

Supplementary Material for “Decoherence-free interaction between giant atoms in waveguide QED”

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In this Supplementary Material, we present detailed derivations of all master equations for small and giant atoms used in the main text. We perform these derivations for the most general cases, where both the coupling strengths for different coupling points and the phases acquired between coupling points may differ. We prove the relations between exchange interaction, individual decays, and collective decay in this general setting. We also give input-output relations for multiple giant atoms with multiple connection points.

S1. SLH BASICS

We derive all master equations for the multi-atom systems in the SLH formalism [S1–S4] for cascaded quantum systems. To make the treatment here self-contained, we first state the basic properties and rules of this formalism. For more details, see the review in Ref. [S3].

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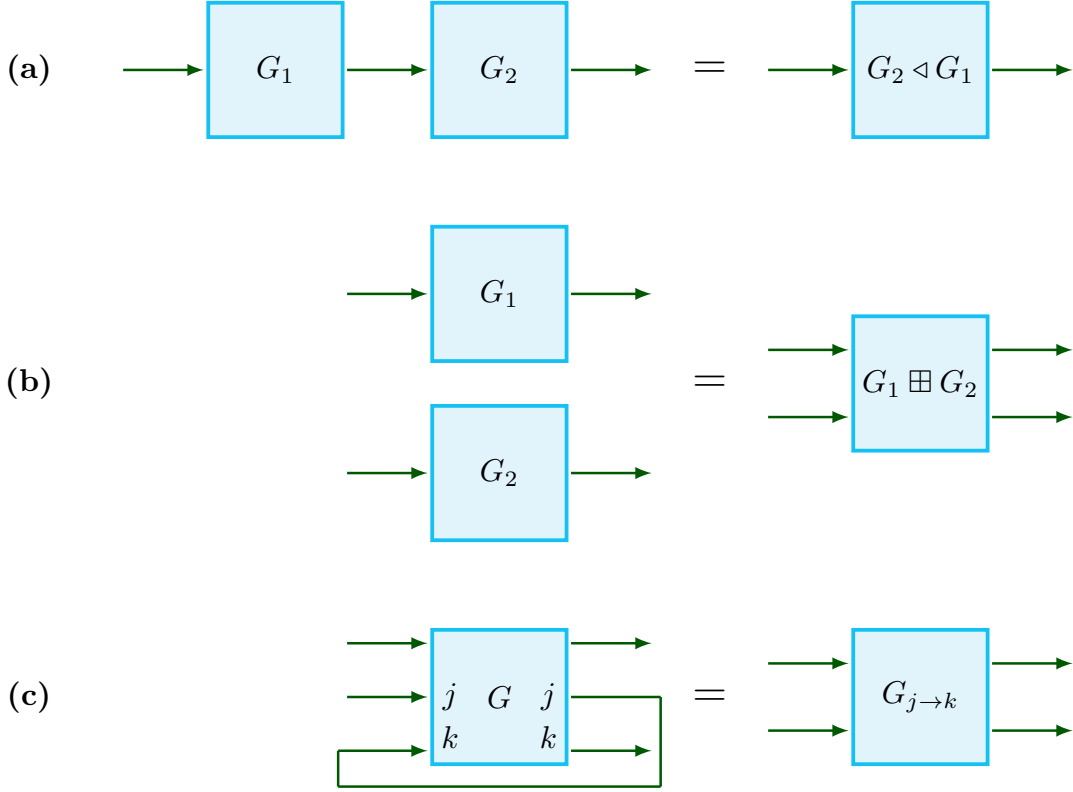


Figure S1. Illustrations of the operations covered by the three basic composition rules in the SLH formalism. (a) The series product $G_2 \triangleleft G_1$. (b) The concatenation product $G_1 \boxplus G_2$. (c) The feedback operation $G_{j \rightarrow k}$.

An open quantum system with n input-output ports can be described by an SLH triplet $G = (\mathbf{S}, \mathbf{L}, H)$, where \mathbf{S} is an $n \times n$ scattering matrix, \mathbf{L} is an $n \times 1$ vector describing the coupling of the system to the environment at the input-output ports (e.g., if a cavity with annihilation operator a is leaking photons at a rate κ through one of its mirrors, and this mirror constitutes the j th input-output port of the system, that will give an entry $L_j = \sqrt{\kappa}a$ in \mathbf{L}), and H is the Hamiltonian of the system.

To combine SLH triplets of cascaded quantum systems into a single triplet describing the entire setup, three composition rules, illustrated in Fig. S1, are used: the series product \triangleleft , the concatenation product \boxplus , and a feedback operation. The series product, shown in Fig. S1(a), is used when the outputs from a system described by an SLH triplet G_1 are used as inputs to another system, described by an SLH triplet G_2 with the same number of input-output ports, such that the j th output from the first system becomes the j th input to the second system. The resulting total SLH triplet is then

$$G_2 \triangleleft G_1 = \left(\mathbf{S}_2 \mathbf{S}_1, \mathbf{S}_2 \mathbf{L}_1 + \mathbf{L}_2, H_1 + H_2 + \frac{1}{2i} [\mathbf{L}_2^\dagger \mathbf{S}_2 \mathbf{L}_1 - \mathbf{L}_1^\dagger \mathbf{S}_2^\dagger \mathbf{L}_2] \right). \quad (\text{S1})$$

If two systems, described by G_1 and G_2 , are combined in parallel, the resulting total SLH triplet is given by the concatenation product

$$G_1 \boxplus G_2 = \left(\begin{bmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{bmatrix}, H_1 + H_2 \right), \quad (\text{S2})$$

shown in Fig. S1(b). Finally, if the j th output of a system described by the triplet $G = (\mathbf{S}, \mathbf{L}, H)$ is fed back as the k th input of the same system, as illustrated in Fig. S1(c), the rule for feedback reduction states that the resulting system is described by the triplet $G_{j \rightarrow k} = (\tilde{\mathbf{S}}, \tilde{\mathbf{L}}, \tilde{H})$, where

$$\tilde{\mathbf{S}} = \mathbf{S}_{j\bar{k}} + \mathbf{S}_{j\bar{k}}(1 - S_{jk})^{-1} \mathbf{S}_{j\bar{k}}, \quad (\text{S3})$$

$$\tilde{\mathbf{L}} = \mathbf{L}_{\bar{j}} + \mathbf{S}_{j\bar{k}}(1 - S_{jk})^{-1} L_j, \quad (\text{S4})$$

$$\tilde{H} = H + \frac{1}{2i} [\mathbf{L}^\dagger \mathbf{S}_{:k}(1 - S_{jk})^{-1} L_j - \text{H.c.}]. \quad (\text{S5})$$

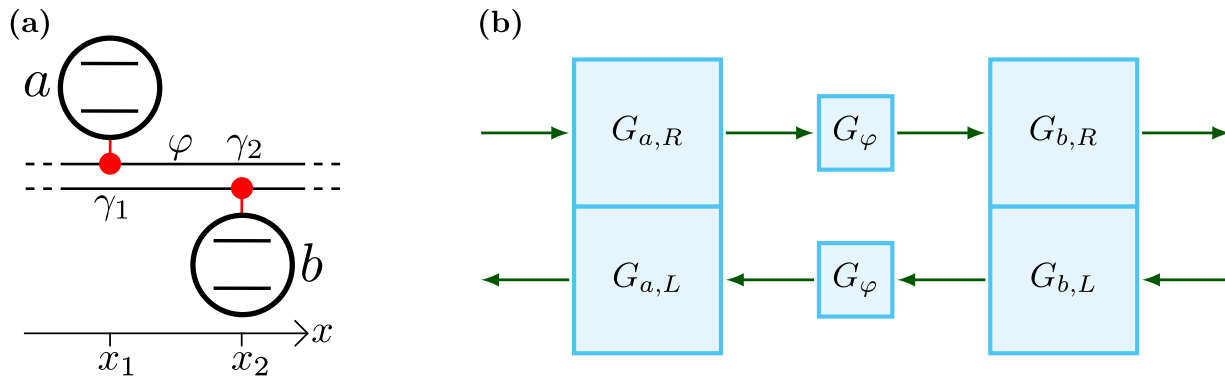


Figure S2. Two small atoms in an open waveguide. (a) A sketch showing the relevant parameters. (b) The input-output flows in the corresponding SLH calculation.

Here, $\mathbf{S}_{j\bar{k}}$ denotes \mathbf{S} with the j th row and k th column removed, $\mathbf{S}_{j\bar{k}}$ denotes the k th column of \mathbf{S} with the j th row removed, $\mathbf{S}_{\bar{j}k}$ denotes the j th row of \mathbf{S} with the k th column removed, S_{jk} is the element in the j th row and k th column of \mathbf{S} , $\mathbf{S}_{\cdot k}$ is the k th column of \mathbf{S} , and H.c. denotes Hermitian conjugate.

When the SLH triplet $G = (\mathbf{S}, \mathbf{L}, H)$ for a system has been found, the master equation for the system is given by

$$\dot{\rho} = -i[H, \rho] + \sum_{j=1}^n \mathcal{D}[L_j]\rho, \quad (\text{S6})$$

where $\mathcal{D}[X]\rho = X\rho X^\dagger - \frac{1}{2}X^\dagger X\rho - \frac{1}{2}\rho X^\dagger X$ are Lindblad operators. The output from port j of the system is simply given by L_j .

Note that the SLH formalism relies on the same physical assumptions as the standard Lindblad master equation, i.e., weak coupling and the Markov approximation. In addition, the SLH formalism as presented so far also requires the fields connecting various systems to propagate in linear, dispersion-less media, and that the propagation time is negligible. Furthermore, it is also assumed that all input fields are in the vacuum state, but non-vacuum inputs can be incorporated by introducing triplets for various sources. For example, if input port j of a system with N input-output ports and triplet $G = (\mathbf{S}, \mathbf{L}, H)$ is driven by a coherent drive supplying $|\alpha|^2$ photons per second, this can be modeled by $G_1 \triangleleft (\mathbb{I}_{j-1} \boxplus G_\alpha \boxplus \mathbb{I}_{N-j})$, where $\mathbb{I}_k = (\mathbf{1}_k, \mathbf{0}, 0)$, $\mathbf{1}_k$ is the $k \times k$ identity matrix, and $G_\alpha = (1, \alpha, 0)$, in the rotating frame of the drive.

S2. MASTER EQUATIONS FOR TWO SMALL ATOMS

As a prelude to the calculations for giant atoms, we first re-derive the known master equation for two small atoms in an open waveguide using the SLH formalism. We then also study the case of two small atoms in a semi-infinite waveguide, which has similarities with giant-atom setups.

A. Open waveguide

We first consider the setup shown in Fig. S2(a), where the two atoms, with resonance frequencies ω_a and ω_b , respectively, have relaxation rates γ_1 and γ_2 , respectively, due to their coupling to the waveguide. The distance between the atoms is such that a signal propagating in the waveguide acquires a phase shift φ when traveling between them. This phase shift is calculated as $\varphi = k|x_2 - x_1|$, where $k = \omega_k/v$ is the wavenumber given by an angular frequency ω_k and the propagation velocity in the waveguide, v . We assume $\omega_k = \omega_a \approx \omega_b$. This is consistent with the assumptions behind the SLH formalism. As long as the relaxation rates γ_j are small compared to the transition frequencies ω_j , and the distance $|x_2 - x_1|$ is not much more than a wavelength, only frequencies in a bandwidth of about γ_j will contribute to the dynamics. This bandwidth is small enough that variations in the phase shift, due to different frequencies within this bandwidth, are negligible. If ω_a and ω_b differ by more than γ_j , the exchange interaction between atoms a and b will be negligible in the rotating-wave approximation.

There are both right- and left-propagating modes in the waveguide. To handle this in the SLH formalism, the easiest approach is to calculate the cascading operations separately for these modes and then concatenate the results.

