 m_k + E_0(N, N_2)] \delta_{\gamma_i, \gamma_k} + \langle j, m_i; N, N_2, \gamma_i | \chi(aS^+ + a^\dagger S^-) | j, m_k; N, N_2, \gamma_k \rangle \quad (\text{S92})$$

$$= [\omega_c \gamma_k + \omega_0 m_k + E_0(N, N_2)] \delta_{\gamma_i, \gamma_k} \\ + \chi \sqrt{\gamma_k + 1} \sqrt{(j + m_k)(j - m_k + 1)} \delta_{\gamma_i, \gamma_k + 1} + \chi \sqrt{\gamma_k} \sqrt{(j - m_k)(j + m_k + 1)} \delta_{\gamma_i, \gamma_k - 1}, \quad (\text{S93})$$

with  $\gamma_k = n - k$ ,  $m_k = -j + n - \gamma_k = -j + k$  where  $k = 0, \dots, n$ . We define  $m_k$  dependent on  $\gamma_k$  because the total number of total excitations in each subspace is fixed and equal to  $n$ . In the previous equation, the Krönecker deltas on  $m$  quantum number translated into Krönecker deltas on  $\gamma$ .

Using the definitions of  $\gamma_j$  and  $m_j$ , we then immediately find the matrix elements

$$\begin{aligned} \bar{H}_{k-1,k}^{j,N,N_2,n} &= \chi \sqrt{n - k + 1} \sqrt{k(2j - k + 1)}, \\ \bar{H}_{k,k}^{j,N,N_2,n} &= (n - k)\omega_c + k\omega_0 + \tilde{E}_0(N, N_2), \\ \bar{H}_{k+1,k}^{j,N,N_2,n} &= \chi \sqrt{n - k} \sqrt{(k + 1)(2j - k)}, \end{aligned} \quad (\text{S94})$$

where

$$\tilde{E}_0 = E_0(N, N_2) - j\omega_0. \quad (\text{S95})$$

Since we are interested in perturbations over the eigenstates of the rotating wave Hamiltonian in Eq. (S80), we can specialize the general notation allowing the label  $\beta$  to specify not only  $j^\beta, N^\beta, N_2^\beta$ , but also the total number of bare excitations  $n^\beta$ .

All of the  $(n^\beta + 1)$  eigenvectors  $u^{(0)\beta}$  of the matrix  $\bar{H}^{j^\beta, N^\beta, N_2^\beta, n^\beta}$ , defined by Eq. (S93), can then be used in Eq. (S84) to compute the matrix elements  $U_{m\gamma}^{(0)\beta}$  in the rotating wave approximation.

We now explicitly compute the eigenstates for the first excited states in the rotating wave approximation. The Hamiltonian kernel for the excited states with  $n = 1$  total excitation takes the form

$$\bar{H}^{j,N,N_2,1} = \begin{pmatrix} \omega_c + \tilde{E}_0 & \chi\sqrt{2j} \\ \chi\sqrt{2j} & \omega_0 + \tilde{E}_0 \end{pmatrix}. \quad (\text{S96})$$

The corresponding first two excited polaritonic states have the form

$$|\beta'_N{}^{(0)}\rangle = \cos\theta_{\beta'}^N |j, -j + 1; N, N_2, 0\rangle + \sin\theta_{\beta'}^N |j, -j; N, N_2, 1\rangle \quad (\text{S97})$$

where  $|\beta'_N{}^{(0)}\rangle = \{|+_N\rangle, |-_N\rangle\}$ , labels the upper and lower polaritonic modes, respectively, with the prime denoting the odd parity of these states. We have

$$\tan\theta_+ = (-\Delta + \sqrt{4g_N^2 + \Delta^2})/2g_N, \quad (\text{S98})$$

$\theta_+ = \theta_- - \pi/2$  with  $\Delta = \omega_0 - \omega_c$ , and as always  $g_N = \chi\sqrt{N}$ . This corresponds to

$$U_{m\gamma}^{+N(0)} = \delta_{m,-j+1}\delta_{\gamma,0} \cos\theta_+ - \delta_{m,-j}\delta_{\gamma,1} \sin\theta_+$$

and

$$U_{m\gamma}^{-N(0)} = \delta_{m,-j+1}\delta_{\gamma,0} \sin\theta_- - \delta_{m,-j}\delta_{\gamma,1} \cos\theta_-$$

in Eq. (S84).

Now that the single-polariton eigenstates involved in the transition matrix elements of the Fermi golden rule rates are explicitly defined, in order to perform the calculation in Eq. (S8), it suffices to calculate the action of the fermionic operators describing the electron scattering processes. We will assess this problem in the next subsection, in which the Clebsch-Gordan formalism will be extended to a fermionic Hilbert space in second quantization.

## B. Macroscopic states and fermionic master equation

The light-matter states of Eq. (S82) retain the full fermionic nonlinearity and, as shown in the precedent subsections, can be used to diagonalize the light-matter Hamiltonian of the system. In order to describe all the different *microscopic* states which constitute a basis for the Hilbert space of the full fermionic Hamiltonian, Eq. (1) one needs to consider *sets* for the single-occupied and double-occupied electron sites,  $\mathbf{N}$  and  $\mathbf{N}_2$ , of cardinality  $N$  and  $N_2$ . For each of those sets, and for each total angular momentum  $j$  and third component  $m$ , we should also take into account the presence of  $d$  inequivalent irreducible representations of  $\text{su}(2)$ . We will use the label  $r = 1, \dots, d$  to characterize this degeneracy. As a consequence, the states for the full fermionic model can be defined as

$$|\alpha\rangle = |j, m, r; \mathbf{N}, \mathbf{N}_2, \gamma\rangle \quad (\text{S99})$$

where  $\gamma$  is the number of photons. The microscopic information contained in these states is much more than what we need to compute macroscopic effects due to the current flowing through the system. For this reason, to describe *electron scattering* processes in full generality, we are interested in defining equivalence classes of these states only characterized by the cardinalities of these sets  $N$  and  $N_2$ . Furthermore, in order to be able to derive a closed analytical form for the rates induced by adding electronic reservoirs to the model, we extend these equivalence classes to all possible inequivalent representations of  $\text{su}(2)$ . The *equivalence classes* can be explicitly defined as

$$|A\rangle = |j, m; N, N_2, \gamma\rangle = \{|\alpha\rangle : N^\alpha = N, N_2^\alpha = N_2, j^\alpha = j, m^\alpha = m; \gamma^\alpha = \gamma\}. \quad (\text{S100})$$

