Tunable Majorana fermion from Landau quantization in 2D topological superconductors

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We study Majorana fermions in a two-dimensional topological superconductor placed in a transverse magnetic-field B. We consider a topological insulator/superconductor heterostructure and a two-dimensional p-wave superconductor. A single field-generated vortex creates two Majorana fermions, one of which is hosted at the vortex core. The wave function of the second Majorana state is localized in the superconductor volume along a circle with a radius of $r^* \propto B^{-1}$ centered at the vortex core. In the case of many vortices, the sensitivity of r^* to the magnetic field B may be used to control the coupling between the Majorana fermions. The latter property could be an asset for quantum computations.

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I. INTRODUCTION

Majorana fermions (MFs) in condensed matter became the focus of many studies [1–5], especially in connection with the future possibility of topologically protected computation [6,7]. A necessary operation for such a computation is braiding, which could be performed [8–10] by tuning the pairwise interaction between the MFs ("nontopological" qubit operations [7] were also discussed in the literature, Refs. [11–16]). Different structures have been proposed as possible hosts for MFs [17–28]. Preliminary experimental hints of Majorana states were observed in a nanowire with strong spin-orbit coupling on top of a superconductor [29], in a ferromagnetic atoms chain [30] and, most recently, in a topological insulator/superconductor (TI/SC) heterostructure [31].

It is generally assumed that MFs localize at some heterogeneity separating media with different topological numbers, which may be a vortex core in a superconductor or superfluid, sample boundaries, interfaces in heterostructures, etc. (see, e.g., Refs. [1,32]). The attachment of the MF to such a physical "defect" may be disadvantageous for a number of reasons: The presence of physical inhomogeneities in the system requires additional effort at the fabrication stage, it introduces extra disorder, and creates interfaces, whose properties are difficult to control. In particular, binding a MF to some point in space limits its control and manipulation options.

However, it is not a general law that MFs must be localized near some "topological defect." Below we consider two models of a two-dimensional (2D) topological superconductor hosting a vortex. Systems of this type have been studied experimentally [33,34], and hints of a Majorana state at the vortex core were reported [31]. From the theoretical standpoint, these models are particularly interesting because, as will be demonstrated below, they may serve as a platform where a confinement mechanism unrelated to a physical heterogeneity is realized. The MFs can arise only in pairs since only a superposition of two MFs has a physical sense [2]. Therefore, in addition to the well-studied MF at the core [22,35–37], a second exterior Majorana state can emerge [28]. The second MF is not necessarily pinned by some interface or sample edge. We show that it can be localized by a finite magnetic field. For a uniform field B, the wave function of the exterior MF can be calculated exactly. Its weight is centered at some field-dependent mesoscopic radius $r^* \propto B^{-1}$, away from the vortex center. In other words, the wave function is localized at a circle with a radius of r^*. Varying the magnetic field, we may control r^*. This feature could be useful for the manipulation of Majorana states. We will analyze the tunability of the pairwise splitting between MFs in the case of two vortices.

The paper is organized as follows. In Sec. II we discuss the generation of the Majorana fermions in the topological insulator-superconductor heterostructure. In Sec. III similar ideas are applied to a p-wave superconductor. The discussion and conclusions are in Secs. IV and V, respectively.

II. TI/SC HETEROSTRUCTURE

Given the recent success [31] in the fabrication of TI/SC heterostructures, let us discuss this system first. Our model describes the 2D states at the surface of the TI. Proximity to the superconductor induces superconducting correlations in these states. An external magnetic field B inserts a vortex with integer vorticity l into the system, which is schematically shown in Fig. 1. The corresponding Hamiltonian [17,20,22,27] is a 4×4 matrix,

$$H = (v_{\rm F}\boldsymbol{\sigma}\cdot\mathbf{p} - U)\tau_z - \frac{ev_{\rm F}}{c}(\boldsymbol{\sigma}\cdot\mathbf{A})\tau_0 + |\Delta(r)|\tau_x e^{il\phi\tau_z}, \quad (1)$$

acting in a space of bispinor wave functions,

$$\psi = (u_\uparrow, u_\downarrow, v_\downarrow, -v_\downarrow)^T. \quad (2)$$

Above, $\mathbf{p}$ is the momentum operator, $\boldsymbol{\sigma}$ is the vector of the three Pauli matrices acting in spin space, τ_j's are the Pauli matrices acting in the charge space, U is the shift of the Fermi level from the Dirac point, and $v_{\rm F}$ is the Fermi velocity in the TI surface. We introduce polar coordinates (r,ϕ) with the origin at the vortex core. In these coordinates, the proximity-induced superconducting order parameter can be expressed as