Explicitly, we divide the system into SLH triplets as shown in Fig. S2(b), separating the triplets for the atoms into two parts with the Hamiltonians residing in the part coupling to the right-propagating modes (we use $\hbar = 1$ throughout this article):

$$G_a = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{\gamma_1}{2}} \sigma_-^a \\ \sqrt{\frac{\gamma_1}{2}} \sigma_-^a \end{bmatrix}, \omega_a \frac{\sigma_z^a}{2} \right) = G_{a,R} \boxplus G_{a,L} = \left(1, \sqrt{\frac{\gamma_1}{2}} \sigma_-^a, \omega_a \frac{\sigma_z^a}{2} \right) \boxplus \left(1, \sqrt{\frac{\gamma_1}{2}} \sigma_-^a, 0 \right), \quad (\text{S7})$$

$$G_b = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{\gamma_2}{2}} \sigma_-^b \\ \sqrt{\frac{\gamma_2}{2}} \sigma_-^b \end{bmatrix}, \omega_b \frac{\sigma_z^b}{2} \right) = G_{b,R} \boxplus G_{b,L} = \left(1, \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, \omega_b \frac{\sigma_z^b}{2} \right) \boxplus \left(1, \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, 0 \right). \quad (\text{S8})$$

The phase shift is included through the triplet $G_\varphi = (e^{i\varphi}, 0, 0)$. From the rule for the series product given in Eq. (S1) we obtain the triplet for the right-propagating modes

$$\begin{aligned} G_R &= G_{b,R} \triangleleft G_\varphi \triangleleft G_{a,R} = \left(e^{i\varphi}, \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, \omega_b \frac{\sigma_z^b}{2} \right) \triangleleft G_{a,R} \\ &= \left(e^{i\varphi}, e^{i\varphi} \sqrt{\frac{\gamma_1}{2}} \sigma_-^a + \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, \omega_a \frac{\sigma_z^a}{2} + \omega_b \frac{\sigma_z^b}{2} + \frac{\sqrt{\gamma_1 \gamma_2}}{4i} [e^{i\varphi} \sigma_-^a \sigma_+^b - e^{-i\varphi} \sigma_+^a \sigma_-^b] \right), \end{aligned} \quad (\text{S9})$$

and, in the same way, the triplet for the left-propagating modes

$$G_L = G_{a,L} \triangleleft G_\varphi \triangleleft G_{b,L} = \left(e^{i\varphi}, \sqrt{\frac{\gamma_1}{2}} \sigma_-^a + e^{i\varphi} \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, \frac{\sqrt{\gamma_1 \gamma_2}}{4i} [e^{i\varphi} \sigma_+^a \sigma_-^b - e^{-i\varphi} \sigma_-^b \sigma_+^a] \right). \quad (\text{S10})$$

Concatenating these triplets, the final result is

$$G_{\text{tot}} = G_R \boxplus G_L = \left(\begin{bmatrix} e^{i\varphi} & 0 \\ 0 & e^{i\varphi} \end{bmatrix}, \begin{bmatrix} e^{i\varphi} \sqrt{\frac{\gamma_1}{2}} \sigma_-^a + \sqrt{\frac{\gamma_2}{2}} \sigma_-^b \\ \sqrt{\frac{\gamma_1}{2}} \sigma_-^a + e^{i\varphi} \sqrt{\frac{\gamma_2}{2}} \sigma_-^b \end{bmatrix}, \omega_a \frac{\sigma_z^a}{2} + \omega_b \frac{\sigma_z^b}{2} + \frac{\sqrt{\gamma_1 \gamma_2}}{2} \sin \varphi [\sigma_-^a \sigma_+^b + \sigma_+^a \sigma_-^b] \right). \quad (\text{S11})$$

From the SLH triplet G_{tot} in Eq. (S11), we can extract the master equation for the system using Eq. (S6). In this calculation, we use the following property of the Lindblad operators:

$$\mathcal{D}[a+b]\rho = \mathcal{D}[a]\rho + \mathcal{D}[b]\rho + a\rho b^\dagger + b\rho a^\dagger - \frac{1}{2} [(a^\dagger b + b^\dagger a)\rho + \rho(a^\dagger b + b^\dagger a)]. \quad (\text{S12})$$

The resulting master equation is

$$\begin{aligned} \dot{\rho} &= -i \left[\omega_a \frac{\sigma_z^a}{2} + \omega_b \frac{\sigma_z^b}{2} + \frac{\sqrt{\gamma_1 \gamma_2}}{2} \sin \varphi (\sigma_-^a \sigma_+^b + \sigma_+^a \sigma_-^b), \rho \right] \\ &\quad + \mathcal{D} \left[e^{i\varphi} \sqrt{\frac{\gamma_1}{2}} \sigma_-^a + \sqrt{\frac{\gamma_2}{2}} \sigma_-^b \right] \rho + \mathcal{D} \left[\sqrt{\frac{\gamma_1}{2}} \sigma_-^a + e^{i\varphi} \sqrt{\frac{\gamma_2}{2}} \sigma_-^b \right] \rho \\ &= -i \left[\omega_a \frac{\sigma_z^a}{2} + \omega_b \frac{\sigma_z^b}{2} + \frac{\sqrt{\gamma_1 \gamma_2}}{2} \sin \varphi (\sigma_-^a \sigma_+^b + \sigma_+^a \sigma_-^b), \rho \right] \\ &\quad + \gamma_1 \mathcal{D}[\sigma_-^a] \rho + \gamma_2 \mathcal{D}[\sigma_-^b] \rho + \sqrt{\gamma_1 \gamma_2} \cos \varphi \left\{ \sigma_-^a \rho \sigma_+^b + \sigma_-^b \rho \sigma_+^a - \frac{1}{2} [(\sigma_+^a \sigma_-^b + \sigma_+^b \sigma_-^a) \rho + \rho(\sigma_+^a \sigma_-^b + \sigma_+^b \sigma_-^a)] \right\}. \end{aligned} \quad (\text{S13})$$

Assuming equal relaxation rates for the two atoms, i.e., setting $\gamma_1 = \gamma_2 \equiv \gamma$, we see that the master equation in Eq. (S13) reduces to Eq. (1) of the main text with the coefficients given in the second row of Table I.

B. Semi-infinite waveguide

Adding a mirror to make the open waveguide, to which the two small atoms couple, semi-infinite instead, we have the setup shown in Fig. S3(a). The phase acquired when traveling from atom a to the mirror is $\varphi_1/2$, so the phase acquired during a roundtrip from atom a to the mirror and back is φ_1 .

To calculate the SLH triplet for this setup, we again start by separating the triplets for the atoms into parts interacting with the left- and right-propagating modes in the transmission line, exactly like in Eqs. (S7)-(S8). All parts needed for the SLH calculation are shown in Fig. S3(b). Unlike in the open-waveguide case, here we do not use the concatenation product to add up the left- and right-moving parts; instead, we combine through the series product due to the presence of the mirror. The triplet for the whole system is

$$G_{\text{tot}} = G_{b,R} \triangleleft G_{\varphi_2} \triangleleft G_{a,R} \triangleleft G_{\varphi_1} \triangleleft G_{a,L} \triangleleft G_{\varphi_2} \triangleleft G_{b,L} \quad (\text{S14})$$

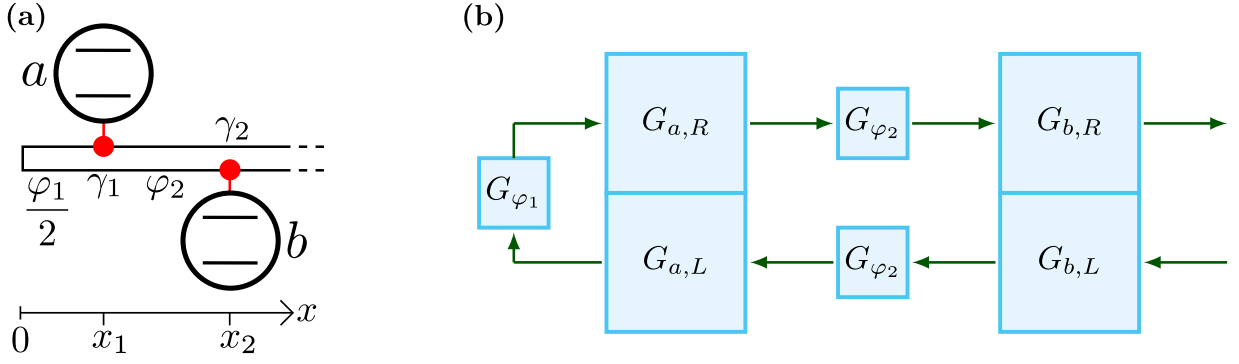


Figure S3. Two small atoms in a semi-infinite waveguide. (a) A sketch showing the relevant parameters. (b) The input-output flows in the corresponding SLH calculation.

Using the results in Eqs. (S9)-(S10) leads to

$$G_{\text{tot}} = \left(e^{i\varphi_2}, e^{i\varphi_2} \sqrt{\frac{\gamma_1}{2}} \sigma_-^a + \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, \omega_a \frac{\sigma_z^a}{2} + \omega_b \frac{\sigma_z^b}{2} + \frac{\sqrt{\gamma_1 \gamma_2}}{4i} [e^{i\varphi_2} \sigma_-^a \sigma_+^b - e^{-i\varphi_2} \sigma_+^a \sigma_-^b] \right) \triangleleft G_{\varphi_1} \triangleleft \left(e^{i\varphi_2}, \sqrt{\frac{\gamma_1}{2}} \sigma_-^a + e^{i\varphi_2} \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, \frac{\sqrt{\gamma_1 \gamma_2}}{4i} [e^{i\varphi_2} \sigma_+^a \sigma_-^b - e^{-i\varphi_2} \sigma_-^b \sigma_+^a] \right) \quad (\text{S15})$$

By repeated application of the series-product rule, we arrive at the three components of G_{tot} :

$$\mathbf{S}_{\text{tot}} = e^{i(\varphi_1 + 2\varphi_2)}, \quad (\text{S16})$$

$$\mathbf{L}_{\text{tot}} = e^{i\varphi_2} (1 + e^{i\varphi_1}) \sqrt{\frac{\gamma_1}{2}} \sigma_-^a + (1 + e^{i(\varphi_1 + 2\varphi_2)}) \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, \quad (\text{S17})$$

$$H_{\text{tot}} = \left(\omega_a + \frac{\gamma_1}{2} \sin \varphi_1 \right) \frac{\sigma_z^a}{2} + \left[\omega_b + \frac{\gamma_2}{2} \sin(\varphi_1 + 2\varphi_2) \right] \frac{\sigma_z^b}{2} + \frac{\sqrt{\gamma_1 \gamma_2}}{2} [\sin \varphi_2 + \sin(\varphi_1 + \varphi_2)] (\sigma_-^a \sigma_+^b + \sigma_+^a \sigma_-^b). \quad (\text{S18})$$

To obtain the final form of H_{tot} , we used the identity $\sigma_+ \sigma_- = (1 + \sigma_z)/2$ and the fact that constant terms can be excluded from the Hamiltonian since they do not contribute to the dynamics.

With the total triplet in hand, we extract the master equation in the same way as for the open-waveguide case:

$$\begin{aligned} \dot{\rho} &= -i[H_{\text{tot}}, \rho] + \mathcal{D} \left[e^{i\varphi_2} (1 + e^{i\varphi_1}) \sqrt{\frac{\gamma_1}{2}} \sigma_-^a + (1 + e^{i(\varphi_1 + 2\varphi_2)}) \sqrt{\frac{\gamma_2}{2}} \sigma_-^b \right] \rho \\ &= -i[H_{\text{tot}}, \rho] + \gamma_1 (1 + \cos \varphi_1) \mathcal{D}[\sigma_-^a] \rho + \gamma_2 [1 + \cos(\varphi_1 + 2\varphi_2)] \mathcal{D}[\sigma_-^b] \rho \\ &\quad + \sqrt{\gamma_1 \gamma_2} [\cos \varphi_2 + \cos(\varphi_1 + \varphi_2)] \left\{ \sigma_-^a \rho \sigma_+^b + \sigma_-^b \rho \sigma_+^a - \frac{1}{2} [(\sigma_+^a \sigma_-^b + \sigma_+^b \sigma_-^a) \rho + \rho (\sigma_+^a \sigma_-^b + \sigma_+^b \sigma_-^a)] \right\}. \quad (\text{S19}) \end{aligned}$$

If we assume equal relaxation rates for the two atoms ($\gamma_1 = \gamma_2 \equiv \gamma$) and equal phases ($\varphi_1 = \varphi_2 \equiv \varphi$), Eq. (S19) reduces to Eq. (1) of the main text with the coefficients given in the third row of Table I.

S3. MASTER EQUATIONS FOR TWO GIANT ATOMS WITH TWO CONNECTION POINTS

We now use the SLH formalism to derive the master equations for all geometries with giant atoms coupled to an open waveguide at two connection points. The remarks about approximations made for the phase shift for two small atoms in the preceding section are valid here as well, and also for more than two giant atoms with more than two connection points.

