Second-Order Topological Phases in Non-Hermitian Systems

Tao Liu,1,* Yu-Ran Zhang,1,2 Qing Ai,1,3 Zongping Gong,4,1 Kohei Kawabata,4,‡ Masahito Ueda,4,5,∥ and Franco Nori1,6,†

1Theoretical Quantum Physics Laboratory, RIKEN Cluster for Pioneering Research, Wako-shi, Saitama 351-0198, Japan
2Department of Physics, Applied Optics Beijing Area Major Laboratory, Beijing Normal University, Beijing 100875, China
3Department of Physics, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
4RIKEN Center for Emergent Matter Science (CEMS), Wako, Saitama 351-0198, Japan
5Department of Physics, University of Michigan, Ann Arbor, Michigan 48109-1040, USA

(Received 9 October 2018; published 20 February 2019)

A $d$-dimensional second-order topological insulator (SOTI) can host topologically protected $(d-2)$-dimensional gapless boundary modes. Here, we show that a 2D non-Hermitian SOTI can host zero-energy modes at its corners. In contrast to the Hermitian case, these zero-energy modes can be localized only at one corner. A 3D non-Hermitian SOTI is shown to support second-order boundary modes, which are localized not along hinges but anomalously at a corner. The usual bulk-corner (hinge) correspondence in the second-order 2D (3D) non-Hermitian system breaks down. The winding number (Chern number) based on complex wave vectors is used to characterize the second-order topological phases in 2D (3D). A possible experimental situation with ultracold atoms is also discussed. Our work lays the cornerstone for exploring higher-order topological phenomena in non-Hermitian systems.

DOI: 10.1103/PhysRevLett.122.076801

Introduction.—Recent years have witnessed a surge of theoretical and experimental interest in studying topological phases [1–3] in insulators [4–9], superconductors [10–12], ultracold atoms [13–18], and classical waves [19–22]. These topologically nontrivial phases are characterized by the topological index of gapped bulk-energy bands and exhibit gapless states on their boundaries. Such gapless boundary states cannot be gapped out by local perturbations that preserve both bulk gap and symmetry.

Topological phases have widely been studied in closed systems, which are described by Hermitian Hamiltonians featuring real eigenenergies and orthogonal eigenstates. Recently, there has been a great deal of effort in exploring topological invariants of open systems governed by non-Hermitian operators [23,24]. Non-Hermitian Hamiltonians can find applications in a wide range of systems including optical and mechanical structures subjected to gain and loss [25–40], and solid-state systems with finite quasiparticle lifetimes [41–45]. In particular, topological phases of non-Hermitian Hamiltonians have recently been investigated in these systems [43–70]. The most prominent feature of non-Hermitian Hamiltonians is the existence of exceptional points (EPs), where more than one eigenstate coalesces [24,71,72]. This coalescence of eigenstates at EPs makes the corresponding eigenspace no longer complete, and the non-Hermitian Hamiltonian becomes nondiagonalizable. These unique features of EPs can lead to rich topological features in non-Hermitian topological systems with no counterpart in Hermitian cases such as Weyl exceptional rings [51], bulk Fermi arcs, and half-integer topological charges [57]. Furthermore, the interplay between non-Hermiticity and topology can lead to the breakdown of the usual bulk-boundary correspondence [50,52,58,63,65–67] due to the non-Bloch-wave behavior of open-boundary eigenstates, where the conventional Bloch wave functions do not precisely describe topological-phase transitions under the open-boundary conditions. The non-Bloch winding (Chern) number defined via complex wave vectors in 1D (2D) has recently been introduced to fill this gap [65,66].

More recently, the concept of topological insulators (TIs) has been generalized to second-order [73–91] and third-order [74,92,93] TIs in Hermitian systems. In contrast to conventional first-order TIs, a $d$-dimensional second-order topological insulator (SOTI) only hosts topologically protected $(d-2)$-dimensional gapless boundary states. For example, a 2D SOTI has zero-energy states localized at its corners, and a 3D SOTI hosts 1D gapless modes along its hinges. Therefore, the conventional bulk-boundary correspondence is no longer applicable to SOTIs. Up to now, studies of the second-order and third-order topological phases have been restricted to Hermitian systems. We now ask: is it possible for a non-Hermitian system to exhibit second-order topological phases? If yes, how can we define a topological invariant to characterize them?