$$\Delta(\mathbf{r}) = |\Delta| f(r) e^{-il\phi}. \quad (3)$$

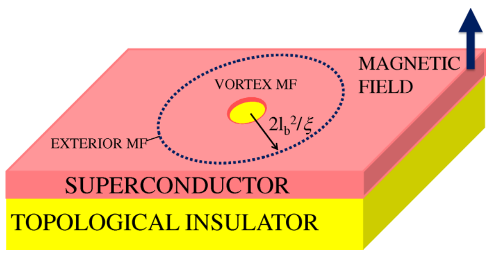

FIG. 1. Topological insulator/superconductor heterostructure with a vortex. A layer of a topological insulator (yellow) is covered by a superconducting film (pink) with a s-wave order parameter. The yellow hole in the superconductor represents the vortex with a core-localized Majorana fermion. The dashed line corresponds to the exterior Majorana fermion.

The phase of $\Delta(\mathbf{r})$ changes by $2\pi l$ around the origin. The absolute value $|\Delta(\mathbf{r})| = |\Delta| f(r)$ varies with r: The function $f(r)$ vanishes if $r \ll \xi_{\rm SC}$ and approaches unity when $r \gg \xi_{\rm SC}$, where $\xi_{\rm SC}$ is the coherence length inside the superconductor. We assume that the magnetic field is homogeneous (this is possible when the superconducting layer is sufficiently thin). Below, the vector potential is chosen as

$$A_\phi = -\frac{Br}{2}, \quad A_r = A_z = 0. \tag{4}$$

The Hamiltonian H possesses particle-hole symmetry: Using the complex conjugation operator K one defines the particle-hole conjugation operator,

$$\Xi = \sigma_y \tau_y K \quad \text{such that} \quad \Xi H \Xi = -H. \tag{5}$$

Thus, for any eigenfunction ψ_ε, satisfying $H\psi_\varepsilon = \varepsilon\psi_\varepsilon$, there is another eigenfunction $\psi_{-\varepsilon} = \Xi\psi_\varepsilon$ with eigenenergy $-\varepsilon$. Since the MF wave function satisfies

$$\Xi\psi_{\rm MF} = \psi_{\rm MF}, \tag{6}$$

the corresponding eigenenergy is zero $\varepsilon_{\rm MF} = 0$.

We introduce a spinor $F^\mu(r) = (f_1^\mu, f_2^\mu, f_3^\mu, -f_4^\mu)^T$ as

$$\psi = \exp[-i\phi(l\tau_z - \sigma_z)/2 + i\mu\phi]F^\mu(r), \tag{7}$$

where the index μ represents the total angular momentum of the state. In the case of a vortex with single vorticity ($l = 1$), the transformation (7) is well defined only when μ is an integer. We can express the equation $H\psi = \varepsilon\psi$ as [28]

$$\begin{aligned}
i\left(\frac{d}{dr} + \frac{2\mu+l+1}{2r} - \frac{r}{2l_{\rm B}^2}\right)f_2^\mu + \frac{\Delta}{\hbar v_{\rm F}}f_3^\mu &= \left(\frac{\varepsilon+U}{\hbar v_{\rm F}}\right)f_1^\mu,\\
i\left(\frac{d}{dr} - \frac{2\mu+l-1}{2r} + \frac{r}{2l_{\rm B}^2}\right)f_1^\mu - \frac{\Delta}{\hbar v_{\rm F}}f_4^\mu &= \left(\frac{\varepsilon+U}{\hbar v_{\rm F}}\right)f_2^\mu,\\
i\left(\frac{d}{dr} + \frac{2\mu-l+1}{2r} + \frac{r}{2l_{\rm B}^2}\right)f_4^\mu + \frac{\Delta}{\hbar v_{\rm F}}f_1^\mu &= \left(\frac{\varepsilon-U}{\hbar v_{\rm F}}\right)f_3^\mu,\\
i\left(\frac{d}{dr} - \frac{2\mu-l-1}{2r} - \frac{r}{2l_{\rm B}^2}\right)f_3^\mu - \frac{\Delta}{\hbar v_{\rm F}}f_2^\mu &= \left(\frac{\varepsilon-U}{\hbar v_{\rm F}}\right)f_4^\mu,
\end{aligned} \tag{8}$$