A. Separate giant atoms

We first consider two separate giant atoms. The setup, with definitions of all coupling strengths and phase shifts, is shown in Fig. S4(a). To calculate the SLH triplet for this setup, we follow the scheme sketched in Fig. S4(b). The

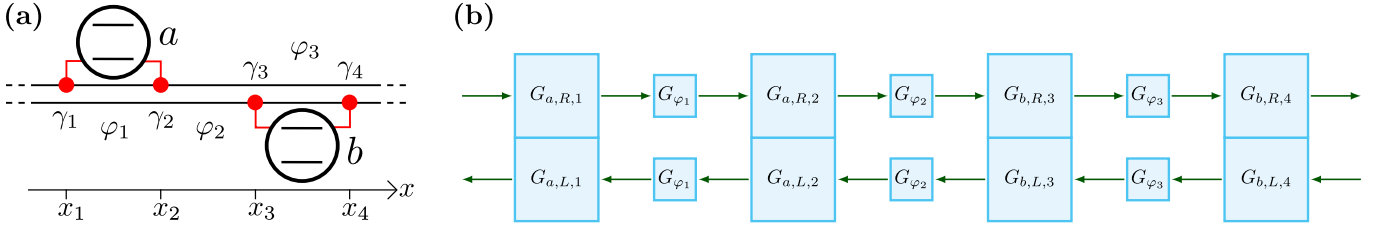


Figure S4. Two separate giant atoms in an open waveguide. (a) A sketch showing the relevant parameters. (b) The input-output flows in the corresponding SLH calculation.

triplets for the atoms are first decomposed into the following parts:

$$G_{a,R,1} = \left(1, \sqrt{\frac{\gamma_1}{2}} \sigma_-^a, \omega_a \frac{\sigma_z^a}{2} \right), \quad (\text{S20})$$

$$G_{a,R,2} = \left(1, \sqrt{\frac{\gamma_2}{2}} \sigma_-^a, 0 \right), \quad (\text{S21})$$

$$G_{a,L,1} = \left(1, \sqrt{\frac{\gamma_1}{2}} \sigma_-^a, 0 \right), \quad (\text{S22})$$

$$G_{a,L,2} = \left(1, \sqrt{\frac{\gamma_2}{2}} \sigma_-^a, 0 \right), \quad (\text{S23})$$

$$G_{b,R,3} = \left(1, \sqrt{\frac{\gamma_3}{2}} \sigma_-^b, \omega_b \frac{\sigma_z^b}{2} \right), \quad (\text{S24})$$

$$G_{b,R,4} = \left(1, \sqrt{\frac{\gamma_4}{2}} \sigma_-^b, 0 \right), \quad (\text{S25})$$

$$G_{b,L,3} = \left(1, \sqrt{\frac{\gamma_3}{2}} \sigma_-^b, 0 \right), \quad (\text{S26})$$

$$G_{b,L,4} = \left(1, \sqrt{\frac{\gamma_4}{2}} \sigma_-^b, 0 \right). \quad (\text{S27})$$

The triplet for the right-moving part is then given by

$$G_R = G_{b,R,4} \triangleleft G_{\varphi_3} \triangleleft G_{b,R,3} \triangleleft G_{\varphi_2} \triangleleft G_{a,R,2} \triangleleft G_{\varphi_1} \triangleleft G_{a,R,1}. \quad (\text{S28})$$

Using the series product rule, the first part of this expression becomes

$$G_{b,R,4} \triangleleft G_{\varphi_3} \triangleleft G_{b,R,3} = \left(e^{i\varphi_3}, \left[e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}} + \sqrt{\frac{\gamma_4}{2}} \right] \sigma_-^b, \left(\omega_b + \frac{\sqrt{\gamma_3 \gamma_4}}{2} \sin \varphi_3 \right) \frac{\sigma_z^b}{2} \right), \quad (\text{S29})$$

where we again removed a constant term in the Hamiltonian. From symmetry, we then immediately obtain

$$G_{a,R,1} \triangleleft G_{\varphi_1} \triangleleft G_{a,R,2} = \left(e^{i\varphi_1}, \left[e^{i\varphi_1} \sqrt{\frac{\gamma_1}{2}} + \sqrt{\frac{\gamma_2}{2}} \right] \sigma_-^a, \left(\omega_a + \frac{\sqrt{\gamma_1 \gamma_2}}{2} \sin \varphi_1 \right) \frac{\sigma_z^a}{2} \right). \quad (\text{S30})$$

These results, together with repeated application of the series product rule, lead to $G_R = (\mathbf{S}_R, \mathbf{L}_R, H_R)$ with

$$\mathbf{S}_R = e^{i(\varphi_1 + \varphi_2 + \varphi_3)}, \quad (\text{S31})$$

$$\mathbf{L}_R = \left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_1}{2}} + e^{i(\varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_2}{2}} \right) \sigma_-^a + \left(e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}} + \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^b, \quad (\text{S32})$$

$$H_R = \left(\omega_a + \frac{\sqrt{\gamma_1 \gamma_2}}{2} \sin \varphi_1 \right) \frac{\sigma_z^a}{2} + \left(\omega_b + \frac{\sqrt{\gamma_3 \gamma_4}}{2} \sin \varphi_3 \right) \frac{\sigma_z^b}{2} + \frac{1}{4i} \left[\left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\gamma_1 \gamma_4} + e^{i(\varphi_2 + \varphi_3)} \sqrt{\gamma_2 \gamma_4} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\gamma_1 \gamma_3} + e^{i\varphi_2} \sqrt{\gamma_2 \gamma_3} \right) \sigma_-^a \sigma_+^b - \text{H.c.} \right]. \quad (\text{S33})$$

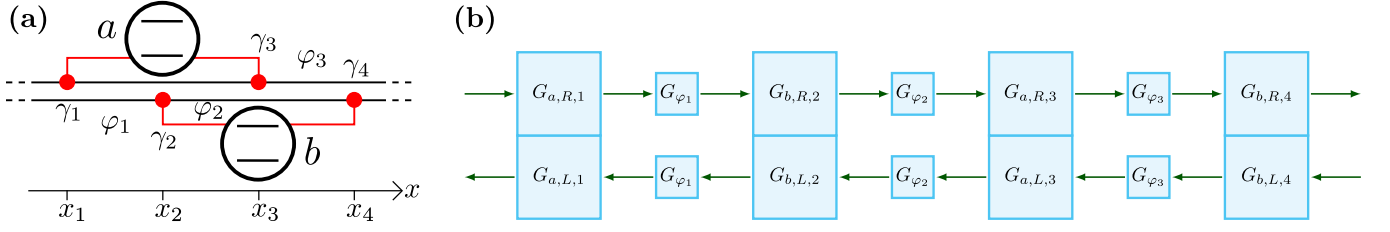


Figure S5. Two braided giant atoms in an open waveguide. (a) A sketch showing the relevant parameters. (b) The input-output flows in the corresponding SLH calculation.

From the symmetry that is apparent in Fig. S4(b), we can immediately deduce that the triplet for the left-moving part, $G_L = (\mathbf{S}_L, \mathbf{L}_L, H_L)$, is given by removing the ω_j parts in the Hamiltonian, making the substitution $\varphi_1 \leftrightarrow \varphi_3$, and changing the non- φ indices according to $a \leftrightarrow b$, $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ in the equations for G_R , i.e.,

$$\mathbf{S}_L = e^{i(\varphi_1 + \varphi_2 + \varphi_3)}, \quad (\text{S34})$$

$$\mathbf{L}_L = \left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_4}{2}} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^b + \left(e^{i\varphi_1} \sqrt{\frac{\gamma_2}{2}} + \sqrt{\frac{\gamma_1}{2}} \right) \sigma_-^a, \quad (\text{S35})$$

$$H_L = \frac{\sqrt{\gamma_1 \gamma_2}}{2} \sin \varphi_1 \frac{\sigma_z^a}{2} + \frac{\sqrt{\gamma_3 \gamma_4}}{2} \sin \varphi_3 \frac{\sigma_z^b}{2} + \frac{1}{4i} \left[\left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\gamma_1 \gamma_4} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\gamma_1 \gamma_3} + e^{i(\varphi_2 + \varphi_3)} \sqrt{\gamma_2 \gamma_4} + e^{i\varphi_2} \sqrt{\gamma_2 \gamma_3} \right) \sigma_+^a \sigma_-^b - \text{H.c.} \right]. \quad (\text{S36})$$

The total triplet for the system is then $G_{\text{tot}} = G_R \boxplus G_L$, with the components

$$\mathbf{S}_{\text{tot}} = \begin{bmatrix} e^{i(\varphi_1 + \varphi_2 + \varphi_3)} & 0 \\ 0 & e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \end{bmatrix}, \quad (\text{S37})$$

$$\mathbf{L}_{\text{tot}} = \begin{bmatrix} \left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_1}{2}} + e^{i(\varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_2}{2}} \right) \sigma_-^a + \left(e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}} + \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^b \\ \left(\sqrt{\frac{\gamma_1}{2}} + e^{i\varphi_1} \sqrt{\frac{\gamma_2}{2}} \right) \sigma_-^a + \left(e^{i(\varphi_1 + \varphi_2)} \sqrt{\frac{\gamma_3}{2}} + e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^b \end{bmatrix}, \quad (\text{S38})$$

$$H_{\text{tot}} = (\omega_a + \sqrt{\gamma_1 \gamma_2} \sin \varphi_1) \frac{\sigma_z^a}{2} + (\omega_b + \sqrt{\gamma_3 \gamma_4} \sin \varphi_3) \frac{\sigma_z^b}{2} + \frac{1}{2} [\sqrt{\gamma_1 \gamma_4} \sin(\varphi_1 + \varphi_2 + \varphi_3) + \sqrt{\gamma_2 \gamma_4} \sin(\varphi_2 + \varphi_3) + \sqrt{\gamma_1 \gamma_3} \sin(\varphi_1 + \varphi_2) + \sqrt{\gamma_2 \gamma_3} \sin \varphi_2] (\sigma_-^a \sigma_+^b + \sigma_+^a \sigma_-^b). \quad (\text{S39})$$

Continuing from the total triplet, we arrive at the master equation in the same way as in previous calculations:

$$\begin{aligned} \dot{\rho} &= -i[H_{\text{tot}}, \rho] + \mathcal{D} \left[\left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_1}{2}} + e^{i(\varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_2}{2}} \right) \sigma_-^a + \left(e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}} + \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^b \right] \rho \\ &\quad + \mathcal{D} \left[\left(e^{i\varphi_1} \sqrt{\frac{\gamma_2}{2}} + \sqrt{\frac{\gamma_1}{2}} \right) \sigma_-^a + \left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_4}{2}} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^b \right] \rho \\ &= -i[H_{\text{tot}}, \rho] + (\gamma_1 + \gamma_2 + 2\sqrt{\gamma_1 \gamma_2} \cos \varphi_1) \mathcal{D}[\sigma_-^a] \rho + (\gamma_3 + \gamma_4 + 2\sqrt{\gamma_3 \gamma_4} \cos \varphi_3) \mathcal{D}[\sigma_-^b] \rho \\ &\quad + [\sqrt{\gamma_1 \gamma_3} \cos(\varphi_1 + \varphi_2) + \sqrt{\gamma_1 \gamma_4} \cos(\varphi_1 + \varphi_2 + \varphi_3) + \sqrt{\gamma_2 \gamma_3} \cos \varphi_2 + \sqrt{\gamma_2 \gamma_4} \cos(\varphi_2 + \varphi_3)] \\ &\quad \times \left\{ \sigma_-^a \rho \sigma_+^b + \sigma_-^b \rho \sigma_+^a - \frac{1}{2} [(\sigma_+^a \sigma_-^b + \sigma_+^b \sigma_-^a) \rho + \rho (\sigma_+^a \sigma_-^b + \sigma_+^b \sigma_-^a)] \right\}. \quad (\text{S40}) \end{aligned}$$

If we assume equal relaxation rates at all coupling points ($\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 \equiv \gamma$) and equal phases ($\varphi_1 = \varphi_2 = \varphi_3 \equiv \varphi$), Eq. (S40) reduces to Eq. (1) of the main text with the coefficients given in the fourth row of Table I.

B. Braided giant atoms

We now consider two braided giant atoms. The setup, with definitions of all coupling strengths and phase shifts, is shown in Fig. S5(a). To calculate the SLH triplet for this setup, we follow the scheme sketched in Fig. S5(b). The

triplets for the atoms are first decomposed into the following parts:

$$G_{a,R,1} = \left(1, \sqrt{\frac{\gamma_1}{2}} \sigma_-^a, \omega_a \frac{\sigma_z^a}{2}\right), \quad (\text{S41})$$

$$G_{a,R,3} = \left(1, \sqrt{\frac{\gamma_3}{2}} \sigma_-^a, 0\right), \quad (\text{S42})$$

$$G_{a,L,1} = \left(1, \sqrt{\frac{\gamma_1}{2}} \sigma_-^a, 0\right), \quad (\text{S43})$$

$$G_{a,L,3} = \left(1, \sqrt{\frac{\gamma_3}{2}} \sigma_-^a, 0\right), \quad (\text{S44})$$

$$G_{b,R,2} = \left(1, \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, \omega_b \frac{\sigma_z^b}{2}\right), \quad (\text{S45})$$

$$G_{b,R,4} = \left(1, \sqrt{\frac{\gamma_4}{2}} \sigma_-^b, 0\right), \quad (\text{S46})$$

$$G_{b,L,2} = \left(1, \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, 0\right), \quad (\text{S47})$$

$$G_{b,L,4} = \left(1, \sqrt{\frac{\gamma_4}{2}} \sigma_-^b, 0\right). \quad (\text{S48})$$

The triplet for the right-moving part is then given by

$$G_R = G_{b,R,4} \triangleleft G_{\varphi_3} \triangleleft G_{a,R,3} \triangleleft G_{\varphi_2} \triangleleft G_{b,R,2} \triangleleft G_{\varphi_1} \triangleleft G_{a,R,1}. \quad (\text{S49})$$