In this Letter, we investigate 2D and 3D SOTIs described by non-Hermitian Hamiltonians. Even though the bulk bands are first-order topologically trivial insulators, there are degenerate second-order bound states. In contrast to the Hermitian case, these zero-energy states in 2D are localized only at one corner protected by mirror-rotation symmetry.
and sublattice symmetry. Moreover, the second-order boundary modes in 3D are localized not along the hinges but anomalously at a corner. The winding number (Chern number) characterizes its second-order topological phase in 2D (3D), where the non-Bloch-wave behavior of open-boundary eigenstates is included due to the breakdown of the usual bulk-corner (hinge) correspondence in second-order non-Hermitian systems. The proposed non-Hermitian model can experimentally be realized in ultracold atoms.

2D SOTI.—We consider a 2D non-Hermitian Hamiltonian $H_{2D}$ that respects both twofold mirror-rotation symmetry $\mathcal{M}_{xy}$ and sublattice symmetry $S$

$$\mathcal{M}_{xy} H_{2D}(k_x, k_y) \mathcal{M}_{xy}^{-1} = H_{2D}(k_x, k_y),$$  

and $[S, \mathcal{M}_{xy}] = 0$. Note that the Hermitian counterpart with the same symmetries was investigated in Ref. [81]. Because of the mirror-rotation symmetry in Eq. (1), we can express the Hamiltonian $H_{2D}$ on the high-symmetry line $k_x = k_y$ as

$$U^{-1} H_{2D}(k, k) U = \begin{pmatrix} H_+(k) & 0 \\ 0 & H_-(k) \end{pmatrix},$$

where $U$ is a unitary operator, and $H_\pm(k)$ acts on the mirror-rotation subspace. Since $H_{\pm}(k)$ respects sublattice symmetry $S'$ defined in each mirror-rotation subspace [note that $S$ in Eq. (2) is defined in the entire lattice space], we can define the winding number as follows:

$$w_\pm = \int_{\text{BZ}} \frac{dk}{4\pi i} \text{Tr} \left[ S' H_{\pm}^{-1}(k) \frac{dH_\pm(k)}{dk} \right].$$

The topological index that characterizes the second-order topological phases in 2D is given by

$$w := w_+ - w_-.$$  

We investigate a concrete model of a 2D SOTI on a square lattice, where each unit cell contains four orbitals and asymmetric particle hopping within each unit cell is introduced, as shown in Fig. 1(a). The Bloch Hamiltonian is written as

$$H_{2D} = [t + \lambda \cos(k_x)] \tau_x - [\lambda \sin(k_x) + iy] \tau_y \sigma_z + [t + \lambda \cos(k_y)] \tau_y \sigma_y + [\lambda \sin(k_y) + iy] \tau_x \sigma_x,$$

where we have set the lattice constant $a_0 = 1$, $\lambda$ is a real-valued intercell hopping amplitude, $t \pm y$ denote real-valued asymmetric intracell hopping amplitudes, and $\sigma_i$ and $\tau_i$ ($i = x, y, z$) are Pauli matrices for the degrees of freedom within a unit cell. The Hamiltonian $H_{2D}$ can be implemented experimentally using ultracold atoms in optical lattices with engineered dissipation [see Fig. 1(b) and Sec. VIII in the Supplemental Material [94] for details]. The Hermitian part of $H_{2D}(k)$ preserves mirror and fourfold rotational symmetries with $\mathcal{M}_x = \tau_z \sigma_x$, $\mathcal{M}_y = \tau_z \sigma_y$, and $C_4 = [\tau_x - i\tau_y] \sigma_0 - (\tau_x + i\tau_y) (i\sigma_0)]/2$. While they are broken by asymmetric hopping, $H_{2D}$ stays invariant under sublattice symmetry $S = \tau_x$ and mirror-rotation symmetry $\mathcal{M}_{xy} = C_4 \mathcal{M}_{xy}$, and $[S, \mathcal{M}_{xy}] = 0$.