In order to find the relation between microscopic and macroscopic rates, it is instructive to write the *microscopic* rate equation for electron scattering

$$\dot{\rho}_\alpha = - \sum_\beta \Gamma_{\text{el}}^{\alpha \rightarrow \beta} \rho_\alpha + \sum_\beta \Gamma_{\text{el}}^{\beta \rightarrow \alpha} \rho_\beta, \quad (\text{S101})$$

and the corresponding *macroscopic* version by summing over the equivalence classes previously defined

$$\dot{\rho}_A = \sum_{\alpha \in A} \dot{\rho}_\alpha = \sum_{\alpha \in A} \sum_{\beta \in B} \left( \Gamma_{\text{el}}^{\beta \rightarrow \alpha} \rho_\beta - \Gamma_{\text{el}}^{\alpha \rightarrow \beta} \rho_\alpha \right). \quad (\text{S102})$$

Note here that  $\rho_\alpha$  ( $\rho_A$ ) is the density matrix on the microscopic Hilbert space whose states are described in Eq. (S99) (Eq. (S100)).

To proceed further we need to make a critical assumption. We suppose that any coherence present in the system is averaged out in the macroscopic representation, as every electron scattering process needs not to be traced to the different microscopic processes. The density matrix of the system is then proportional to the identity within each set of states which only differ by the representation label  $r$ , i.e., we suppose that  $\rho_\alpha = \rho_{\tilde{\alpha}}/d_\alpha$ , where we denoted with  $\tilde{\alpha} = \{j, m, N, N_2, \gamma\}$  the remaining degrees of freedom. This allows to write

$$\dot{\rho}_A = - \sum_B \sum'_{r_\alpha, r_\beta} \sum_{r_\alpha, r_\beta} \Gamma_{\text{el}}^{\alpha \rightarrow \beta} \frac{\rho_{\tilde{\alpha}}}{d_\alpha} + \sum_B \sum'_{r_\alpha, r_\beta} \sum_{r_\alpha, r_\beta} \Gamma_{\text{el}}^{\beta \rightarrow \alpha} \frac{\rho_{\tilde{\beta}}}{d_\beta}, \quad (\text{S103})$$

where we used the short-hand notation

$$\sum' := \sum_{N^\alpha = N^A, N_2^\alpha = N_2^A} \sum_{N^\beta = N^B, N_2^\beta = N_2^B}. \quad (\text{S104})$$

## C. General transition rates in the fermionic model

In this subsection we finally evaluate the rates for transitions between the states  $\alpha$  and  $\beta$  of the system. By inserting Eq. (S4) in Eq. (S103) we see that this task requires to calculate objects of the form

$$\sum_{r_\alpha, r_\beta} \Gamma_{\text{el}, n}^{\alpha \rightarrow \beta} \propto Q_n, \quad (\text{S105})$$

where

$$Q_n = \sum_{r_\alpha, r_\beta} |\langle \alpha | \hat{O}_n | \beta \rangle|^2, \quad (\text{S106})$$



	$J = j_2 + \frac{1}{2}$	$J = j_2 - \frac{1}{2}$
$m_1 = \frac{1}{2}$	$\sqrt{\frac{J+M}{2J}}$	$-\sqrt{\frac{J-M+1}{2J+2}}$
$m_2 = -\frac{1}{2}$	$\sqrt{\frac{J-M}{2J}}$	$\sqrt{\frac{J+M+1}{2J+2}}$

TABLE I: Values of the Clebsch-Gordan coefficients  $C_{j_1, m_1; j_2, m_2}^{J, M}$  for  $j_1 = 1/2$ .

where  $\hat{O}_n = (c_{1,n} + c_{2,n})$  or its hermitian conjugate. To proceed, we consider how to define a basis within the degeneracies labelled by  $r$ . To this goal, we first write explicitly the following standard basis transformation for the Dicke states introduced in Eq. (S81)

$$|j_1, j_2; J, M\rangle = \sum_{m_1, m_2} C_{j_1, m_1; j_2, m_2}^{J, M} |j_1, m_1\rangle \otimes |j_2, m_2\rangle, \quad (\text{S107})$$

where  $m_i = -j_i, -j_i + 1, \dots, j_i - 1, j_i$ , for  $i = 1, 2$ , and where we introduced the Clebsch-Gordan coefficients  $C_{j_1, m_1; j_2, m_2}^{J, M} = (\langle j_1, m_1 | \otimes \langle j_2, m_2 | J, M \rangle)$ , reported also in Table I.

For clarity, we also write the inverse transformation which reads

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_J \sum_M C_{j_1, m_1; j_2, m_2}^{J, M} |j_1, j_2; J, M\rangle, \quad (\text{S108})$$

where  $J = |j_1 - j_2|, \dots, j_1 + j_2$  and  $M = -J, \dots, J$ .

In particular, we will only be using  $j_1 = 1/2$  in Eq. (S107) in which case the Clebsch-Gordan coefficients are explicitly given by and the condition  $C_{j_1, m_1; j_2, m_2}^{J, M} = 0$ , if  $m_1 + m_2 \neq M$ .