Comparing to the calculation for two small atoms in an open waveguide in Eq. (S9), we see that the first and last parts of this expression become

$$G_{b,R,4} \triangleleft G_{\varphi_3} \triangleleft G_{a,R,3} = \left(e^{i\varphi_3}, e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}} \sigma_-^a + \sqrt{\frac{\gamma_4}{2}} \sigma_-^b, \frac{\sqrt{\gamma_3 \gamma_4}}{4i} [e^{i\varphi_3} \sigma_-^a \sigma_+^b - e^{-i\varphi_3} \sigma_+^a \sigma_-^b]\right), \quad (\text{S50})$$

$$G_{b,R,2} \triangleleft G_{\varphi_1} \triangleleft G_{a,R,1} = \left(e^{i\varphi_1}, e^{i\varphi_1} \sqrt{\frac{\gamma_1}{2}} \sigma_-^a + \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, \omega_a \frac{\sigma_z^a}{2} + \omega_b \frac{\sigma_z^b}{2} + \frac{\sqrt{\gamma_1 \gamma_2}}{4i} [e^{i\varphi_1} \sigma_-^a \sigma_+^b - e^{-i\varphi_1} \sigma_+^a \sigma_-^b]\right), \quad (\text{S51})$$

and further application of the series product rule leads to $G_R = (\mathbf{S}_R, \mathbf{L}_R, H_R)$ with

$$\mathbf{S}_R = e^{i(\varphi_1 + \varphi_2 + \varphi_3)}, \quad (\text{S52})$$

$$\mathbf{L}_R = \left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_1}{2}} + e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}}\right) \sigma_-^a + \left(e^{i(\varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_2}{2}} + \sqrt{\frac{\gamma_4}{2}}\right) \sigma_-^b, \quad (\text{S53})$$

$$H_R = \left[\omega_a + \frac{\sqrt{\gamma_1 \gamma_3}}{2} \sin(\varphi_1 + \varphi_2)\right] \frac{\sigma_z^a}{2} + \left[\omega_b + \frac{\sqrt{\gamma_2 \gamma_4}}{2} \sin(\varphi_2 + \varphi_3)\right] \frac{\sigma_z^b}{2} + \frac{1}{4i} \left\{ \left[\left(e^{i\varphi_1} \sqrt{\gamma_1 \gamma_2} + e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\gamma_1 \gamma_4} + e^{i\varphi_3} \sqrt{\gamma_3 \gamma_4} \right) \sigma_-^a \sigma_+^b + e^{i\varphi_2} \sqrt{\gamma_2 \gamma_3} \sigma_+^a \sigma_-^b \right] - \text{H.c.} \right\}. \quad (\text{S54})$$

From the symmetry that is apparent in Fig. S5(b), we can immediately deduce that the triplet for the left-moving part, $G_L = (\mathbf{S}_L, \mathbf{L}_L, H_L)$, is given by removing the ω_j parts in the Hamiltonian, making the substitution $\varphi_1 \leftrightarrow \varphi_3$, and changing the non- φ indices according to $a \leftrightarrow b$, $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ in the equations for G_R , i.e.,

$$\mathbf{S}_L = e^{i(\varphi_1 + \varphi_2 + \varphi_3)}, \quad (\text{S55})$$

$$\mathbf{L}_L = \left(\sqrt{\frac{\gamma_1}{2}} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\frac{\gamma_3}{2}}\right) \sigma_-^a + \left(e^{i\varphi_1} \sqrt{\frac{\gamma_2}{2}} + e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_4}{2}}\right) \sigma_-^b, \quad (\text{S56})$$

$$H_L = \left[\frac{\sqrt{\gamma_1 \gamma_3}}{2} \sin(\varphi_1 + \varphi_2)\right] \frac{\sigma_z^a}{2} + \left[\frac{\sqrt{\gamma_2 \gamma_4}}{2} \sin(\varphi_2 + \varphi_3)\right] \frac{\sigma_z^b}{2} + \frac{1}{4i} \left\{ \left[\left(e^{i\varphi_1} \sqrt{\gamma_1 \gamma_2} + e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\gamma_1 \gamma_4} + e^{i\varphi_3} \sqrt{\gamma_3 \gamma_4} \right) \sigma_+^a \sigma_-^b + e^{i\varphi_2} \sqrt{\gamma_2 \gamma_3} \sigma_-^a \sigma_+^b \right] - \text{H.c.} \right\}. \quad (\text{S57})$$

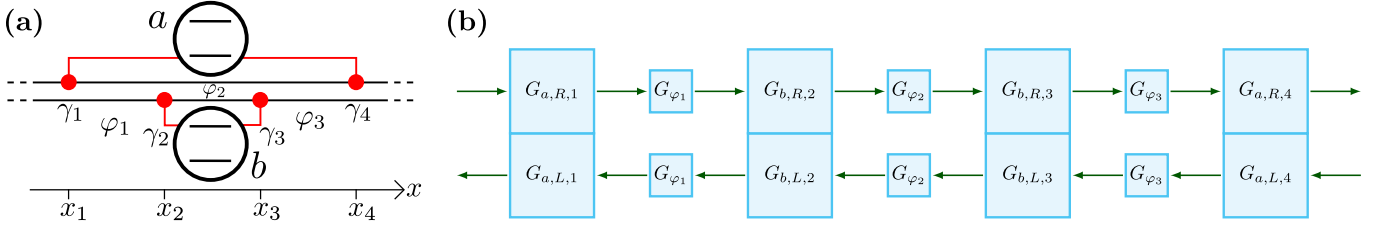


Figure S6. Two nested giant atoms in an open waveguide. (a) A sketch showing the relevant parameters. (b) The input-output flows in the corresponding SLH calculation.

The total triplet for the system is then $G_{\text{tot}} = G_{\text{R}} \boxplus G_{\text{L}}$, with the components

$$\mathbf{S}_{\text{tot}} = \begin{bmatrix} e^{i(\varphi_1 + \varphi_2 + \varphi_3)} & 0 \\ 0 & e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \end{bmatrix}, \quad (\text{S58})$$

$$\mathbf{L}_{\text{tot}} = \begin{bmatrix} \left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_1}{2}} + e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^a + \left(e^{i(\varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_2}{2}} + \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^b \\ \left(\sqrt{\frac{\gamma_1}{2}} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^a + \left(e^{i\varphi_1} \sqrt{\frac{\gamma_2}{2}} + e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^b \end{bmatrix}, \quad (\text{S59})$$

$$H_{\text{tot}} = [\omega_a + \sqrt{\gamma_1 \gamma_3} \sin(\varphi_1 + \varphi_2)] \frac{\sigma_z^a}{2} + [\omega_b + \sqrt{\gamma_2 \gamma_4} \sin(\varphi_2 + \varphi_3)] \frac{\sigma_z^b}{2} + \frac{1}{2} [\sqrt{\gamma_1 \gamma_2} \sin \varphi_1 + \sqrt{\gamma_2 \gamma_3} \sin \varphi_2 + \sqrt{\gamma_3 \gamma_4} \sin \varphi_3 + \sqrt{\gamma_1 \gamma_4} \sin(\varphi_1 + \varphi_2 + \varphi_3)] (\sigma_-^a \sigma_+^b + \sigma_+^a \sigma_-^b). \quad (\text{S60})$$

Continuing from the total triplet, we arrive at the master equation in the same way as in previous calculations:

$$\begin{aligned} \dot{\rho} = & -i[H_{\text{tot}}, \rho] + \mathcal{D} \left[\left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_1}{2}} + e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^a + \left(e^{i(\varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_2}{2}} + \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^b \right] \rho \\ & + \mathcal{D} \left[\left(\sqrt{\frac{\gamma_1}{2}} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^a + \left(e^{i\varphi_1} \sqrt{\frac{\gamma_2}{2}} + e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^b \right] \rho \\ = & -i[H_{\text{tot}}, \rho] + [\gamma_1 + \gamma_3 + 2\sqrt{\gamma_1 \gamma_3} \cos(\varphi_1 + \varphi_2)] \mathcal{D}[\sigma_-^a] \rho + [\gamma_2 + \gamma_4 + 2\sqrt{\gamma_2 \gamma_4} \cos(\varphi_2 + \varphi_3)] \mathcal{D}[\sigma_-^b] \rho \\ & + [\sqrt{\gamma_1 \gamma_2} \cos \varphi_1 + \sqrt{\gamma_2 \gamma_3} \cos \varphi_2 + \sqrt{\gamma_3 \gamma_4} \cos \varphi_3 + \sqrt{\gamma_1 \gamma_4} \cos(\varphi_1 + \varphi_2 + \varphi_3)] \\ & \times \left\{ \sigma_-^a \rho \sigma_+^b + \sigma_-^b \rho \sigma_+^a - \frac{1}{2} [(\sigma_+^a \sigma_-^b + \sigma_+^b \sigma_-^a) \rho + \rho (\sigma_+^a \sigma_-^b + \sigma_+^b \sigma_-^a)] \right\}. \quad (\text{S61}) \end{aligned}$$

If we assume equal relaxation rates at all coupling points ($\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 \equiv \gamma$) and equal phases ($\varphi_1 = \varphi_2 = \varphi_3 \equiv \varphi$), Eq. (S61) reduces to Eq. (1) of the main text with the coefficients given in the fifth row of Table I.

C. Nested giant atoms

Finally, we consider two nested giant atoms. The setup, with definitions of all coupling strengths and phase shifts, is shown in Fig. S6(a). To calculate the SLH triplet for this setup, we follow the scheme sketched in Fig. S6(b). The

triplets for the atoms are first decomposed into the following parts:

$$G_{a,R,1} = \left(1, \sqrt{\frac{\gamma_1}{2}} \sigma_-^a, \omega_a \frac{\sigma_z^a}{2} \right), \quad (\text{S62})$$

$$G_{a,R,4} = \left(1, \sqrt{\frac{\gamma_4}{2}} \sigma_-^a, 0 \right), \quad (\text{S63})$$

$$G_{a,L,1} = \left(1, \sqrt{\frac{\gamma_1}{2}} \sigma_-^a, 0 \right), \quad (\text{S64})$$

$$G_{a,L,4} = \left(1, \sqrt{\frac{\gamma_4}{2}} \sigma_-^a, 0 \right), \quad (\text{S65})$$

$$G_{b,R,2} = \left(1, \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, \omega_b \frac{\sigma_z^b}{2} \right), \quad (\text{S66})$$

$$G_{b,R,3} = \left(1, \sqrt{\frac{\gamma_3}{2}} \sigma_-^b, 0 \right), \quad (\text{S67})$$

$$G_{b,L,2} = \left(1, \sqrt{\frac{\gamma_2}{2}} \sigma_-^b, 0 \right), \quad (\text{S68})$$

$$G_{b,L,3} = \left(1, \sqrt{\frac{\gamma_3}{2}} \sigma_-^b, 0 \right). \quad (\text{S69})$$

The triplet for the right-moving part is then given by

$$G_R = G_{a,R,4} \triangleleft G_{\varphi_3} \triangleleft G_{b,R,3} \triangleleft G_{\varphi_2} \triangleleft G_{b,R,2} \triangleleft G_{\varphi_1} \triangleleft G_{a,R,1}. \quad (\text{S70})$$

Repeated application of the series product rule, aided by comparison with results from previous calculations, leads to $G_R = (\mathbf{S}_R, \mathbf{L}_R, H_R)$ with

$$\mathbf{S}_R = e^{i(\varphi_1 + \varphi_2 + \varphi_3)}, \quad (\text{S71})$$

$$\mathbf{L}_R = \left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_1}{2}} + \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^a + \left(e^{i(\varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_2}{2}} + e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^b, \quad (\text{S72})$$

$$H_R = \left[\omega_a + \frac{\sqrt{\gamma_1 \gamma_4}}{2} \sin(\varphi_1 + \varphi_2 + \varphi_3) \right] \frac{\sigma_z^a}{2} + \left(\omega_b + \frac{\sqrt{\gamma_2 \gamma_3}}{2} \sin \varphi_2 \right) \frac{\sigma_z^b}{2} + \frac{1}{4i} \left\{ \left[\left(e^{i\varphi_1} \sqrt{\gamma_1 \gamma_2} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\gamma_1 \gamma_3} \right) \sigma_-^a \sigma_+^b + \left(e^{i(\varphi_2 + \varphi_3)} \sqrt{\gamma_2 \gamma_4} + e^{i\varphi_3} \sqrt{\gamma_3 \gamma_4} \right) \sigma_+^a \sigma_-^b \right] - \text{H.c.} \right\}. \quad (\text{S73})$$

From the symmetry that is apparent in Fig. S6(b), we can immediately deduce that the triplet for the left-moving part, $G_L = (\mathbf{S}_L, \mathbf{L}_L, H_L)$, is given by removing the ω_j parts in the Hamiltonian, making the substitution $\varphi_1 \leftrightarrow \varphi_3$, and changing the non- φ indices according to $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ in the equations for G_R , i.e.,

$$\mathbf{S}_L = e^{i(\varphi_1 + \varphi_2 + \varphi_3)}, \quad (\text{S74})$$

$$\mathbf{L}_L = \left(\sqrt{\frac{\gamma_1}{2}} + e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^a + \left(e^{i\varphi_1} \sqrt{\frac{\gamma_2}{2}} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^b, \quad (\text{S75})$$

$$H_L = \frac{\sqrt{\gamma_1 \gamma_4}}{2} \sin(\varphi_1 + \varphi_2 + \varphi_3) \frac{\sigma_z^a}{2} + \frac{\sqrt{\gamma_2 \gamma_3}}{2} \sin \varphi_2 \frac{\sigma_z^b}{2} + \frac{1}{4i} \left\{ \left(e^{i(\varphi_2 + \varphi_3)} \sqrt{\gamma_2 \gamma_4} + e^{i\varphi_3} \sqrt{\gamma_3 \gamma_4} \right) \sigma_-^a \sigma_+^b + \left[\left(e^{i\varphi_1} \sqrt{\gamma_1 \gamma_2} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\gamma_1 \gamma_3} \right) \sigma_+^a \sigma_-^b \right] - \text{H.c.} \right\}. \quad (\text{S76})$$