Bulk and edge states.—The upper and lower bands $E_{\pm}(k)$ of $H(k)$ are twofold degenerate [94], and these bands coalesce at EPs with $E_{\pm}(k_{EP}) = 0$ for $t = \pm \lambda \pm y$ or $\pm \sqrt{\gamma^2 - \lambda^2}$. Figure 2 shows the complex energy spectra with open and periodic boundaries along the $x$ and $y$ directions, respectively. The non-Hermitian system supports gapped complex edge states for $|t| < |\gamma| + |\lambda|$, as shown in the red curves in Figs. 2(a) and 2(b). On the other hand, for $|t| > |\gamma| + |\lambda|$, there are no edge states [see Figs. 2(c) and 2(d)]. In spite of their existence, edge states can

![FIG. 1. Non-Hermitian SOTI in 2D. (a) Tight-binding representation of the model [Eq. (6)] on a square lattice. Each unit cell contains four orbitals (blue solid circles). The orange lines denote intercell coupling, and the red and black lines with arrows represent asymmetric intracell hopping. The dashed lines indicate hopping terms with a negative sign, accounting for a flux of $\pi$ piercing each plaquette. (b) Schematic illustration of a proposed experimental setup using ultracold atoms [94]. The primary lattice together with a pair of Raman lasers gives rise to a Hermitian SOTI, where the Raman lasers are used for inducing effective particle hopping. The asymmetric hopping amplitudes are introduced via coherent coupling to a dissipative auxiliary lattice.](image)

![FIG. 2. Complex energy spectra of the non-Hermitian SOTI described by Eq. (6) with open boundaries along the x direction and periodic boundaries along the y direction. The edge states (red curves) are gapped for (a),(b) $t = 0.6$. No edge states exist for (c),(d) $t = 2.0$. An EP exists for $t = \lambda + \gamma = 1.9$, where a phase transition occurs. The number of unit cells along the x direction is $N = 20$ with $\lambda = 1.5$ and $\gamma = 0.4$.](image)
As shown in Figs. 3(b) and 3(c), the bulk eigenenergies in the lower-left corner due to the non-Hermiticity caused by the imaginary parts of the bulk eigenenergies of a finite-size sample are complex in the case of the periodic boundaries. Furthermore, we find that, in contrast to the Hermitian Hermiticity of the open-boundary system [94], while they are suggested for defining the topological index of non-Hermitian systems [65,66]. Here, we generalize this idea to the non-Hermitian SOTI (see Sec. VII in Ref. [94] for details). After replacing real wave vectors \( \mathbf{k} \) with complex ones

\[
\mathbf{k} = (k_x, k_y) \rightarrow \mathbf{\tilde{k}} = (k_x - i \ln(\beta_0), k_y - i \ln(\beta_0)),
\]

with \( \beta_0 = \sqrt{|(t - \gamma)/(t + \gamma)|} \), the Hamiltonian \( H_{\pm} \) for \( H_{2D} \) in Eq. (3) has the following forms:

\[
\frac{\hat{\mathbf{H}}_{\pm}}{\sqrt{2}} = (t - \gamma + \lambda(\beta_0 e^{ik})\sigma_\pm + (t + \gamma + \lambda\beta_0 e^{-ik})\sigma_\mp,
\]

where \( \sigma_\pm = (\sigma_x \pm i\sigma_y)/2 \). Note that the location of the midgap corner modes depends on \( \beta_0 \): they are localized at the lower-left corners for \( \beta_0 < 1 \), and at the upper-right corners for \( \beta_0 > 1 \). Figure 4(a) shows the topological-phase diagram. The number of zero-energy corner modes is counted as \( 2|w| \). Furthermore, the phase boundaries are determined by \( t^2 = \lambda^2 + \gamma^2 \) and \( \beta^2 = \gamma^2 - \lambda^2 \).

**FIG. 3.** Corner states in the non-Hermitian SOTI described by the Hamiltonian (6). (a) Probability density distributions \( \sum_{i=1}^4 |\psi_{R,i,n}|^2 \) (n is the index of an eigenstate and R specifies a unit cell) of four zero-energy states under the open-boundary condition along the x and y directions. The zero-energy modes are localized only at the lower-left corner. (b), (c) Real and imaginary parts of complex eigenenergies around zero energy. The red dots represent eigenenergies of the corner modes. The imaginary parts of the bulk eigenenergies of a finite-size sample vanish over a wide range of parameters. (d) Probability density distribution of a typical bulk state under the open-boundary condition along the x and y directions. The bulk state is exponentially localized at the lower-left corner. The number of unit cells is \( 20 \times 20 \) with \( t = 0.6 \), \( \lambda = 1.5 \), and \( \gamma = 0.4 \).
diagram contains the trivial phase \((w = 0)\) and the second-order topological phase \((w = -2)\).