As mentioned in Sections V, VB, there can exist inequivalent representations of  $\text{su}(2)$  with the same quantum numbers  $j, m$  and  $N$ . The basis within each of these representations can be explicitly obtained by iteratively applying Eq. (S108). However, the Hamiltonian has the same action in all these inequivalent representations. We would then have a problem in defining which basis should be used in the Fermi Golden rule. The solution of this problem comes from our previous assumption of describing these degeneracies with a density matrix proportional to the identity. This led us to consider a sum over  $r_\alpha$  and  $r_\beta$  in Eq. (S103). Thanks to this assumption, Eq. (S106) is valid for any basis in this subspace. In particular, this allows us to fix a preferential order by which electrons populate the system in the calculations without any loss of generality. Plugging Eq. (S108) into Eq. (S99) allows to fully specify the degeneracy  $r_\alpha$  of a state with unchanged number of photons  $\gamma^\alpha$ ,

$$|\alpha\rangle = |j_1^\alpha, j_2^\alpha; J^\alpha, M^\alpha, r_\alpha; \mathbf{N}^\alpha, \mathbf{N}_2^\alpha, \gamma^\alpha\rangle \quad (\text{S109})$$

as a function of  $J^\alpha, j_2^\alpha$  and the label  $r_{\alpha,2}$  for the degeneracy of the representation with total angular momentum  $j_2^\alpha$  (we do not need consider the same for  $j_1^\alpha = 1/2$  since it is the fundamental representation for the additional electron). Otherwise stated, we have that  $r_\alpha = r(J^\alpha, j_2^\alpha, r_{\alpha,2})$ .

With this in mind, we can then consider the following identity

$$|X\rangle = (c_{1,n} + c_{2,n})|j_1, j_2; J, M, r_X; \mathbf{N}, \mathbf{N}_2, \gamma\rangle =$$

$$\bullet n \in \mathbf{N}$$

$$\bullet j_2 = J + \frac{1}{2}$$

$$\begin{aligned} |X\rangle &= C_{\frac{1}{2}, -\frac{1}{2}; J+\frac{1}{2}, M+\frac{1}{2}}^{JM} (c_{1,n} + c_{2,n}) \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \otimes |j_2, M + \frac{1}{2}, r_{X,2}; \mathbf{N} - n, \mathbf{N}_2, \gamma\rangle \\ &\quad + C_{\frac{1}{2}, \frac{1}{2}; J+\frac{1}{2}, M-\frac{1}{2}}^{JM} (c_{1,n} + c_{2,n}) \left| \frac{1}{2}, \frac{1}{2} \right\rangle \otimes |j_2, M - \frac{1}{2}, r_{X,2}; \mathbf{N} - n, \mathbf{N}_2, \gamma\rangle \\ &= C_{\frac{1}{2}, -\frac{1}{2}; J+\frac{1}{2}, M+\frac{1}{2}}^{JM} |j_2, M + \frac{1}{2}, r_{X,2}; \mathbf{N} - n, \mathbf{N}_2, \gamma\rangle \\ &\quad + C_{\frac{1}{2}, \frac{1}{2}; J+\frac{1}{2}, M-\frac{1}{2}}^{JM} |j_2, M - \frac{1}{2}, r_{X,2}; \mathbf{N} - n, \mathbf{N}_2, \gamma\rangle \end{aligned}$$

$$\bullet j_2 = J - \frac{1}{2}$$

$$\begin{aligned} |X\rangle &= C_{\frac{1}{2}, -\frac{1}{2}; J-\frac{1}{2}, M+\frac{1}{2}}^{JM} (c_{1,n} + c_{2,n}) \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \otimes |j_2, M + \frac{1}{2}, r_{X,2}; \mathbf{N} - n, \mathbf{N}_2, \gamma\rangle \\ &\quad + C_{\frac{1}{2}, \frac{1}{2}; J-\frac{1}{2}, M-\frac{1}{2}}^{JM} (c_{1,n} + c_{2,n}) \left| \frac{1}{2}, \frac{1}{2} \right\rangle \otimes |j_2, M - \frac{1}{2}, r_{X,2}; \mathbf{N} - n, \mathbf{N}_2, \gamma\rangle \\ &= C_{\frac{1}{2}, -\frac{1}{2}; J-\frac{1}{2}, M+\frac{1}{2}}^{JM} |j_2, M + \frac{1}{2}, r_{X,2}; \mathbf{N} - n, \mathbf{N}_2, \gamma\rangle \\ &\quad + C_{\frac{1}{2}, \frac{1}{2}; J-\frac{1}{2}, M-\frac{1}{2}}^{JM} |j_2, M - \frac{1}{2}, r_{X,2}; \mathbf{N} - n, \mathbf{N}_2, \gamma\rangle \end{aligned}$$

$$\bullet n \in \mathbf{N}_2$$

$$\begin{aligned} |X\rangle &= \left| \frac{1}{2}, \frac{1}{2} \right\rangle \otimes |J, M, r_X; \mathbf{N}, \mathbf{N}_2 - n, \gamma\rangle + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \otimes |J, M, r_X; \mathbf{N}, \mathbf{N}_2 - n, \gamma\rangle \\ &= C_{\frac{1}{2}, \frac{1}{2}; J, M}^{J-\frac{1}{2}, M+\frac{1}{2}} \left| \frac{1}{2}, J; J - \frac{1}{2}, M + \frac{1}{2}, r(J - \frac{1}{2}, J, r_X); \mathbf{N} + n, \mathbf{N}_2 - n, \gamma \right\rangle \\ &\quad + C_{\frac{1}{2}, \frac{1}{2}; J, M}^{J+\frac{1}{2}, M+\frac{1}{2}} \left| \frac{1}{2}, J; J + \frac{1}{2}, M + \frac{1}{2}, r(J + \frac{1}{2}, J, r_X); \mathbf{N} + n, \mathbf{N}_2 - n, \gamma \right\rangle \\ &\quad + C_{\frac{1}{2}, -\frac{1}{2}; J, M}^{J-\frac{1}{2}, M-\frac{1}{2}} \left| \frac{1}{2}, J; J - \frac{1}{2}, M - \frac{1}{2}, r(J - \frac{1}{2}, J, r_X); \mathbf{N} + n, \mathbf{N}_2 - n, \gamma \right\rangle \\ &\quad + C_{\frac{1}{2}, -\frac{1}{2}; J, M}^{J+\frac{1}{2}, M-\frac{1}{2}} \left| \frac{1}{2}, J; J + \frac{1}{2}, M - \frac{1}{2}, r(J + \frac{1}{2}, J, r_X); \mathbf{N} + n, \mathbf{N}_2 - n, \gamma \right\rangle \end{aligned}$$