The total triplet for the system is then $G_{\text{tot}} = G_R \boxplus G_L$, with the components

$$\mathbf{S}_{\text{tot}} = \begin{bmatrix} e^{i(\varphi_1 + \varphi_2 + \varphi_3)} & 0 \\ 0 & e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \end{bmatrix}, \quad (\text{S77})$$

$$\mathbf{L}_{\text{tot}} = \begin{bmatrix} \left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_1}{2}} + \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^a + \left(e^{i(\varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_2}{2}} + e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^b \\ \left[\left(\sqrt{\frac{\gamma_1}{2}} + e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^a + \left(e^{i\varphi_1} \sqrt{\frac{\gamma_2}{2}} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^b \right] \end{bmatrix}, \quad (\text{S78})$$

$$H_{\text{tot}} = \left[\omega_a + \sqrt{\gamma_1 \gamma_4} \sin(\varphi_1 + \varphi_2 + \varphi_3) \right] \frac{\sigma_z^a}{2} + \left(\omega_b + \sqrt{\gamma_2 \gamma_3} \sin \varphi_2 \right) \frac{\sigma_z^b}{2} + \frac{1}{2} \left[\sqrt{\gamma_1 \gamma_2} \sin \varphi_1 + \sqrt{\gamma_1 \gamma_3} \sin(\varphi_1 + \varphi_2) + \sqrt{\gamma_2 \gamma_4} \sin(\varphi_2 + \varphi_3) + \sqrt{\gamma_3 \gamma_4} \sin \varphi_3 \right] (\sigma_-^a \sigma_+^b + \sigma_+^a \sigma_-^b). \quad (\text{S79})$$

Continuing from the total triplet, we arrive at the master equation in the same way as in previous calculations:

$$\begin{aligned}
\dot{\rho} &= -i[H_{\text{tot}}, \rho] + \mathcal{D} \left[\left(e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_1}{2}} + \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^a + \left(e^{i(\varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_2}{2}} + e^{i\varphi_3} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^b \right] \rho \\
&\quad + \mathcal{D} \left[\left(\sqrt{\frac{\gamma_1}{2}} + e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \sqrt{\frac{\gamma_4}{2}} \right) \sigma_-^a + \left(e^{i\varphi_1} \sqrt{\frac{\gamma_2}{2}} + e^{i(\varphi_1 + \varphi_2)} \sqrt{\frac{\gamma_3}{2}} \right) \sigma_-^b \right] \rho \\
&= -i[H_{\text{tot}}, \rho] + [\gamma_1 + \gamma_4 + 2\sqrt{\gamma_1\gamma_4} \cos(\varphi_1 + \varphi_2 + \varphi_3)] \mathcal{D}[\sigma_-^a] \rho + [\gamma_2 + \gamma_3 + 2\sqrt{\gamma_2\gamma_3} \cos \varphi_2] \mathcal{D}[\sigma_-^b] \rho \\
&\quad + [\sqrt{\gamma_1\gamma_2} \cos \varphi_1 + \sqrt{\gamma_1\gamma_3} \cos(\varphi_1 + \varphi_2) + \sqrt{\gamma_2\gamma_4} \cos(\varphi_2 + \varphi_3) + \sqrt{\gamma_3\gamma_4} \cos \varphi_3] \\
&\quad \times \left\{ \sigma_-^a \rho \sigma_+^b + \sigma_-^b \rho \sigma_+^a - \frac{1}{2} [(\sigma_+^a \sigma_-^b + \sigma_+^b \sigma_-^a) \rho + \rho (\sigma_+^a \sigma_-^b + \sigma_+^b \sigma_-^a)] \right\}. \tag{S80}
\end{aligned}$$

If we assume equal relaxation rates at all coupling points ($\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 \equiv \gamma$) and equal phases ($\varphi_1 = \varphi_2 = \varphi_3 \equiv \varphi$), Eq. (S80) reduces to Eq. (1) of the main text with the coefficients given in the sixth row of Table I.

D. Comparison of the setups

To enable a detailed comparison of the properties of the various setups for two small atoms and two giant atoms, we here reproduce Fig. 2 from the main text in a larger version as Fig. S7.

S4. MASTER EQUATION FOR MULTIPLE GIANT ATOMS

We now turn to the general case of N giant atoms, where each giant atom j is coupled to the waveguide at M_j points. The strength with which atom j couples to the waveguide at connection point j_n (coordinate x_{j_n}) is characterized by the relaxation rate γ_{j_n} . The positive phase acquired when moving from j_n to k_m is denoted φ_{j_n, k_m} . From the derivations for two giant atoms above, it is straightforward to generalize the SLH calculations to more atoms with more connection points. We first decompose the system into triplets for individual coupling points and phase shifts, interacting with right- and left-moving waves.

Looking at the right-moving part,

$$G_{\text{R}} = (\mathbf{S}_{\text{R}}, \mathbf{L}_{\text{R}}, H_{\text{R}}) = G_{N_{M_N}, \text{R}} \triangleleft G_{\varphi_{\dots, N_{M_N}}} \triangleleft \dots \triangleleft G_{\varphi_{1, \dots}} \triangleleft G_{1, \text{R}}, \tag{S81}$$

we first note that the scattering matrix simply picks up a phase factor as we go from one connection point to the next. The total phase acquired from the first connection point, 1_1 , to the last connection point, N_{M_N} , is $\varphi_{1_1, N_{M_N}}$. For \mathbf{L}_{R} , each connection point j_n contributes $\sqrt{\gamma_{j_n}/2} \sigma_-^{(j)}$, which then is multiplied by phase factors adding up to $\exp(i\varphi_{j_n, N_{M_N}})$ as we reach the last connection point. In H_{R} , each pair of connection points j_n and j_m ($x_{j_n} < x_{j_m}$, i.e., $n < m$) belonging to the same atom j contributes a term $\sqrt{\gamma_{j_n}\gamma_{j_m}} \sin(\varphi_{j_n, j_m}) \sigma_z^{(j)}/4$, and each pair of connection points j_n and k_m ($x_{j_n} < x_{k_m}$) belonging to different atoms j and k contributes a term $\frac{1}{4i} [\exp(i\varphi_{j_n, k_m}) \sqrt{\gamma_{j_n}\gamma_{k_m}} \sigma_-^{(j)} \sigma_+^{(k)} - \text{H.c.}]$. The result is

$$\mathbf{S}_{\text{R}} = \exp(i\varphi_{1_1, N_{M_N}}), \tag{S82}$$

$$\mathbf{L}_{\text{R}} = \sum_{j=1}^N \sum_{n=1}^{M_j} \exp(i\varphi_{j_n, N_{M_N}}) \sqrt{\frac{\gamma_{j_n}}{2}} \sigma_-^{(j)}, \tag{S83}$$

$$\begin{aligned}
H_{\text{R}} &= \sum_{j=1}^N \left(\omega_j + \sum_{n=1}^{M_j-1} \sum_{m=n+1}^{M_j} \frac{\sqrt{\gamma_{j_n}\gamma_{j_m}}}{2} \sin \varphi_{j_n, j_m} \right) \frac{\sigma_z^{(j)}}{2} \\
&\quad + \sum_{j=1}^N \sum_{k \neq j} \sum_{n=1}^{M_j} \sum_{\substack{m=1 \\ x_{j_n} < x_{k_m}}}^{M_k} \frac{1}{4i} \sqrt{\gamma_{j_n}\gamma_{k_m}} [\exp(i\varphi_{j_n, k_m}) \sigma_-^{(j)} \sigma_+^{(k)} - \text{H.c.}]. \tag{S84}
\end{aligned}$$

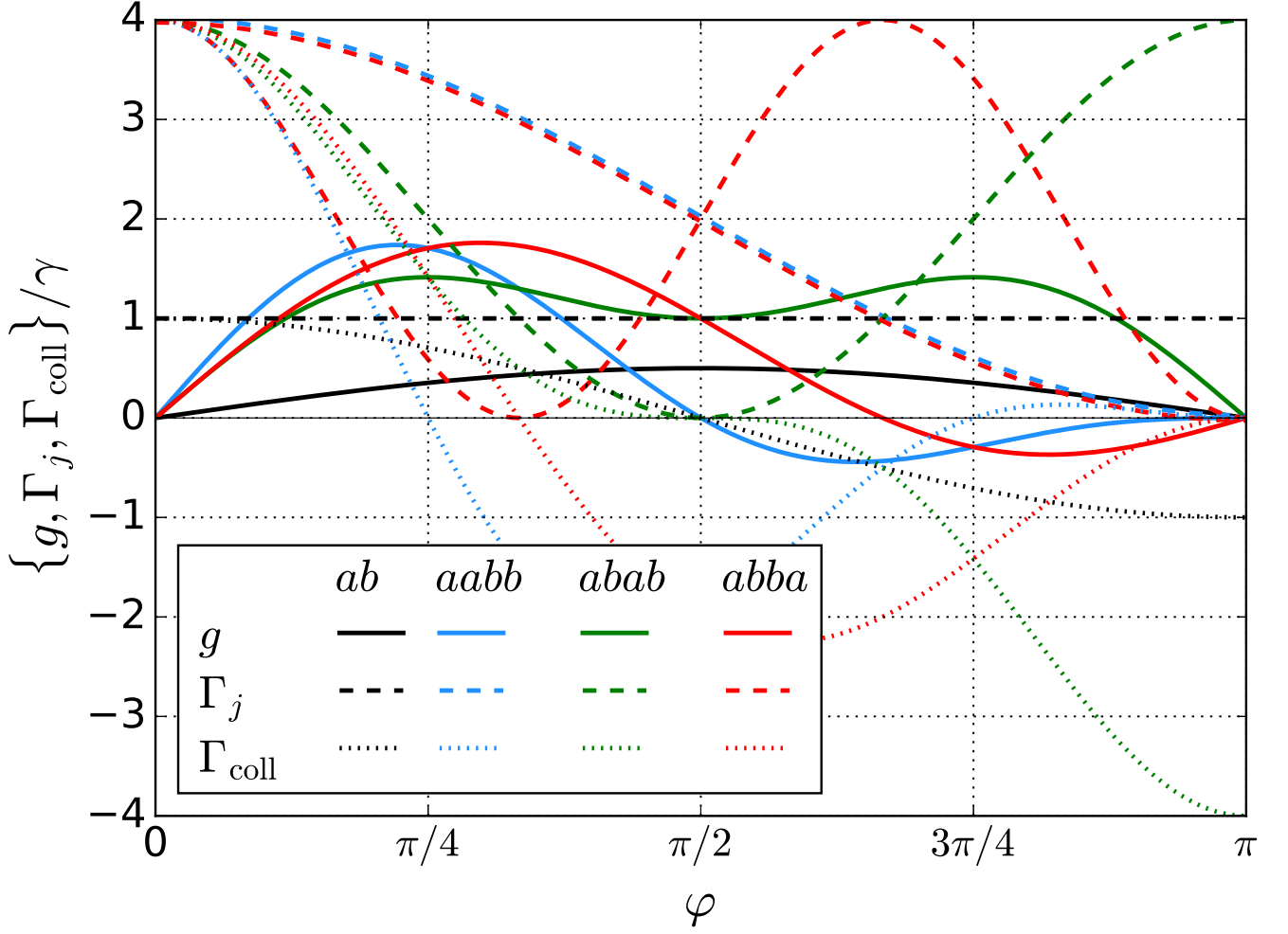


Figure S7. Exchange interaction (g , solid lines) and decoherence rates (Γ_j , individual: dashed lines; Γ_{coll} , collective: dotted lines) as a function of φ for the setups in Fig. 1 of the main text. The corresponding expressions are given in Table I of the main text. The labels ab (small atoms in an open waveguide, black), $aabb$ (separate giant atoms, blue), $abab$ (braided giant atoms, green), and $abba$ (nested giant atoms, red) correspond to the ordering of connection points for the two atoms. The case of small atoms in a semi-infinite waveguide [Fig. 1(b) in the main text] is not plotted separately, since it is qualitatively equivalent to the case of nested giant atoms. Note that there are two red dashed lines, one for Γ_a (overlapping the blue dashed line) and one for Γ_b .