3D SOTI.—We now consider a 3D non-Hermitian Hamiltonian \(H_{3D}\) that respects twofold mirror-rotation symmetry

\[
\mathcal{M}_{xy} H_{3D}(k_x, k_y, k_z) \mathcal{M}_{xy}^{-1} = H_{3D}(k_x, k_y, k_z). \quad (9)
\]

Note that the Hermitian counterpart was investigated in Ref. [82]. As in the 2D case, due to the mirror-rotation symmetry in Eq. (9), we can express the Hamiltonian \(H_{3D}\) along the high-symmetry line \(k_x = k_y\) as

\[
U^{-1} H_{3D}(k, k, k_z) U = \begin{pmatrix} H_+(k, k_z) & 0 \\ 0 & H_-(k, k_z) \end{pmatrix}, \quad (10)
\]

where \(H_\pm(k, k_z)\) acts on the corresponding mirror-rotation subspace. We can define the Chern number

\[
C_\pm := \frac{1}{2\pi} \int_{BZ} \text{Tr}[d A_\pm + i A_\pm \wedge A_\pm], \quad (11)
\]

where \(A_\alpha^{ab} = i \langle \phi_\alpha^a(k, k_z) | d \phi_\beta^b(k, k_z) \rangle\) with \(\alpha\) and \(\beta\) taken over the filled bands, and \(| \phi_\alpha^a \rangle (| \phi_\alpha^b \rangle)\) is a right (left) eigenstate of \(H_\pm(k, k_z)\). This formula is a natural generalization of the single-band non-Hermitian Chern number discussed in Ref. [53] to multiple bands. Then the topological index that characterizes the second-order topological phases in 3D is

\[
C := C_+ - C_- . \quad (12)
\]

We investigate a concrete model of a 3D non-Hermitian SOTI on a cubic lattice described by

\[
H_{3D} = \left( m + t \sum_i \cos k_i \right) \tau_z + \sum_i (\Delta_1 \sin k_i + i \gamma) \sigma_i \tau_x + \Delta_2 (\cos k_x - \cos k_y) \tau_y, \quad (13)
\]

where \(i\) runs over \(x, y,\) and \(z,\) and \(\gamma_x = \gamma_y = \gamma_0\). This Hamiltonian \(H_{3D}\) only preserves mirror-rotation symmetry \(\mathcal{M}_{xy}\) (see Sec. IX in Ref. [94]).

When the bulk bands of \(H_{3D}\) are gapped and first-order topologically trivial, it does not support gapless surface states, as shown by energy spectra with open boundaries along the \(y\) direction in Figs. 5(a) and 5(b). However, the system with open boundaries in both \(x\) and \(y\) directions hosts fourfold degenerate second-order boundary modes, as shown in Figs. 5(c) and 5(d). In contrast to the Hermitian case [82], these second-order boundary modes under the open-boundary condition along all the directions are localized not along the hinge but anomalously localized at one corner [see Fig. 5(e)]. This indicates that the usual bulk-hinge correspondence is broken for the 3D non-Hermitian SOTI. Moreover, these second-order boundary modes are only localized at the corners on the \(x = y\) plane due to the mirror-rotation symmetry \(\mathcal{M}_{xy}\) (see Fig. S10 in Ref. [94]). In addition, the second-order boundary modes can be localized at more than one corner when the mirror-rotation symmetry is broken or there exists the balanced gain and loss (see Sec. IX in Ref. [94]).