$$\bullet n \notin \mathbf{N}, \mathbf{N}_2$$

$$|X\rangle = 0,$$

(S110)

where we highlighted in blue (red) the terms arising from the scattering of a lower (upper) fermion. As it can be seen above, in Eq. (S110) there are four possible results of the action of the fermionic destruction operators at position  $n$ , depending on the state of the system at site  $n$ . If the site  $n$  is occupied by a single electron,  $n \in \mathbf{N}$ , then the population of the electrons effectively coupled to light diminishes by one. The ket becomes the state  $|j_2, M \pm \frac{1}{2}, r_{X,2}; \mathbf{N} - n, \mathbf{N}_2, \gamma\rangle$  by the action of  $c_{1,n}$  ( $c_{2,n}$ ), with a different Clebsch-Gordan coefficient depending on the value of  $j_2$ . If the site  $n$  is occupied by two electrons,  $n \in \mathbf{N}_2$ , then the population of the electrons effectively coupled to light *increases* by one. The ket then becomes of the form of Eq. (S109) and the averaging of the processes involved in the macroscopic state scattering eventually leads to a ket of the form of Eq. (S99). If the site  $n$  is not occupied by any electron, no scattering occurs.

We thus can appreciate how Eq. (S110) is fundamental for the use of Dicke state formalism in the open dynamics considered here, bridging a connection with the second-quantization formalism and which can be used also for the study of other processes. Note that Ref. [4] contains related independent work in first quantization, in the context of superradiant lasing.

By inserting these results in Eq. (S106), we find

$$\begin{aligned}
Q_n &= \sum_{r_\alpha, r_\beta} |\langle \beta | (c_{1,n} + c_{2,n}) | \alpha \rangle|^2 \\
&= \sum_{r_\alpha, r_\beta} \left| \left[ \delta_{\mathbf{N}^\beta, \mathbf{N}^\alpha - n} \delta_{\mathbf{N}_2^\beta, \mathbf{N}_2^\alpha} \delta_{r_\beta, r_\alpha, 2} \delta_{J^\beta, J^\alpha + \frac{1}{2}} \left( \delta_{m_\beta, m_\alpha + \frac{1}{2}} C_{J^\alpha + \frac{1}{2}, m_\alpha + \frac{1}{2}}^{J^\alpha m_\alpha} + \delta_{m_\beta, m_\alpha - \frac{1}{2}} D_{J^\alpha + \frac{1}{2}, m_\alpha - \frac{1}{2}}^{J^\alpha m_\alpha} \right) \right. \right. \\
&\quad + \delta_{\mathbf{N}^\beta, \mathbf{N}^\alpha - n} \delta_{\mathbf{N}_2^\beta, \mathbf{N}_2^\alpha} \delta_{J^\beta, J^\alpha - \frac{1}{2}} \delta_{r_\beta, r_\alpha, 2} \left( \delta_{m_\beta, m_\alpha + \frac{1}{2}} C_{J^\alpha - \frac{1}{2}, m_\alpha + \frac{1}{2}}^{J^\alpha m_\alpha} + \delta_{m_\beta, m_\alpha - \frac{1}{2}} D_{J^\alpha - \frac{1}{2}, m_\alpha - \frac{1}{2}}^{J^\alpha m_\alpha} \right) \\
&\quad + \delta_{\mathbf{N}^\beta, \mathbf{N}^\alpha + n} \delta_{\mathbf{N}_2^\beta, \mathbf{N}_2^\alpha - n} \delta_{J^\beta, J^\alpha - \frac{1}{2}} \delta_{r_\beta, r_\alpha} \left( \delta_{m_\beta, m_\alpha + \frac{1}{2}} D_{J^\beta, m_\alpha}^{J^\alpha - \frac{1}{2} m_\alpha + \frac{1}{2}} + \delta_{m_\beta, m_\alpha - \frac{1}{2}} C_{J^\alpha, m_\alpha}^{J^\alpha - \frac{1}{2} m_\alpha - \frac{1}{2}} \right) \\
&\quad \left. + \delta_{\mathbf{N}^\beta, \mathbf{N}^\alpha + n} \delta_{\mathbf{N}_2^\beta, \mathbf{N}_2^\alpha - n} \delta_{J^\beta, J^\alpha + \frac{1}{2}} \delta_{r_\beta, r_\alpha} \left( \delta_{m_\beta, m_\alpha + \frac{1}{2}} D_{J^\alpha, m_\alpha}^{J^\alpha + \frac{1}{2} m_\alpha + \frac{1}{2}} + \delta_{m_\beta, m_\alpha - \frac{1}{2}} C_{J^\alpha, m_\alpha}^{J^\alpha + \frac{1}{2} m_\alpha - \frac{1}{2}} \right) \right] \right|^2, \tag{S112}
\end{aligned}$$

which, in turn, allows to perform the two sums over the representation indexes to obtain