where $l_{\rm B}^2 = \hbar c/eB$ is the magnetic length. Majorana states exist only when [27,28] $\varepsilon = 0$ and $\mu = 0$. If ε and μ vanish, we introduce the following linear combinations:

$$\begin{aligned}
X_1 &= if_1^0 + f_4^0, \quad X_2 = if_1^0 - f_4^0,\\
Y_1 &= if_2^0 + f_3^0, \quad Y_2 = if_2^0 - f_3^0
\end{aligned} \tag{9}$$

for which Eqs. (8) decouple into two independent systems of equations for (X_1,Y_2) and (X_2,Y_1). Each of these systems can be reduced to the following equation:

$$\chi'' + \frac{\chi'}{\bar r} + \chi\left(\bar U^2 - \frac{1}{\bar r^2} - \frac{\bar r^2}{4}\right) = 0, \tag{10}$$

where the dimensionless variables,

$$\bar r = \frac{r}{l_{\rm B}}, \quad \bar U = \frac{U}{\hbar\omega_{\rm c}} \tag{11}$$

are used, and the quantity,

$$\omega_{\rm c} = \frac{v_{\rm F}}{l_{\rm B}} \tag{12}$$

may be viewed as the cyclotron frequency for a massless relativistic particle. Function χ is connected to $Y_{1,2}$ as follows:

$$Y_1 = \chi(\bar r)\exp\left[-\alpha\int_0^{\bar r} f(\bar r')d\bar r'\right], \tag{13}$$

$$Y_2 = \chi(\bar r)\exp\left[\alpha\int_0^{\bar r} f(\bar r')d\bar r'\right], \tag{14}$$

where

$$\alpha = \frac{l_{\rm B}}{\xi} \quad \text{and} \quad \xi = \frac{\hbar v_{\rm F}}{|\Delta|}. \tag{15}$$

The length scale ξ is the familiar coherence length due to the proximity effect. The substitution,

$$\chi(\bar r) = \frac{g(\bar r^2/2)}{\bar r} \tag{16}$$

transforms Eq. (10) into a one-dimensional Schrödinger equation describing a quantum particle in an attractive Coulomb potential. Using this analogy, it it easy to check that Eq. (10) has a normalizable solution, provided that

$$\bar U^2 = 2N \geqslant 0, \quad N \text{ is any non-negative integer.} \tag{17}$$

When these conditions hold, we solve Eq. (10), and using Eqs. (7)–(9) we obtain the solution for the MF localized near the vortex core in the form

$$\psi_{\rm v} = B_{\rm v}\exp\left[-\frac{i\pi}{4} - \frac{r^2}{4l_{\rm B}^2} - \int_0^r f(r')\frac{dr'}{\xi}\right]\Psi_{\rm MF}, \tag{18}$$

$$\Psi_{\rm MF}(r,\phi) = \begin{bmatrix} L_N^{(0)}(\bar r^2/2)\\ -i(\bar r/\bar U)L_{N-1}^{(1)}(\bar r^2/2)e^{i\phi}\\ (\bar r/\bar U)L_{N-1}^{(1)}(\bar r^2/2)e^{-i\phi}\\ iL_N^{(0)}(\bar r^2/2)\end{bmatrix}, \tag{19}$$

where $L_m^{(n)}(y)$'s are generalized Laguerre polynomials [38] and the normalization coefficient $B_{\rm v}$ is real. Note, if

$$\bar U = N = 0, \quad \text{then } L_{-1}^{(1)}/\bar U \equiv 0. \tag{20}$$

To prove that $\psi_{\rm v}$ is a MF, it is enough to check that $\Xi\psi_{\rm v} = \psi_{\rm v}$.

The solution ψ_v of Eq. (18) corresponds to Eq. (13). The solution that corresponds to Eq. (14) describes yet another MF state,

$$\psi_e = B_e \exp\left[\frac{i\pi}{4} - \frac{r^2}{4l_B^2} + \int_0^r f(r')\frac{dr'}{\xi}\right]\tau_z \Psi_{MF}. \quad (21)$$

In this expression Ψ_{MF} is defined by Eq. (19), and the coefficient B_e is real. If N is not too large, the maximum of the wave function weight $|\psi_e|^2$ is at

$$r \sim r^* = \frac{2l_B^2}{\xi} = \frac{2c\Delta}{ev_F B}. \quad (22)$$

This means that Majorana fermion ψ_e localizes along a circle centered at the vortex core with field-dependent radius $r^*(B)$. We will refer to this state as the exterior MF (thus, the subscript "e").