From symmetry it follows that the components of the triplet G_L for the left-moving part are

$$\mathbf{S}_L = \exp(i\varphi_{1_1, N_{M_N}}), \quad (\text{S85})$$

$$\mathbf{L}_L = \sum_{j=1}^N \sum_{n=1}^{M_j} \exp(i\varphi_{1_1, j_n}) \sqrt{\frac{\gamma_{j_n}}{2}} \sigma_-^{(j)}, \quad (\text{S86})$$

$$\begin{aligned} H_L = & \sum_{j=1}^N \sum_{n=1}^{M_j-1} \sum_{m=n+1}^{M_j} \frac{\sqrt{\gamma_{j_n} \gamma_{j_m}}}{2} \sin \varphi_{j_n, j_m} \frac{\sigma_z^{(j)}}{2} \\ & + \sum_{j=1}^N \sum_{k \neq j} \sum_{n=1}^{M_j} \sum_{\substack{m=1 \\ x_{j_n} < x_{k_m}}}^{M_k} \frac{1}{4i} \sqrt{\gamma_{j_n} \gamma_{k_m}} \left[\exp(i\varphi_{j_n, k_m}) \sigma_+^{(j)} \sigma_-^{(k)} - \text{H.c.} \right]. \end{aligned} \quad (\text{S87})$$

The total triplet for the system is then $G_{\text{tot}} = G_{\text{R}} \boxplus G_{\text{L}}$, with the components

$$\mathbf{S}_{\text{tot}} = \begin{bmatrix} \exp(i\varphi_{1_1, N_{M_N}}) & 0 \\ 0 & \exp(i\varphi_{1_1, N_{M_N}}) \end{bmatrix}, \quad (\text{S88})$$

$$\mathbf{L}_{\text{tot}} = \begin{bmatrix} \sum_{j=1}^N \sum_{n=1}^{M_j} \exp(i\varphi_{j_n, N_{M_N}}) \sqrt{\frac{\gamma_{j_n}}{2}} \sigma_-^{(j)} \\ \sum_{j=1}^N \sum_{n=1}^{M_j} \exp(i\varphi_{1_1, j_n}) \sqrt{\frac{\gamma_{j_n}}{2}} \sigma_-^{(j)} \end{bmatrix}, \quad (\text{S89})$$

$$\begin{aligned} H_{\text{tot}} &= \sum_{j=1}^N \left(\omega_j + \sum_{n=1}^{M_j-1} \sum_{m=n+1}^{M_j} \sqrt{\gamma_{j_n} \gamma_{j_m}} \sin \varphi_{j_n, j_m} \right) \frac{\sigma_z^{(j)}}{2} \\ &\quad + \sum_{j=1}^{N-1} \sum_{k=j+1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \frac{\sqrt{\gamma_{j_n} \gamma_{k_m}}}{2} \sin \varphi_{j_n, k_m} \left(\sigma_-^{(j)} \sigma_+^{(k)} + \sigma_+^{(j)} \sigma_-^{(k)} \right). \end{aligned} \quad (\text{S90})$$

This leads to the master equation

$$\begin{aligned} \dot{\rho} &= -i[H_{\text{tot}}, \rho] + \mathcal{D} \left[\sum_{j=1}^N \sum_{n=1}^{M_j} \exp(i\varphi_{j_n, N_{M_N}}) \sqrt{\frac{\gamma_{j_n}}{2}} \sigma_-^{(j)} \right] \rho + \mathcal{D} \left[\sum_{j=1}^N \sum_{n=1}^{M_j} \exp(i\varphi_{1_1, j_n}) \sqrt{\frac{\gamma_{j_n}}{2}} \sigma_-^{(j)} \right] \rho \\ &= -i[H_{\text{tot}}, \rho] + \sum_{j=1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_j} \frac{\sqrt{\gamma_{j_n} \gamma_{j_m}}}{2} \left[\exp(i\varphi_{j_n, N_{M_N}}) \exp(-i\varphi_{j_m, N_{M_N}}) + \exp(i\varphi_{1_1, j_n}) \exp(-i\varphi_{1_1, j_m}) \right] \mathcal{D} \left[\sigma_-^{(j)} \right] \rho \\ &\quad + \sum_{j=1}^N \sum_{k \neq j}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \frac{\sqrt{\gamma_{j_n} \gamma_{k_m}}}{2} \left[\exp(i\varphi_{j_n, N_{M_N}}) \exp(-i\varphi_{k_m, N_{M_N}}) + \exp(i\varphi_{1_1, j_n}) \exp(-i\varphi_{1_1, k_m}) \right] \\ &\quad \times \left(\sigma_-^{(j)} \rho \sigma_+^{(k)} - \frac{1}{2} \left\{ \sigma_+^{(j)} \sigma_-^{(k)}, \rho \right\} \right) \\ &= -i[H_{\text{tot}}, \rho] + \sum_{j=1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_j} \sqrt{\gamma_{j_n} \gamma_{j_m}} \cos \varphi_{j_n, j_m} \mathcal{D} \left[\sigma_-^{(j)} \right] \rho \\ &\quad + \sum_{j=1}^{N-1} \sum_{k=j+1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \sqrt{\gamma_{j_n} \gamma_{k_m}} \cos \varphi_{j_n, k_m} \left[\left(\sigma_-^{(j)} \rho \sigma_+^{(k)} - \frac{1}{2} \left\{ \sigma_+^{(j)} \sigma_-^{(k)}, \rho \right\} \right) + \text{H.c.} \right], \end{aligned} \quad (\text{S91})$$

which is Eq. (2) of the main text. Note that the individual and collective decay terms also could be written together in a more compact form:

$$\begin{aligned} &\sum_{j=1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_j} \sqrt{\gamma_{j_n} \gamma_{j_m}} \cos \varphi_{j_n, j_m} \mathcal{D} \left[\sigma_-^{(j)} \right] \rho \\ &+ \sum_{j=1}^{N-1} \sum_{k=j+1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \sqrt{\gamma_{j_n} \gamma_{k_m}} \cos \varphi_{j_n, k_m} \left[\left(\sigma_-^{(j)} \rho \sigma_+^{(k)} - \frac{1}{2} \left\{ \sigma_+^{(j)} \sigma_-^{(k)}, \rho \right\} \right) + \text{H.c.} \right] \\ &= \sum_{j=1}^N \sum_{k=1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \sqrt{\gamma_{j_n} \gamma_{k_m}} \cos \varphi_{j_n, k_m} \left(\sigma_-^{(j)} \rho \sigma_+^{(k)} - \frac{1}{2} \left\{ \sigma_+^{(j)} \sigma_-^{(k)}, \rho \right\} \right). \end{aligned} \quad (\text{S92})$$

This should be compared to the master equation for N *small* atoms, each with individual relaxation rate γ_j , where the decay can be written as

$$\begin{aligned} &\sum_{j,k=1}^N \sqrt{\gamma_j \gamma_k} \cos \varphi_{j,k} \left(\sigma_-^{(j)} \rho \sigma_+^{(k)} - \frac{1}{2} \left\{ \sigma_+^{(j)} \sigma_-^{(k)}, \rho \right\} \right) \\ &= \sum_{j=1}^N \gamma_j \mathcal{D} \left[\sigma_-^{(j)} \right] \rho + \sum_{j \neq k} \sqrt{\gamma_j \gamma_k} \cos \varphi_{j,k} \left(\sigma_-^{(j)} \rho \sigma_+^{(k)} - \frac{1}{2} \left\{ \sigma_+^{(j)} \sigma_-^{(k)}, \rho \right\} \right). \end{aligned} \quad (\text{S93})$$

S5. INPUT-OUTPUT RELATIONS

For completeness, we note here that having access to the SLH components in Eqs. (S88)-(S90) makes it straightforward to derive the Hamiltonian and input-output relations in case a coherent drive or a Fock-state pulse is sent towards the giant atoms through the waveguide. For example, if a coherent drive at frequency ω_d with $|\alpha|^2$ photons per second is sent in from the left, the resulting triplet is

$$G = (\mathbf{S}, \mathbf{L}, H) = G_{\text{tot}} \triangleleft (G_\alpha \boxplus \mathbb{I}_1) = G_{\text{tot}} \triangleleft \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, 0 \right), \quad (\text{S94})$$

which leads to

$$\mathbf{S} = \begin{bmatrix} \exp(i\varphi_{11, N_{M_N}}) & 0 \\ 0 & \exp(i\varphi_{11, N_{M_N}}) \end{bmatrix}, \quad (\text{S95})$$

$$\mathbf{L} = \begin{bmatrix} \alpha \exp(i\varphi_{11, N_{M_N}}) + \sum_{j=1}^N \sum_{n=1}^{M_j} \exp(i\varphi_{j_n, N_{M_N}}) \sqrt{\frac{\gamma_{j_n}}{2}} \sigma_-^{(j)} \\ \sum_{j=1}^N \sum_{n=1}^{M_j} \exp(i\varphi_{11, j_n}) \sqrt{\frac{\gamma_{j_n}}{2}} \sigma_-^{(j)} \end{bmatrix}, \quad (\text{S96})$$

$$\begin{aligned} H = & \sum_{j=1}^N \left(\Delta_j + \sum_{n=1}^{M_j-1} \sum_{m=n+1}^{M_j} \sqrt{\gamma_{j_n} \gamma_{j_m}} \sin \varphi_{j_n, j_m} \right) \frac{\sigma_z^{(j)}}{2} \\ & + \sum_{j=1}^{N-1} \sum_{k=j+1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \frac{\sqrt{\gamma_{j_n} \gamma_{k_m}}}{2} \sin \varphi_{j_n, k_m} \left(\sigma_-^{(j)} \sigma_+^{(k)} + \sigma_+^{(j)} \sigma_-^{(k)} \right) \\ & + \frac{1}{2i} \left[\alpha \exp(i\varphi_{11, N_{M_N}}) \sum_{j=1}^N \sum_{n=1}^{M_j} \exp(-i\varphi_{j_n, N_{M_N}}) \sqrt{\frac{\gamma_{j_n}}{2}} \sigma_+^{(j)} - \text{H.c.} \right], \end{aligned} \quad (\text{S97})$$

where we have moved to a frame rotating with ω_d , introducing the notation $\Delta_j = \omega_j - \omega_d$. The master equation for the driven system then becomes

$$\begin{aligned} \dot{\rho} = & -i[H_{\text{driven}}, \rho] + \sum_{j=1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_j} \sqrt{\gamma_{j_n} \gamma_{j_m}} \cos \varphi_{j_n, j_m} \mathcal{D}[\sigma_-^{(j)}] \rho \\ & + \sum_{j=1}^{N-1} \sum_{k=j+1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \sqrt{\gamma_{j_n} \gamma_{k_m}} \cos \varphi_{j_n, k_m} \left[\left(\sigma_-^{(j)} \rho \sigma_+^{(k)} - \frac{1}{2} \{ \sigma_+^{(j)} \sigma_-^{(k)}, \rho \} \right) + \text{H.c.} \right], \end{aligned} \quad (\text{S98})$$

where

$$\begin{aligned} H_{\text{driven}} = & \sum_{j=1}^N \left(\Delta_j + \sum_{n=1}^{M_j-1} \sum_{m=n+1}^{M_j} \sqrt{\gamma_{j_n} \gamma_{j_m}} \sin \varphi_{j_n, j_m} \right) \frac{\sigma_z^{(j)}}{2} \\ & + \sum_{j=1}^{N-1} \sum_{k=j+1}^N \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \frac{\sqrt{\gamma_{j_n} \gamma_{k_m}}}{2} \sin \varphi_{j_n, k_m} \left(\sigma_-^{(j)} \sigma_+^{(k)} + \sigma_+^{(j)} \sigma_-^{(k)} \right) \\ & - i \sum_{j=1}^N \sum_{n=1}^{M_j} \sqrt{\frac{\gamma_{j_n}}{2}} \left[\alpha \exp(i\varphi_{11, j_n}) \sigma_+^{(j)} - \alpha^* \exp(-i\varphi_{11, j_n}) \sigma_-^{(j)} \right]. \end{aligned} \quad (\text{S99})$$

The output traveling to the right from the atoms is given by the upper row of \mathbf{L} in Eq. (S96) and the output traveling to the left is given by the lower row of \mathbf{L} in the same equation. From this, we can obtain the amplitude transmission coefficient t and the amplitude reflection coefficient r :

$$t = \frac{\langle L_1 \rangle}{\alpha} = \exp(i\varphi_{11, N_{M_N}}) + \frac{1}{\alpha} \sum_{j=1}^N \sum_{n=1}^{M_j} \exp(i\varphi_{j_n, N_{M_N}}) \sqrt{\frac{\gamma_{j_n}}{2}} \langle \sigma_-^{(j)} \rangle, \quad (\text{S100})$$

$$r = \frac{\langle L_2 \rangle}{\alpha} = \frac{1}{\alpha} \sum_{j=1}^N \sum_{n=1}^{M_j} \exp(i\varphi_{11, j_n}) \sqrt{\frac{\gamma_{j_n}}{2}} \langle \sigma_-^{(j)} \rangle. \quad (\text{S101})$$

S6. CONNECTION BETWEEN EXCHANGE INTERACTION, INDIVIDUAL DECAY, AND COLLECTIVE DECAY FOR GIANT ATOMS

In this section, we give detailed proofs for the statements in the main text regarding the connections between the exchange interaction g , the individual decays Γ_j , and the collective decays $\Gamma_{\text{coll},j,k}$. We first consider two giant atoms with two connection points each and then generalize to multiple atoms with multiple connections points.

A. Two atoms, two connection points

For two atoms with two connection points, there are three distinct geometries, shown in Fig. 1(c)-(e). Before treating these cases one by one, we note that the individual decay rates in all these setups can be written

$$\Gamma_j = \gamma_{j,1} + \gamma_{j,2} + 2\sqrt{\gamma_{j,1}\gamma_{j,2}} \cos \varphi_j, \quad (\text{S102})$$

where $\gamma_{j,1}$ and $\gamma_{j,2}$ are the relaxation rates characterizing the coupling of atom j to the waveguide at connection points 1 and 2, respectively, and φ_j is the phase acquired traveling between these two connection points. From the inequality between the arithmetic and geometric means,

$$\frac{\gamma_{j,1} + \gamma_{j,2}}{2} \geq \sqrt{\gamma_{j,1}\gamma_{j,2}}, \quad (\text{S103})$$

we see that Γ_j can only become zero if $\gamma_{j,1} = \gamma_{j,2}$. Since we are interested in exactly those cases where the decay rates for different setups go to zero, we will use $\gamma_{j,1} = \gamma_{j,2} \equiv \gamma_j$ throughout this subsection.