$$\begin{aligned}
Q_n &= d_\alpha \left| \left[ \delta_{\mathbf{N}^\beta, \mathbf{N}^\alpha - n} \delta_{\mathbf{N}_2^\beta, \mathbf{N}_2^\alpha} \delta_{J^\beta, J^\alpha + \frac{1}{2}} \left( \delta_{m_\beta, m_\alpha + \frac{1}{2}} C_{J^\alpha + \frac{1}{2}, m_\alpha + \frac{1}{2}}^{J^\alpha m_\alpha} + \delta_{m_\beta, m_\alpha - \frac{1}{2}} D_{J^\alpha + \frac{1}{2}, m_\alpha - \frac{1}{2}}^{J^\alpha m_\alpha} \right) \right. \right. \\
&\quad + \delta_{\mathbf{N}^\beta, \mathbf{N}^\alpha - n} \delta_{\mathbf{N}_2^\beta, \mathbf{N}_2^\alpha} \delta_{J^\beta, J^\alpha - \frac{1}{2}} \left( \delta_{m_\beta, m_\alpha + \frac{1}{2}} C_{J^\alpha - \frac{1}{2}, m_\alpha + \frac{1}{2}}^{J^\alpha m_\alpha} + \delta_{m_\beta, m_\alpha - \frac{1}{2}} D_{J^\alpha - \frac{1}{2}, m_\alpha - \frac{1}{2}}^{J^\alpha m_\alpha} \right) \\
&\quad + \delta_{\mathbf{N}^\beta, \mathbf{N}^\alpha + n} \delta_{\mathbf{N}_2^\beta, \mathbf{N}_2^\alpha - n} \delta_{J^\beta, J^\alpha - \frac{1}{2}} \left( \delta_{m_\beta, m_\alpha + \frac{1}{2}} D_{J^\beta, m_\alpha}^{J^\alpha - \frac{1}{2} m_\alpha + \frac{1}{2}} + \delta_{m_\beta, m_\alpha - \frac{1}{2}} C_{J^\alpha, m_\alpha}^{J^\alpha - \frac{1}{2} m_\alpha - \frac{1}{2}} \right) \\
&\quad \left. + \delta_{\mathbf{N}^\beta, \mathbf{N}^\alpha + n} \delta_{\mathbf{N}_2^\beta, \mathbf{N}_2^\alpha - n} \delta_{J^\beta, J^\alpha + \frac{1}{2}} \left( \delta_{m_\beta, m_\alpha + \frac{1}{2}} D_{J^\alpha, m_\alpha}^{J^\alpha + \frac{1}{2} m_\alpha + \frac{1}{2}} + \delta_{m_\beta, m_\alpha - \frac{1}{2}} C_{J^\alpha, m_\alpha}^{J^\alpha + \frac{1}{2} m_\alpha - \frac{1}{2}} \right) \right] \right|^2. \tag{S113}
\end{aligned}$$

The coefficients  $C$  and  $D$  appearing in Eqs. (S112),(S113) are the Clebsch-Gordan coefficients defined as

$$\begin{aligned}
C_{j,m}^{J,M} &= \left( \delta_{J,j+\frac{1}{2}} \sqrt{\frac{J-M}{2J}} + \delta_{J,j-\frac{1}{2}} \sqrt{\frac{J+M+1}{2J+2}} \right) \delta_{M,m-\frac{1}{2}}, \\
D_{j,m}^{J,M} &= \left( \delta_{J,j+\frac{1}{2}} \sqrt{\frac{J+M}{2J}} - \delta_{J,j-\frac{1}{2}} \sqrt{\frac{J-M+1}{2J+2}} \right) \delta_{M,m+\frac{1}{2}}, \tag{S114}
\end{aligned}$$

for which the different coefficients connecting different pseudo-spin states are summarized in Table I.

Having performed the sum over the representation index allows to write Eq. (S103) as

$$\begin{aligned}
\dot{\rho}_A &= - \sum_B \sum_{N^\alpha=N^A, N_2^\alpha=N_2^A} \sum_n \sum_{N^\beta=N^B, N_2^\beta=N_2^B} \sum_i \lambda_{\alpha\beta}^i Q_n \frac{\rho_{\mathbf{N}^\alpha, \mathbf{N}_2^\alpha; j_\alpha, m_\alpha, \gamma_\alpha}}{d_\alpha} \\
&\quad + \sum_B \sum_{N^\alpha=N^A, N_2^\alpha=N_2^A} \sum_n \sum_{N^\beta=N^B, N_2^\beta=N_2^B} \sum_i \lambda_{\alpha\beta}^i Q_n \frac{\rho_{\mathbf{N}^\beta, \mathbf{N}_2^\beta; j_\beta, m_\beta, \gamma_\beta}}{d_\beta}, \tag{S115}
\end{aligned}$$

where  $i = \text{in, out}$ , following the definition inspired by Eq. (S6),

$$\lambda_n^{\text{out}} = 2\pi\lambda^2(1 - \bar{n}(-\Delta_{\alpha\beta}))\nu(-\Delta_{\alpha\beta}) \tag{S116}$$

and

$$\lambda_n^{\text{in}} = 2\pi\lambda^2(\bar{n}(\Delta_{\alpha\beta}))\nu(\Delta_{\alpha\beta}). \tag{S117}$$

We also arranged the sums in the order we will perform them. We now give the arguments to simplify the calculations on the first line of the above formula. Analogous considerations can be applied to the second line.

For a fixed  $\mathbf{N}^\alpha$  and  $\mathbf{N}_2^\alpha$ , and for each  $n$  the sum over  $\mathbf{N}^\beta$  and  $\mathbf{N}_2^\beta$  can be performed using the deltas in the expression for  $Q_n$ . The net effect of this operation is to limit the range of the sum over  $n$  to those electrons which can be added or removed to turn the sets  $\mathbf{N}^\alpha$  into  $\mathbf{N}^\beta$  or  $\mathbf{N}_2^\alpha$  into  $\mathbf{N}_2^\beta$ . Since nothing depends explicitly on such a specific  $n$ , the sum over  $n$  will return the number of microscopic ways  $\kappa_{A \rightarrow B}$  to go from a specific set characterized by  $N^A, N_2^A$  to one characterized by  $N^B$  and  $N_2^B$ . The last sum over  $\{N^\beta\}$  and  $\{N_2^\beta\}$  can now be used to sum over the probability density

$$\sum_{N^\alpha=N^A, N_2^\alpha=N_2^A} \rho_{\mathbf{N}^\alpha, \mathbf{N}_2^\alpha; j^\alpha, m^\alpha} \tag{S118}$$