There is an important distinction between these two MFs. The state at the vortex core ψ_v is localized by "the vorticity." That is, as long as the vortex is present, the wave function ψ_v remains normalizable even for a zero magnetic field ($l_B \to \infty$) with its weight mostly confined within a circle of a radius of $\sim\xi$, the latter quantity being field independent. Because of this, it is permissible to neglect $\mathbf{A}$ in the model Hamiltonian. This is a common approximation used in the literature dedicated to the core-bound Majorana fermion [20,22]. However, including the magnetic field B into the model is of central importance to study the exterior MF ψ_e since the MF weight concentrates mostly at a field-dependent radius of $r^* \sim 1/B$. If B decreases, the radius r^* grows. Eventually, at some very weak field, the exterior state recedes to the outer boundaries of the system.

When the value of $\bar{U}$ violates condition (17), strictly speaking, the Majorana states disappear. Let us consider a weak violation of (17): $\bar{U} = \sqrt{2N} + \delta\bar{U}$, where $\delta\bar{U} = \delta U/\hbar\omega_c$. Treating the term $\delta U \tau_z$ as a perturbation, one can evaluate the matrix element $\delta U\langle\psi_v|\tau_z|\psi_e\rangle$ and obtain the corresponding energy splitting,

$$\delta E \approx \delta U \sqrt{\frac{l_B}{\xi}} \exp\left(-\frac{l_B^2}{\xi^2}\right). \quad (23)$$

This value characterizes the hybridization between the core and the exterior MF states for nonzero $\delta\bar{U}$. Because of this hybridization, two MFs are replaced by a single Dirac fermion with energy δE. The eigenenergy is an oscillating function of U, vanishing each time when condition (17) is met. In the limit $\xi \ll r^*$ ($\xi \ll l_B$), the overlap between the Majorana wave functions is exponentially small, and the hybridization may be neglected.

These overlap oscillations are not uncommon in the Majorana fermion physics: Similar phenomena were discussed in other systems as well [39,40]. It is interesting that single-parameter tuning is sufficient to nullify the hybridization between the MFs. This is unlike a common two-level system, which avoids level crossing unless multiparameter fine-tuning is performed. Such a deviation from a generic behavior occurs because a single (real) parameter t_{12} is enough for the complete specification of the most general Hamiltonian,

$$H_{12} = it_{12}\hat{\gamma}_1\hat{\gamma}_2, \quad (24)$$

describing two coupled Majorana fermions $\hat{\gamma}_{1,2}$. Thus, the single-parameter fine-tuning condition $t_{12} = 0$ is sufficient to guarantee the nullification of the Hamiltonian and the resultant level crossing at zero eigenenergy.

III. SPINLESS TWO-DIMENSIONAL *p*-WAVE SUPERCONDUCTOR

The two-dimensional p-wave superconductor [41] Sr_2RuO_4 is another system where an exterior MF can emerge. The relevant description is very similar to the calculations presented above. We write down the model's Hamiltonian in the form [1,39]

$$H = \begin{pmatrix} \frac{(i\hbar\nabla - e\mathbf{A}/c)^2}{2m} - U & -\frac{i\hbar}{p_F}\{\Delta,\partial_z\} \\ \frac{i\hbar}{p_F}\{\Delta^*,\partial_z^*\} & -\frac{(i\hbar\nabla + e\mathbf{A}/c)^2}{2m} + U \end{pmatrix}. \quad (25)$$

Here m is the electron mass, $p_F = mv_F$ is the Fermi momentum, and the anticommutator is defined in the usual manner: $\{\Delta,\partial_z\} = \Delta\partial_z + \partial_z\Delta$, where

$$\partial_z = e^{i\phi}(\partial_r + ir^{-1}\partial_\phi). \quad (26)$$

The order parameter, as before, is given by Eq. (3). For the case of a single vortex ($l = 1$) we seek zero-energy eigenfunctions in the form

$$\psi_{MF} = (e^{i(\phi-\pi/4)}u_{MF}, e^{-i(\phi-\pi/4)}v_{MF})^T. \quad (27)$$

Similar to previous considerations, we derive a system of differential equations for the radial part of ψ_{MF}.