FIG. 5. Three-dimensional non-Hermitian SOTI described by Eq. (13). (a),(b) Complex energy spectrum under the open-boundary condition along the \(y\) direction. (c),(d) Complex energy spectrum under the open-boundary condition along the \(x\) and \(y\) directions. Red curves denote fourfold degenerate second-order boundary modes. (e) Probability density distribution \(|\Phi_{n,R}|^2\) \((n\) is the index of an eigenstate and \(R\) specifies a lattice site) of midgap modes with open boundaries along the \(x, y,\) and \(z\) directions. The midgap states (with eigenenergies of 0.035) are localized only at one corner. The number of unit cells is \(20 \times 20 \times 30\) with \(t = 1, \gamma_0 = 0.7, \gamma = -0.2, m = -2, \Delta_1 = 1.2,\) and \(\Delta_2 = 1.2.\) (f) Second-order topological-phase diagram characterized by the nonzero Chern number.
Because of mirror-rotation symmetry, the second-order topological phase in 3D can be characterized by the Chern number $C$ [see Eqs. (9)–(12)]. To generalize the bulk-boundary correspondence in 3D non-Hermitian SOTIs, we take into account the exponential-decay behavior of non-Hermitian eigenstates with open boundaries along all the directions. After considering a low-energy continuum model of the Hamiltonian $H_{3D}$ to capture the essential physics of the 3D non-Hermitian SOTI with analytical results, and replacing real wave vectors $k$ with complex ones (see Sec. IX in Ref. [94] for details), the Hamiltonian $H_\pm$ for $H_{3D}$ in Eq. (10) can be expressed as

$$H_\pm (k, z) = \left[ m + 3t - t(k - i\alpha_0)^2 - \frac{t}{2} (k_z - i\alpha_z)^2 \right] \sigma_z$$

$$\pm \sqrt{2} \left[ \Delta_1 (k - i\alpha_0) + i\gamma_0 \right] \sigma_y$$

$$- \Delta_1 (k_z - i\alpha_z) \sigma_x,$$

(14)

where

$$\alpha_0 = \frac{\gamma_0}{\Delta_1}, \quad \alpha_z = \frac{\gamma_z}{\Delta_1}.$$  

(15)

Figure 5(f) shows the topological-phase diagram, where the second-order topological phases are characterized by the nonzero Chern number ($C = -2$). The number of hinge states is counted as $2 |C|$.

**Conclusions.**—In this Letter, we have analyzed 2D and 3D SOTIs in the presence of non-Hermiticity. In spite of their first-order topologically trivial bulk bands, second-order boundary modes exist in both 2D and 3D SOTIs. In contrast to the Hermitian cases, the midgap states in 2D are localized only at one corner protected by mirror-rotation symmetry and sublattice symmetry, and the second-order boundary modes are anomalously localized at a corner in 3D. The winding number (Chern number) defined by complex wave vectors is used to determine their second-order topological phases in 2D (3D). An experimental realization with ultracold atoms is also discussed. Our study provides a framework to explore richer non-Hermitian physics in higher-order topological phases.

T. L. thanks James Jun He for discussions, and Yi Peng for technical assistance. T. L. acknowledges support from a JSPS Postdoctoral Fellowship (P18023). Y. R. Z. was partially supported by China Postdoctoral Science Foundation (Grant No. 2018M640055). Z. G. was supported by MEXT. K. K. acknowledges support from the JSPS through the Program for Leading Graduate Schools (ALPS). M. U. acknowledges support by KAKENHI Grant No. JP18H01145 and a Grant-in-Aid for Scientific Research on Innovative Areas “Topological Materials Science (KAKENHI Grant No. JP15H05855) from the JSPS. F. N. is supported in part by the: MURI Center for Dynamic Magneto-Optics via the Air Force Office of Scientific Research (AFOSR) (Grant No. FA9550-14-1-0040), Army Research Office (ARO) (Grant No. W911NF-18-1-0358), Asian Office of Aerospace Research and Development (AOARD) (Grant No. FA9550-18-1-0405), Japan Science and Technology Agency (JST) (Q-LEAP program, ImpACT program, and CREST Grant No. JPMJCR1676), JSPS (JSFR-RFBR Grant No. 17-52-50023, and JSPS-FWO Grant No. V5.059.18N), RIKEN-AIST Challenge Research Fund, and the John Templeton Foundation.

**Note added.**—After this work was submitted, a related preprint [108] appeared, which focuses on the interplay between topological modes and skin boundary modes induced by nonreciprocity in non-Hermitian higher-order topological phases.


076801-7


[99] P. Cappellaro, Quantum Theory of Radiation Interactions (MIT OpenCourseWare, Cambridge, Massachusetts, 2012).