to give  $\rho_A$  to obtain

$$\dot{\rho}_A = - \sum_B \Gamma_{L/R}^{A \rightarrow B} \rho_A + \sum_B \Gamma_{L/R}^{B \rightarrow A} \rho_B, \quad (\text{S119})$$

where

$$\begin{aligned} \Gamma_{L/R, \text{in/out}}^{A \rightarrow B} &= \theta_{L/R, \text{in/out}}^{\text{AB}} \kappa_{A \rightarrow B} \left| \sum_{\bar{m}, m, \gamma} \left[ \delta_{N^B, N^A - 1} \delta_{N_2^B, N_2^A}^{\text{in/out}} \delta_{J^B, J^A + \frac{1}{2}} (U_{\bar{m}\gamma}^B)^* U_{m\gamma}^A \right. \right. \\ &\times \left( \delta_{\bar{m}, m + \frac{1}{2}} C_{J^A + \frac{1}{2}, m + \frac{1}{2}}^{J^A m} + \delta_{\bar{m}, m - \frac{1}{2}} D_{J^A + \frac{1}{2}, m - \frac{1}{2}}^{J^A m} \right) \\ &+ \delta_{N^B, N^A - 1} \delta_{N_2^B, N_2^A}^{\text{in/out}} \delta_{J^B, J^A - \frac{1}{2}} (U_{\bar{m}\gamma}^B)^* U_{m\gamma}^A \left( \delta_{\bar{m}, m + \frac{1}{2}} C_{J^A - \frac{1}{2}, m + \frac{1}{2}}^{J^A m} + \delta_{\bar{m}, m - \frac{1}{2}} D_{J^A - \frac{1}{2}, m - \frac{1}{2}}^{J^A m} \right) \\ &+ \delta_{N^B, N^A + 1} \delta_{N_2^B, N_2^A - 1}^{\text{in/out}} \delta_{J^B, J^A - \frac{1}{2}} (U_{\bar{m}\gamma}^B)^* U_{m\gamma}^A \left( \delta_{\bar{m}, m + \frac{1}{2}} D_{J^B, m}^{J^A - \frac{1}{2} m + \frac{1}{2}} + \delta_{\bar{m}, m - \frac{1}{2}} C_{J^A, m}^{J^A - \frac{1}{2} m - \frac{1}{2}} \right) \\ &\left. \left. + \delta_{N^B, N^A + 1} \delta_{N_2^B, N_2^A - 1}^{\text{in/out}} \delta_{J^B, J^A + \frac{1}{2}} (U_{\bar{m}\gamma}^B)^* U_{m\gamma}^A \left( \delta_{\bar{m}, m + \frac{1}{2}} D_{J^A, m}^{J^A + \frac{1}{2} m + \frac{1}{2}} + \delta_{\bar{m}, m - \frac{1}{2}} C_{J^A, m}^{J^A + \frac{1}{2} m - \frac{1}{2}} \right) \right] \right|^2, \quad (\text{S120}) \end{aligned}$$

where

$$\theta_{L/R, \text{out}}^{\text{AB}} = \Gamma_{\text{el}} [1 + \theta(\mu_{L/R} + \Delta_{AB})], \quad (\text{S121})$$

$$\theta_{L/R, \text{in}}^{\text{AB}} = \Gamma \theta(\mu_{L/R} - \Delta_{AB}), \quad (\text{S122})$$

$\delta_{a,b}^{\text{out}} = \delta_{a,b}$ , and  $\delta_{a,b}^{\text{in}} = \delta_{a,b+1}$  and where

$$\begin{aligned} \kappa_{(N, N_2) \rightarrow (N+1, N_2)} &= N_T - N - N_2, \\ \kappa_{(N, N_2) \rightarrow (N-1, N_2)} &= N, \\ \kappa_{(N, N_2) \rightarrow (N-1, N_2+1)} &= N, \\ \kappa_{(N, N_2) \rightarrow (N+1, N_2-1)} &= N_2, \end{aligned} \quad (\text{S123})$$

where  $N_T$  is the number of total electron sites.

While Eq. (S120) contains all the information needed to compute the transition rates, it is worth to proceed a little further to obtain an expression which more clearly highlights its physical content. To achieve this, we decompose the coefficients  $U_{m\gamma}^B$  as

$$U_{m\gamma}^B = \sum_n u_\gamma^B(n) \delta_{m, -j^B + n - \gamma}, \quad (\text{S124})$$

where the coefficients  $u_\gamma^B(n)$  account for the coherences of the state  $B$  within the subspace with  $n$  bare excitations. For example, at first order in the perturbation potential  $V$ , we have

$$\begin{aligned} u_\gamma^B(n^B) &= u_\gamma^{(0)B} \\ u_\gamma^B(n^B \pm 2) &= - \sum_B c_\pm^{B\bar{B}} u_\gamma^{(0)\bar{B}}, \end{aligned} \quad (\text{S125})$$

while all other coefficients are zero. Note that the range of  $\gamma$  is constrained inside each  $u_\gamma^{(0)B}$  so that, for each  $n$ , we have  $0 \leq \gamma \leq n$ . However, both Eq. (S120) and Eq. (S125) are general and can, in principle, be applied to higher order cases. Using this notation in Eq. (S120) leads to quantities of the form

$$\bar{Q} = \sum_{m, \bar{m}, \gamma} (U_{\bar{m}, m}^B)^* U_{m, \gamma}^A \delta_{\bar{m}, m+a} \delta_{j^B j^A + b} F(m), \quad (\text{S126})$$

where  $a, b \in \mathbb{R}$  and  $F$  a generic function. We have

$$\begin{aligned} \bar{Q} &= \sum_{n, \bar{n}} \sum_{m, \gamma} \bar{u}_\gamma^B(\bar{n}) u_\gamma^A(n) F(m) \delta_{m+a, -j^B + \bar{n} - \gamma} \delta_{m, -j^A + n - \gamma} \delta_{j^B j^A + b} \\ &= \sum_{n, \gamma} \bar{u}_\gamma^B(n+a+b) u_\gamma^A(n) F(-j^A + n - \gamma) \delta_{j^B j^A + b} = \delta_{j^B j^A + b} \langle A, B \rangle_F^{a+b}, \end{aligned} \quad (\text{S127})$$