1. Separate giant atoms

For separate giant atoms, the master equation is given in Eq. (S40), from which we read off

$$g \propto \sqrt{\gamma_a \gamma_b} [\sin(\varphi_1 + \varphi_2) + \sin(\varphi_1 + \varphi_2 + \varphi_3) + \sin \varphi_2 + \sin(\varphi_2 + \varphi_3)] \quad (\text{S104})$$

$$\Gamma_a \propto \gamma_a (1 + \cos \varphi_1), \quad (\text{S105})$$

$$\Gamma_b \propto \gamma_b (1 + \cos \varphi_3), \quad (\text{S106})$$

$$\Gamma_{\text{coll}} \propto \sqrt{\gamma_a \gamma_b} [\cos(\varphi_1 + \varphi_2) + \cos(\varphi_1 + \varphi_2 + \varphi_3) + \cos \varphi_2 + \cos(\varphi_2 + \varphi_3)]. \quad (\text{S107})$$

If we set the individual decay terms to zero, this implies $\varphi_1 = (2n + 1)\pi$ and $\varphi_3 = (2m + 1)\pi$ with $n, m \in \mathbb{Z}_{\geq 0}$. Inserting this into the expression for the collective decay gives

$$\begin{aligned} \Gamma_{\text{coll}} &\propto \{\cos[(2n + 1)\pi + \varphi_2] + \cos[(2n + 1)\pi + \varphi_2 + (2m + 1)\pi] + \cos \varphi_2 + \cos[\varphi_2 + (2m + 1)\pi]\} \\ &= \cos(\varphi_2 + \pi) + \cos \varphi_2 + \cos \varphi_2 + \cos(\varphi_2 + \pi) = 2[\cos \varphi_2 + \cos(\varphi_2 + \pi)] = 2(\cos \varphi_2 - \cos \varphi_2) = 0. \end{aligned} \quad (\text{S108})$$

For the exchange interaction, we similarly obtain

$$\begin{aligned} g &\propto \{\sin[(2n + 1)\pi + \varphi_2] + \sin[(2n + 1)\pi + \varphi_2 + (2m + 1)\pi] + \sin \varphi_2 + \sin[\varphi_2 + (2m + 1)\pi]\} \\ &= \sin(\varphi_2 + \pi) + \sin \varphi_2 + \sin \varphi_2 + \sin(\varphi_2 + \pi) = 2[\sin \varphi_2 + \sin(\varphi_2 + \pi)] = 2(\sin \varphi_2 - \sin \varphi_2) = 0. \end{aligned} \quad (\text{S109})$$

Thus we conclude that if the individual decays are set to zero for separate giant atoms, *both* the collective decay and the exchange interaction will also be zero.

If we instead set the exchange interaction to zero, neither the individual nor the collective decays need become zero. A counter-example is provided by making all φ_j integer multiples of 2π ; this maximizes the individual and collective decays while the exchange interaction becomes zero.

If we set the collective decay to zero, neither the individual decays nor the exchange interaction need become zero. A counter-example is $\varphi_1 = \varphi_3 = 2\pi$ and $\varphi_2 = \pi/2$, which maximizes both the individual decays and the exchange interaction while making the collective decay zero.

2. Braided giant atoms

For separate giant atoms, the master equation is given in Eq. (S61), from which we read off

$$g \propto \sqrt{\gamma_a \gamma_b} [\sin \varphi_1 + \sin \varphi_2 + \sin \varphi_3 + \sin(\varphi_1 + \varphi_2 + \varphi_3)] \quad (\text{S110})$$

$$\Gamma_a \propto \gamma_a [1 + \cos(\varphi_1 + \varphi_2)], \quad (\text{S111})$$

$$\Gamma_b \propto \gamma_b [1 + \cos(\varphi_2 + \varphi_3)], \quad (\text{S112})$$

$$\Gamma_{\text{coll}} \propto \sqrt{\gamma_a \gamma_b} [\cos \varphi_1 + \cos \varphi_2 + \cos \varphi_3 + \cos(\varphi_1 + \varphi_2 + \varphi_3)]. \quad (\text{S113})$$

If we set the individual decay terms to zero, this implies $\varphi_1 + \varphi_2 = (2n+1)\pi$ and $\varphi_2 + \varphi_3 = (2m+1)\pi$ with $n, m \in \mathbb{Z}_{\geq 0}$. Solving in terms of φ_2 , we obtain $\varphi_1 = (2n+1)\pi - \varphi_2$ and $\varphi_3 = (2m+1)\pi - \varphi_2$, which inserted into the collective decay leads to

$$\Gamma_{\text{coll}} \propto [\cos(\pi - \varphi_2) + \cos(-\varphi_2) + \cos \varphi_2 + \cos(\pi - \varphi_2)] = 0. \quad (\text{S114})$$

For the exchange interaction, we similarly obtain

$$g \propto [\sin(\pi - \varphi_2) + \sin(-\varphi_2) + \sin \varphi_2 + \sin(\pi - \varphi_2)] = 2 \sin \varphi_2. \quad (\text{S115})$$

This is possibly *the most important result* of the present work. Since φ_2 is a free parameter (we only fix φ_1 and φ_3 to make the individual decays zero), we can engineer a *large* exchange interaction, and *choose its sign*, even though both the individual decays and the collective decay are zero.

If we instead set the exchange interaction to zero for the setup with braided giant atoms, neither the individual nor the collective decays need become zero. The same counter-example as for separate giant atoms works here as well: making all φ_j integer multiples of 2π maximizes the individual and collective decays while the exchange interaction becomes zero.

If we set the collective decay to zero, this does *not* mean that the exchange interaction or all of the individual decay will be zero. A counter-example is $\varphi_1 = 2\pi/3$, $\varphi_2 = \pi/2$, and $\varphi_3 = \pi/3$ which makes the collective decay zero, but the exchange interaction and the individual decays all become nonzero.

3. Nested giant atoms

For nested giant atoms, the master equation is given in Eq. (S80), from which we read off

$$g \propto \sqrt{\gamma_a \gamma_b} [\sin \varphi_1 + \sin(\varphi_1 + \varphi_2) + \sin(\varphi_2 + \varphi_3) + \sin \varphi_3] \quad (\text{S116})$$

$$\Gamma_a \propto \gamma_a [1 + \cos(\varphi_1 + \varphi_2 + \varphi_3)], \quad (\text{S117})$$

$$\Gamma_b \propto \gamma_b (1 + \cos \varphi_2), \quad (\text{S118})$$

$$\Gamma_{\text{coll}} \propto \sqrt{\gamma_a \gamma_b} [\cos \varphi_1 + \cos(\varphi_1 + \varphi_2) + \cos(\varphi_2 + \varphi_3) + \cos \varphi_3]. \quad (\text{S119})$$

If we set the individual decay terms to zero, this implies $\varphi_2 = (2n+1)\pi$ and $\varphi_1 + \varphi_2 + \varphi_3 = (2m+1)\pi$ with $n, m \in \mathbb{Z}_{\geq 0}$. Using these two constraints to express φ_3 in terms of φ_1 , the collective decay becomes

$$\Gamma_{\text{coll}} \propto [\cos \varphi_1 + \cos(\varphi_1 + \pi) + \cos(\pi - \varphi_1) + \cos(-\varphi_1)] = 0. \quad (\text{S120})$$

For the exchange interaction, we similarly obtain

$$g \propto [\sin \varphi_1 + \sin(\varphi_1 + \pi) + \sin(\pi - \varphi_1) + \sin(-\varphi_1)] = 0. \quad (\text{S121})$$

Thus we conclude that if the individual decays are set to zero for nested giant atoms, *both* the collective decay and the exchange interaction will also be zero.

If we instead set the exchange interaction to zero for the setup with nested giant atoms, neither the individual nor the collective decays need become zero. The same counter-example as for separate and braided giant atoms works here as well: making all φ_j integer multiples of 2π maximizes the individual and collective decays while the exchange interaction becomes zero.

If we set the collective decay to zero, this does not mean that the exchange interaction, nor all of the individual decay terms, will be zero. A simple counter-example is $\varphi_1 = \varphi_3 = \pi/2$ and $\varphi_2 = 2\pi$, which makes the collective decay zero, maximizes the exchange interaction, maximizes the individual decay of atom *b* (the inner atom) and makes the individual decay of atom *a* zero.

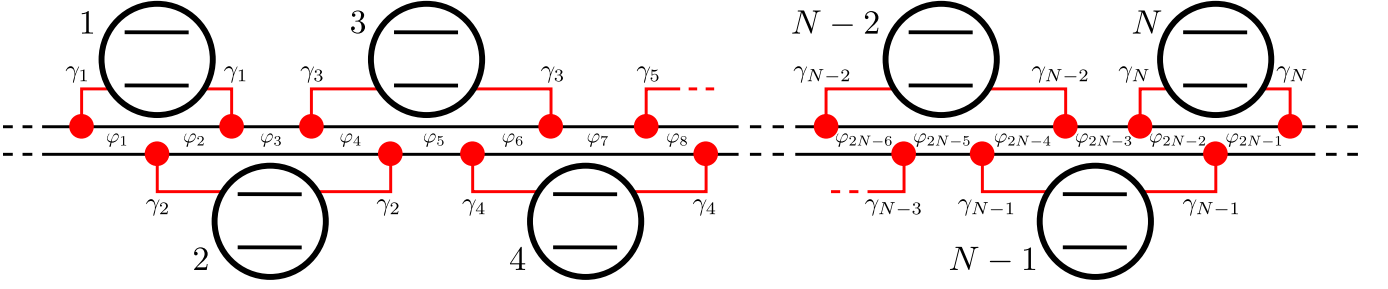


Figure S8. A sketch showing the relevant parameters for a 1D chain of giant atoms with nearest-neighbor couplings protected from relaxation.

B. Multiple atoms, two connection points

From Eq. (2) in the main text, we know that all interactions occur between pairs of atoms. If we have multiple atoms with two connection points each, we can consider two atoms at a time and check whether they are separate, braided or nested. When this classification has been determined, the conclusions from the previous subsection applies to the master-equation terms for this pair of atoms.

For the case of two braided atoms in the previous subsection, it was clear that g could be *set arbitrarily* through the free parameter φ_2 while using φ_1 and φ_3 to ensure that all the individual decay terms, and thus also the collective decay, remained zero. For more than two braided atoms, setting all decay terms to zero may introduce too many constraints to also allow for arbitrary control of all $g_{j,k}$. Below, we explicitly show whether this is the case for the two setups in Figs. 3 and 4 of the main text, i.e., for atoms connected in a 1D chain with nearest-neighbor couplings or in a setup allowing pairwise couplings between all atoms.

We briefly remark here on how our schemes for connecting multiple atoms compare to the setup with a quantum bus discussed in the introduction of the main text. In the scheme with a cavity bus, the atoms are all detuned from the cavity. Two atoms will interact with each other via virtual photons in the cavity if they are on resonance with each other. If three atoms are all on resonance with each other, they will have all possible pairwise interactions, similar to the scheme in Fig. 4 of the main text. However, when these three atoms are all on resonance, we cannot chose to only have certain pairwise interactions and not others (e.g., atom 1 interacting with atom 2, but not with atom 3). With our setup with giant atoms, however, this is possible, as shown by the scheme in Fig. 3 in the main text. We thus have greater freedom in designing the connectivity between atoms in our scheme with giant atoms than in the cavity-bus setup. Another way to put this is that a cavity bus erases information about the spatial ordering of the atoms, while our scheme does not, which means that we can realize a greater variety of connection geometries. We also have greater freedom in setting the strength of each pairwise coupling.

1. 1D chain with nearest-neighbor couplings

The setup for a 1D chain of N braided giant atoms is shown in Fig. S8 with all phases and coupling strengths marked. Just as before, we note that the individual decay rate Γ_j for the j th atom only can become zero if the couples to the waveguide with equal strengths, denoted γ_j , at each of its two connection points. The N constraints that make all individual decay rates zero are then

$$\varphi_1 + \varphi_2 = (2n_1 + 1)\pi, \quad (\text{S122})$$

$$\varphi_2 + \varphi_3 + \varphi_4 = (2n_2 + 1)\pi, \quad (\text{S123})$$

$$\varphi_4 + \varphi_5 + \varphi_6 = (2n_3 + 1)\pi, \quad (\text{S124})$$

\vdots

$$\varphi_{2N-6} + \varphi_{2N-5} + \varphi_{2N-4} = (2n_{N-2} + 1)\pi, \quad (\text{S125})$$

$$\varphi_{2N-4} + \varphi_{2N-3} + \varphi_{2N-2} = (2n_{N-1} + 1)\pi, \quad (\text{S126})$$

$$\varphi_{2N-2} + \varphi_{2N-1} = (2n_N + 1)\pi, \quad (\text{S127})$$

where $n_1, n_2, \dots, n_N \in \mathbb{Z}_{\geq 0}$. From the calculation for two braided giant atoms in Eq. (S115), we see that the nearest-neighbor couplings can be expressed as

$$g_{j,j+1} = \sqrt{\gamma_j \gamma_{j+1}} \sin \varphi_{2j}. \quad (\text{S128})$$

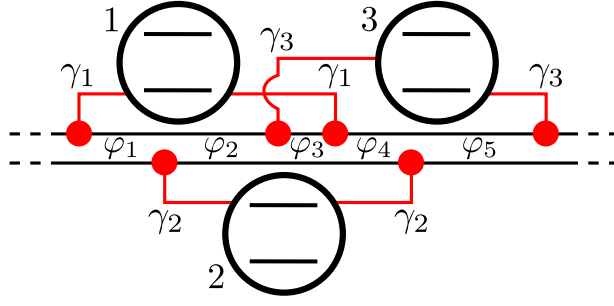


Figure S9. A sketch showing the relevant parameters for three giant atoms braided such that all possible pairs of atoms can achieve decoherence-free exchange interaction.