Introducing the particle-hole conjugation operator for the p-wave superconductor as $\Xi = \tau_x K$, one can prove that the Hamiltonian in Eq. (25) possesses particle-hole symmetry: $\Xi H \Xi = -H$. This property, together with the fact that the differential equations for the radial part of the wave function are real, implies that the MF radial wave function satisfies the conditions $u_{MF} = \lambda v_{MF}$, where $\lambda = \pm 1$. Using these relations we derive the differential equation for function χ,

$$\chi'' + \frac{\chi'}{\bar{r}} + \chi\left(\bar{U} - \alpha^2 f^2 - \frac{1}{\bar{r}^2} - \frac{\bar{r}^2}{4} - \frac{1}{2}\right) = 0, \quad (28)$$

which is connected to the MF wave function as follows:

$$u_{MF} = \chi(\bar{r})\exp\left[\lambda\alpha\int_0^{\bar{r}} f(\bar{r}')d\bar{r}'\right], \quad (29)$$

where

$$\bar{r} = \frac{r}{l_B}, \quad \bar{U} = \frac{2ml_B^2 U}{\hbar^2} = \frac{2U}{\hbar\omega_c}, \quad (30)$$

and the cyclotron frequency for a massive particle equals $\omega_c = eB/mc$.

Approximating $f(r) \approx 1$ we notice that Eqs. (28) and (10) have the same structure. Exploiting this, one can prove that the Hamiltonian (25) admits two MF solutions if

$$\bar{U} - \alpha^2 - \frac{1}{2} = \frac{2U}{\hbar\omega_c} - \frac{l_B^2}{\xi^2} - \frac{1}{2} = 2N, \quad (31)$$

where N is a non-negative integer. From Eq. (29) it is easy to see that the core-localized MF corresponds to $\lambda = -1$ whereas the exterior MF corresponds to $\lambda = 1$. Up to a normalization coefficient, the MF wave functions are

$$\psi_{\mathrm{v,e}} = \frac{r}{2l_{\mathrm{B}}^2} \exp\left(\frac{\lambda\Delta}{v_{\mathrm{F}}}\int_0^r f\, dr' - \frac{r^2}{4l_{\mathrm{B}}^2}\right) L_N^{(-1)} \times \begin{pmatrix} e^{i[\phi-(\pi/4)]} \\ \lambda e^{-i[\phi-(\pi/4)]} \end{pmatrix}. \quad (32)$$

One can verify that $\Xi\psi_{\mathrm{v,e}} = \lambda\psi_{\mathrm{v,e}}$ and both MF wave functions are orthogonal to each other. Hence, $\psi_{\mathrm{v,e}}$'s are two independent MFs. When the condition (31) is violated, the MFs are hybridized, forming a single Dirac electron as in the previously discussed case of the TI/SC heterostructure.

We assumed above that $f(r) \approx 1$ whereas the function f deviates from unity near the core. This deviation may be accounted for by using perturbation theory. As a result, the condition (31) and the wave functions (32) will be slightly corrected.

IV. DISCUSSION

It is typically assumed that, in order to localize a MF, one needs a boundary separating two parts of the system with different topological numbers or some topological defect. The two models considered above serve as counterexamples to this statement: We have shown that the exterior Majorana wave function is not "latched" to any inhomogeneity. Instead, the localization radius r^* is a magnetic-field-dependent quantity and may be manipulated in real time.

The latter feature allows us to control the coupling between the MFs. Let two vortices be pinned at a distance R from each other. Each vortex hosts a core-localized state and exterior Majorana states. Straightforward calculations show that the splitting between the exterior MF of the first vortex and the core MF of the second vortex is zero. However, the splitting between the exterior MFs of different vortices is nonzero and depends on B. When the magnetic field is high such that $R \gg r^* = 2l_{\mathrm{B}}^2/\xi \sim 1/B$, the coupling between two exterior MFs is exponentially weak and can be neglected. With the decrease in B the localization radius r^* grows and so does the coupling between the MFs.