where we defined the following pseudo-inner product

$$\langle A, B \rangle_F^x = \sum_n \sum_\gamma \bar{u}_\gamma^B(n+x) u_\gamma^A(n) F(-j^A + n - \gamma), \quad (\text{S128})$$

which quantifies an effective overlap between coherent components of  $A$  and  $B$  which belongs to subspaces having total number of bare excitations which differ by  $x$ . With this notation at hand, we can rewrite Eq. (S120) as

$$\begin{aligned} \Gamma_{L/R, \text{in/out}}^{A \rightarrow B} = \theta_{L/R, \text{in/out}}^{AB} \kappa_{A \rightarrow B} & \left[ \delta_{\Delta N, -1} \delta_{N_2^B, N_2^A}^{\text{in/out}} \delta_{\Delta J, -1} \left( \langle A, B \rangle_{C_{AB}^\downarrow}^1 + \langle A, B \rangle_{D_{AB}^\downarrow}^0 \right) \right. \\ & + \delta_{\Delta N, -1} \delta_{N_2^B, N_2^A}^{\text{in/out}} \delta_{\Delta J, 0} \left( \langle A, B \rangle_{C_{AB}^\downarrow}^0 + \langle A, B \rangle_{D_{AB}^\downarrow}^{-1} \right) + \delta_{\Delta N, 1} \delta_{N_2^B, N_2^A}^{\text{in/out}} \delta_{\Delta J, 0} \left( \langle A, B \rangle_{D_{BA}^\uparrow}^1 + \langle A, B \rangle_{C_{BA}^\uparrow}^0 \right) \\ & \left. + \delta_{\Delta N, 1} \delta_{N_2^B, N_2^A}^{\text{in/out}} \delta_{\Delta J, 1} \left( \langle A, B \rangle_{D_{BA}^\uparrow}^0 + \langle A, B \rangle_{C_{BA}^\uparrow}^{-1} \right) \right]^2, \quad (\text{S129}) \end{aligned}$$

where  $C_{AB}^\downarrow(m) = C_{j^B, m+\frac{1}{2}}^{j^A, m}$ ,  $C_{AB}^\uparrow(m) = C_{j^B, m}^{j^A, m-\frac{1}{2}}$ ,  $D_{AB}^\downarrow(m) = D_{j^B, m-\frac{1}{2}}^{j^A, m}$ ,  $D_{AB}^\uparrow(m) = D_{j^B, m}^{j^A, m+\frac{1}{2}}$ . We also defined  $\Delta N = N_B - N_A$  and changed the notation to highlight the quantity

$$\Delta_J^N = (j_{N^B} - j^B) - (j_{N^A} - j^A) = (N^B - N^A)/2 - (j^B - j^A), \quad (\text{S130})$$

which identifies changes in the symmetry of the state. For example,  $\Delta_J^N < 0$  indicates a transition towards a state more symmetric than the original one.

This notation highlights some of the physical content of these expressions. For example, transitions between states with equal (different) parity take contributions only from factors proportional to  $\langle A, B \rangle^n$  with an even (odd)  $n$ .

#### D. Emission rate for ground state electroluminescence

In this subsection we analytically estimate the population for the lowest energy states as current passes through the system. Each of the results will be given at lowest non-trivial order in the normalized light-matter coupling  $\eta = g_{\bar{N}}/\omega_0$  and the light-matter coupling  $\eta/\omega_0$ . To simplify the analysis we will further assume that  $N_2 = 0$ , which allow a closed analytical treatment. Hereafter, the notation  $B'$  ( $B''$ ) will indicate one-polariton (two-polariton) eigenstates.

Essential in the discussion, the expression for the ground state with  $N$  electrons is identified by

$$u_\gamma^{G^N}(n) = \delta_{\gamma, 0} \delta_{n, 0} - \delta_{n, 2} \sum_{\bar{B}''} \frac{g_N}{\Delta E_{B''}} \bar{u}_1^{(0)B''} u_\gamma^{(0)\bar{B}''}, \quad (\text{S131})$$

This formula shows explicitly that the ground state is a coherent superposition of states with different number of bare excitations. These are the coherences necessary for ground state electroluminescence.

The virtual polaritonic population present in the ground state highlighted in Eq. (S131) allows for new non-zero out-rates. As a consequence, in the case  $\Gamma_{\text{cav}} \gg \Gamma_{\text{el}}$ , we will estimate the total emission rate from the decay of the polariton  $B = B', B''$  using the shorthand notation

$$\begin{aligned} \langle G, B' \rangle &= \langle G_N, B'_{N-1} \rangle_{D_{G_N B'_{N-1}}^\downarrow}^{-1}, \\ \langle G, B'' \rangle &= \langle G_N, B''_{N-1} \rangle_{C_{G_N B''_{N-1}}^\downarrow}^0, \end{aligned} \quad (\text{S132})$$

where as elsewhere,  $B'$  and  $B''$  indicate the one-polariton and two-polariton eigenstates, respectively. For clarity, we stop a moment to anticipate some future results. The behaviour of the pseudo-inner products in the previous expression are, at lowest non-trivial order, given by

$$\langle G, B' \rangle \propto D_{G_N, B_{N-1}}^\downarrow = O\left(\frac{1}{N}\right) \quad (\text{S133})$$

and

$$\langle G, B'' \rangle \propto \partial_N g_N C_{G_N, B''_{N-1}}^\downarrow = O\left(\frac{1}{N^2}\right). \quad (\text{S134})$$