This implies that all the nearest-neighbor couplings can be tuned individually by choosing φ_{2j} for $j = 1, 2, \dots, N-1$. Since there are in total $2N-1$ phases, this leaves N free parameters, φ_{2j-1} for $j = 1, 2, \dots, N$, which is exactly what is needed to satisfy the constraints in Eqs. (S122)-(S127) and make all the individual decay rates zero.

2. Pairwise coupling between all atoms

If we braid more giant atoms (with two connection points each) than just the nearest neighbors in a 1D chain, as shown for three giant atoms in Fig. 4 in the main text and again with more detailed notation in Fig. S9, the number of constraints that need to be satisfied to ensure that all decay rates are zero remains unchanged (N constraints for N atoms). The number of connection points is still $2N$, so the number of phases that can be set is still $2N-1$. However, if all atoms are to be pairwise connected, the number of exchange couplings becomes $N(N-1)/2$, which, for $N \geq 3$, exceeds the number of free parameters, $N-1$, remaining after satisfying the constraints for zero relaxation. Thus, we will not be able to set all exchange-coupling strengths arbitrarily when the connectivity exceeds that of the 1D chain treated in the previous subsection.

For completeness, we here show explicitly some coupling-strength configurations that can be achieved for the setup with three pairwise connected atoms, shown in Fig. S9. The constraints ensuring zero relaxation through the waveguide are

$$\varphi_1 + \varphi_2 + \varphi_3 = (2n_1 + 1)\pi, \quad (\text{S129})$$

$$\varphi_2 + \varphi_3 + \varphi_4 = (2n_2 + 1)\pi, \quad (\text{S130})$$

$$\varphi_3 + \varphi_4 + \varphi_5 = (2n_3 + 1)\pi, \quad (\text{S131})$$

where $n_1, n_2, n_3 \in \mathbb{Z}_{\geq 0}$. The pairwise couplings are

$$g_{1,2} = \sqrt{\gamma_1 \gamma_2} \sin(\varphi_2 + \varphi_3), \quad (\text{S132})$$

$$g_{1,3} = \sqrt{\gamma_1 \gamma_3} \sin \varphi_3, \quad (\text{S133})$$

$$g_{2,3} = \sqrt{\gamma_2 \gamma_3} \sin(\varphi_3 + \varphi_4). \quad (\text{S134})$$

The constraints in Eqs. (S129) and (S131) can be satisfied without affecting the exchange interactions by choosing suitable φ_1 and φ_5 , respectively. If $\gamma_1 = \gamma_2 = \gamma_3$, we can make all pairwise couplings equal and satisfy the constraint in Eq. (S130) at the same time by choosing $\varphi_2 = \varphi_3 = \varphi_4 = \pi/3$. We can also satisfy the constraint in Eq. (S130) and engineer the couplings to obey $g_{1,2} = g_{2,3} = -g_{1,3}$ by choosing $\varphi_2 = \varphi_4 = 4\pi/3$ and $\varphi_3 = \pi/3$.

C. Multiple atoms, multiple connection points

For the general case, with N giant atoms, where atom j has M_j connection points, we can still divide each pair of atoms j and k into one of the three categories separate, braided, and nested. Explicitly, the atoms are separate if $x_{j_{M_j}} < x_{k_1}$, i.e., if all connection points of atom j are situated to the left of all connection points of atom k . The atoms are nested if, for some n , $x_{j_n} < x_{k_m} < x_{j_{n+1}} \forall m$, i.e., if all connection points of atom k are situated in-between two subsequent connection points of atom j . All other cases are braided atoms, where some of the connection points of atom j are situated in-between some of the connection points of atom k .

To prove the connections between exchange interaction, individual decay, and collective decay for these general cases, it is convenient to look back at the derivation of the general master equation in Sec. S4. We observe that the individual decay rate Γ_j of atom j can be expressed as [S5]

$$\Gamma_j = \left| \sum_{n=1}^{M_j} \sqrt{\gamma_{j_n}} \exp(i\varphi_{j_1, j_n}) \right|^2 = \left| \sum_{n=1}^{M_j} \sqrt{\gamma_{j_n}} \exp(i\varphi_{j_n, j_{M_j}}) \right|^2, \quad (\text{S135})$$

since

$$\sum_{n=1}^{M_j} \sqrt{\gamma_{j_n}} \exp(i\varphi_{j_1, j_n}) = \left\{ \exp(-i\varphi_{j_1, j_{M_j}}) \sum_{n=1}^{M_j} \sqrt{\gamma_{j_n}} \exp(i\varphi_{j_n, j_{M_j}}) \right\}^*. \quad (\text{S136})$$

This means that $\Gamma_j = 0$ implies $\sum_{n=1}^{M_j} \sqrt{\gamma_{j_n}} \exp(i\varphi_{j_1, j_n}) = 0$ and $\sum_{n=1}^{M_j} \sqrt{\gamma_{j_n}} \exp(i\varphi_{j_n, j_{M_j}}) = 0$.

From the derivation in Sec. S4, we see that the collective coupling can be written as

$$\begin{aligned} \Gamma_{\text{coll}, j, k} &= \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \frac{\sqrt{\gamma_{j_n} \gamma_{k_m}}}{2} \left[\exp(i\varphi_{j_n, N_{M_N}}) \exp(-i\varphi_{k_m, N_{M_N}}) + \exp(i\varphi_{1_1, j_n}) \exp(-i\varphi_{1_1, k_m}) \right] \\ &= \frac{1}{2} \exp(i\varphi_{j_{M_j}, N_{M_N}}) \underbrace{\sum_{n=1}^{M_j} \sqrt{\gamma_{j_n}} \exp(i\varphi_{j_n, j_{M_j}})}_{=0 \text{ if } \Gamma_j=0} \sum_{m=1}^{M_k} \sqrt{\gamma_{k_m}} \exp(-i\varphi_{k_m, N_{M_N}}) \\ &\quad + \frac{1}{2} \exp(i\varphi_{1_1, j_1}) \underbrace{\sum_{n=1}^{M_j} \sqrt{\gamma_{j_n}} \exp(i\varphi_{j_1, j_n})}_{=0 \text{ if } \Gamma_j=0} \sum_{m=1}^{M_k} \sqrt{\gamma_{k_m}} \exp(-i\varphi_{1_1, k_m}), \end{aligned} \quad (\text{S137})$$

and hence we conclude that $\Gamma_j = 0$ implies $\Gamma_{\text{coll}, j, k} = 0 \forall k$.

Finally, we study the exchange interaction $g_{j, k}$ for a pair of atoms where, without loss of generality, we assume

$x_{j_1} < x_{k_1}$. Using similar manipulations as above, it can be written

$$\begin{aligned}
g_{j,k} &= \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \frac{\sqrt{\gamma_{j_n} \gamma_{k_m}}}{2} \sin(\varphi_{j_n, k_m}) = \frac{1}{4i} \sum_{n=1}^{M_j} \sum_{m=1}^{M_k} \sqrt{\gamma_{j_n} \gamma_{k_m}} [\exp(i\varphi_{j_n, k_m}) - \exp(-i\varphi_{j_n, k_m})] \\
&= \frac{1}{4i} \sum_{n=1}^{M_j} \sum_{\substack{m=1 \\ x_{j_{M_j}} < x_{k_m}}}^{M_k} \left[\sqrt{\gamma_{j_n}} \exp(i\varphi_{j_n, j_{M_j}}) \sqrt{\gamma_{k_m}} \exp(i\varphi_{j_{M_j}, k_m}) - \text{H.c.} \right] \\
&\quad + \frac{1}{4i} \sum_{n=1}^{M_j} \sum_{\substack{m=1 \\ x_{j_{M_j}} > x_{k_m} > x_{j_n}}}^{M_k} \left[\sqrt{\gamma_{j_n}} \exp(i\varphi_{j_n, j_{M_j}}) \sqrt{\gamma_{k_m}} \exp(-i\varphi_{j_{M_j}, k_m}) - \text{H.c.} \right] \\
&\quad + \frac{1}{4i} \sum_{n=1}^{M_j} \sum_{\substack{m=1 \\ x_{j_n} > x_{k_m}}}^{M_k} \left[\sqrt{\gamma_{j_n}} \exp(-i\varphi_{j_n, j_{M_j}}) \sqrt{\gamma_{k_m}} \exp(i\varphi_{j_{M_j}, k_m}) - \text{H.c.} \right] \\
&= \frac{1}{4i} \sum_{n=1}^{M_j} \sum_{\substack{m=1 \\ x_{j_{M_j}} < x_{k_m}}}^{M_k} \left[\sqrt{\gamma_{j_n}} \exp(i\varphi_{j_n, j_{M_j}}) \sqrt{\gamma_{k_m}} \exp(i\varphi_{j_{M_j}, k_m}) - \text{H.c.} \right] \\
&\quad + \frac{1}{4i} \sum_{n=1}^{M_j} \sum_{\substack{m=1 \\ x_{j_{M_j}} > x_{k_m}}}^{M_k} \left[\sqrt{\gamma_{j_n}} \exp(i\varphi_{j_n, j_{M_j}}) \sqrt{\gamma_{k_m}} \exp(-i\varphi_{j_{M_j}, k_m}) - \text{H.c.} \right] \\
&\quad + 2 \times \frac{1}{4i} \sum_{n=1}^{M_j} \sum_{\substack{m=1 \\ x_{j_n} > x_{k_m}}}^{M_k} \left[\sqrt{\gamma_{j_n}} \exp(-i\varphi_{j_n, j_{M_j}}) \sqrt{\gamma_{k_m}} \exp(i\varphi_{j_{M_j}, k_m}) - \text{H.c.} \right] \\
&= \frac{1}{4i} \underbrace{\sum_{n=1}^{M_j} \sqrt{\gamma_{j_n}} \exp(i\varphi_{j_n, j_{M_j}})}_{=0 \text{ if } \Gamma_j=0} \left[\sum_{\substack{m=1 \\ x_{j_{M_j}} < x_{k_m}}}^{M_k} \sqrt{\gamma_{k_m}} \exp(i\varphi_{j_{M_j}, k_m}) - \text{H.c.} + \sum_{\substack{m=1 \\ x_{j_{M_j}} > x_{k_m}}}^{M_k} \sqrt{\gamma_{k_m}} \exp(-i\varphi_{j_{M_j}, k_m}) - \text{H.c.} \right] \\
&\quad + \sum_{n=1}^{M_j} \sum_{\substack{m=1 \\ x_{j_n} > x_{k_m}}}^{M_k} \sqrt{\gamma_{j_n} \gamma_{k_m}} \sin(\varphi_{j_n, k_m}). \tag{S138}
\end{aligned}$$

If the atoms are separate, $x_{j_{M_j}} < x_{k_1}$, and thus the second line of the final expression in Eq. (S138) contain no terms, which means that $\Gamma_j = 0$ implies $g_{j,k} = 0$ in this case.

If the atoms are nested, $x_{j_p} < x_{k_1} < x_{k_{M_k}} < x_{j_{p+1}}$ ($p < M_j$), the second line of the final expression in Eq. (S138) can be rewritten as

$$\sum_{n=1}^{M_j} \sum_{\substack{m=1 \\ x_{j_n} > x_{k_m}}}^{M_k} \sqrt{\gamma_{j_n} \gamma_{k_m}} \sin(\varphi_{j_n, k_m}) = \sum_{n=p+1}^{M_j} \sum_{m=1}^{M_k} \sqrt{\gamma_{j_n} \gamma_{k_m}} \sin(\varphi_{j_n, k_m}), \tag{S139}$$

which we recognize as the expression for the interaction between two separate giant atoms, where the first giant atom is atom k and the second giant atom is atom j only connected at the connection points from j_{p+1} to j_{M_j} . We can thus conclude that $\Gamma_j = 0$ and $\Gamma_k = 0$ together imply that $g_{j,k} = 0$ for the case of nested atoms.

If the atoms are braided, the second line of the final expression in Eq. (S138) may be nonzero. The implications of setting all individual decay terms to zero in the three different geometries are thus the same for giant atoms with an arbitrary number of connection points as for giant atoms with two connection points.

As for the implications of setting the collective decay or the exchange interaction to zero, we have already shown for the case of giant atoms with two connection points that such a condition does not necessarily have to make any

other terms zero. Thus, that is also true for giant atoms with an arbitrary number of connection points.

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