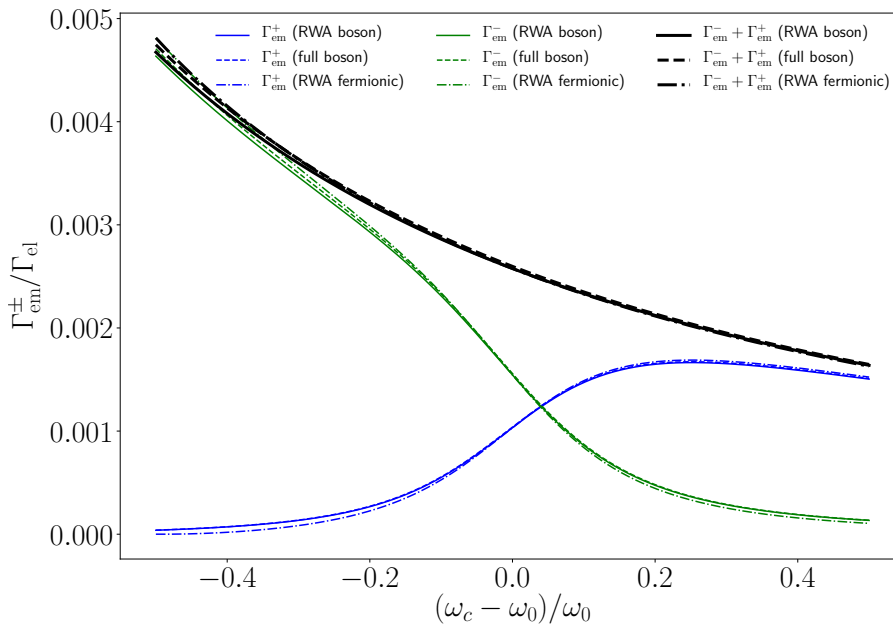


FIG. S5: **Single-polariton emission rates in the fermionic and bosonic models.** Emission rates  $\Gamma_{\text{em}}^{B'}$  in units of the emission rate  $\Gamma_{\text{el}}$  for  $B' = +$  (monotonically increasing blue curves) and  $B' = -$  (monotonically decreasing green curves), for the upper and lower polariton, respectively, and the sum of the two contributions (black thick curves) as a function of the normalized detuning  $(\omega_c - \omega_0)/\omega_0$  for  $g_N/\omega_0 = 0.1$ , with  $\Delta = \omega_0 - \omega_c$ . The solid curves correspond to the perturbative bosonic model, in terms of the Jaynes-Cummings polaritons, used in the main text. The dashed curves correspond to the full bosonic model, where the Hamiltonian is directly diagonalized. The dot-dashed curves represent the rates obtained from the fermionic model, with a perturbative calculation of the eigenstates in terms of the fermionic RWA Hamiltonian. The bosonic approximations fit very well the fermionic model to an extent such that the two curves are barely distinguishable in the scale used for the plot, up to large detuning.

More explicitly, the pseudo-inner products can be calculated as

$$\langle G_N, B'_{N-1} \rangle = \sum_{n,\gamma} \bar{u}_\gamma^{B'_{N-1}} (n-1) u_\gamma^{G_N} (n) D_{j^{G_N-\frac{1}{2}}, m_\gamma^{G_N-\frac{1}{2}}}^{j^{G_N}, m_\gamma^{G_N}} = \sum_{\gamma=0}^1 \bar{u}_\gamma^{B'_{N-1}} (1) u_\gamma^{G_N} (2) \sqrt{\frac{2-\gamma}{N}}. \quad (\text{S135})$$

where

$$m_{\gamma,n}^B = -j^B + n - \gamma. \quad (\text{S136})$$

Interestingly, the suppression  $O(1/N)$  in the pseudo-inner product between states with different parity has a statistical origin encapsulated in the Clebsch-Gordan coefficients. We can then obtain, assuming the chemical potential to be sufficiently low

$$\Gamma^{G_N \rightarrow B'_{N-1}} = \Gamma_{\text{el}} \left[ g_N \frac{\omega_0 (\omega_c \cos \theta_{B'} + g_N \sin \theta_{B'})}{(\omega_0 + \omega_c)(\omega_0 \omega_c - g_N^2)} \right]^2, \quad (\text{S137})$$

with  $\tan \theta_+ = (-\Delta + \sqrt{4g_N^2 + \Delta^2})/(2g_N)$ , given by Eq. (S98), and  $\theta_+ = \theta_- - \pi/2$  with  $\Delta = \omega_0 - \omega_c$ , and  $g = g_N = \chi\sqrt{N}$ . This expression for the single-polariton rates is plotted as a function of the cavity-matter detuning, in Figure S5 valid in the fermionic case (dot-dashed curves), where it is compared with the corresponding estimates obtained in the full bosonic model (dashed curves) and perturbative bosonic model (solid curves), showing an excellent quantitative agreement between the three models up to large detuning,  $|\omega_c - \omega_0| = 0.5\omega_0$ . The values for the plot is  $g = g_N = 0.1\omega_0$ , with  $I \equiv \Gamma_{\text{el}}$ .

### E. Model comparisons and limitations

In order to ascertain the validity of the perturbative bosonic approximation, we derived the full bosonic Hamiltonian (non-perturbative in light-matter coupling) and we developed a fully fermionic theory, which allows to retain the results

for the double-polariton states. To test the quantitative predictions gained with such simple and intuitive bosonic model, we have also developed a full second-quantization fermionic theory which clearly shows that the results hold even if this more refined approach is employed. Both approaches are in good qualitative and quantitative agreement with the perturbative bosonic theory.

The fermionic transport analysis performed here is not limited to solid-state semiconductor devices, but it can be mapped to an open Dicke model with scattering. However, in that setting, which has so far been realized experimentally in lattices of trapped atoms, it could be challenging to engineer and modulate a process equivalent to the electron current we describe herein.

For simplicity, we have here assumed that the electron and photonic reservoirs were unstructured. It is an open question how ground-state electroluminescence could be affected by structured baths. Our calculations show that the effect is visible already in a dilute-current regime, yet at higher current flows, more complex effects could arise. The effect of electron-electron scattering, a second-order process in the dilute regime could also be addressed. The effect of a multi-mode model for the cavity might give rise to nonlinear quantum phenomena.

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