Changing the hybridization between the MFs allows one to perform braiding without moving the vortices. Although the topological computations are not the main topic of this paper, we briefly outline a possible braiding protocol similar to the one proposed in Ref. [9]. Consider first the motion of a single MF. Initially ($t = 0$) we have one separate MF γ_1 and one Dirac fermion, the latter consisting of two coupled MFs γ_2 and γ_3. Transportation of MF γ_1 from its initial position to the position of γ_3 can be described by the Hamiltonian,

$$H(t) = \zeta_{12}\alpha(t)\gamma_1\gamma_2 + \zeta_{23}[1-\alpha(t)]\gamma_2\gamma_3, \quad (33)$$

where ζ_{ij}'s are the tunneling amplitudes between the ith and the jth MFs. The coefficient $\alpha(t)$ changes adiabatically from $\alpha(0) = 0$ to $\alpha(t_1) = 1$. The equation of motion,

$$\dot{\gamma}_i = i[H(t),\gamma_i(t)] \quad (34)$$

can be written as

$$\dot{\gamma}_a = 2\epsilon_{abc}B_b\gamma_c, \quad (35)$$

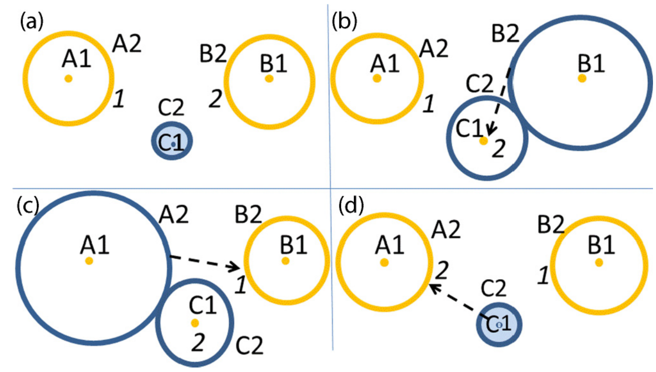

FIG. 2. The braiding of two MFs. The consecutive steps are shown in panels (a)–(d). Each step or shift is a transportation of a single MF between two positions [see the discussion for Eqs. (33) and (37)]. This move (or shift) is represented by a dashed arrow. Orange circles (points) correspond to an exterior (core) MF. When two MFs couple and form a single Dirac fermion, the color is changed to blue. Initially [panel (a)] we have two exterior MFs (1 and 2) on vortices A and B. The MFs of vortex C are coupled into a single Dirac state. In panel (d) everything is the same except that MFs 1 and 2 switched their positions.

where vector $\mathbf{B}$ is equal to

$$\mathbf{B}(t) = [1-\alpha(t)]\zeta_{23}(1,0,0) + \alpha(t)\zeta_{12}(0,0,1), \quad (36)$$

and ϵ_{abc} is the antisymmetric Levi-Civita tensor. Straightforward calculations give the solution,

$$\gamma_3(t_1) = \mathrm{sgn}(\zeta_{12}\zeta_{23})\gamma_1(0). \quad (37)$$

Thus, the MF can be transported by tuning the interaction between MFs. For exterior MFs, this interaction may be varied by adjusting the local magnetic field.

The procedure described above can be used to implement the braiding of the two exterior MFs. As a result of the braiding protocol execution, the two exterior Majorana fermions swap their positions. The protocol requires three pinned vortices. One vortex is auxiliary, and two others serve as the starting and ending locations for the Majorana fermions (denoted below as MF 1 and MF 2) participating in the braiding, see Fig. 2. Initially, MF 1 (MF 2) is in position $A2$ ($B2$), whereas the MFs associated with vortex C are coupled to form a single Dirac fermion. Next, MF 2 is transported to $C1$. After that MF 1 goes to $B2$. Finally, MF 2 moves to $A2$, while MF 1 remains in $B2$.

In our consideration of the TI/SC heterostructure, we assumed that the magnetic field is uniform whereas the suppression of the superconductivity by this field is negligible. Therefore, the applied field must be much smaller than the second critical field of the superconducting layer, that is, $l_{\mathrm{B}} \gg \xi_{\mathrm{SC}}$. Under this condition, the regime $l_{\mathrm{B}} \sim \xi$ can be achieved if $\xi \gg \xi_{\mathrm{SC}}$. Taking the characteristic values $|\Delta| = 1$ meV and $v_{\mathrm{F}} = 5 \times 10^7$ cm/s, we obtain $\xi \approx 400$ nm. The condition $l_{\mathrm{B}} \sim \xi$ corresponds to $B \sim 4$ mT, which is much smaller than the second critical magnetic field for Pb or Nb. Even in the case of smaller coherence length $\xi \approx 100$ nm, we have $B \sim 64$ mT, which is still smaller than the critical magnetic field for Pb or Nb. For the latter case, we can estimate the value of the Zeeman splitting as $V_z = g\mu_{\mathrm{B}}B/2 \sim 0.1$ meV (where $g = 50$ is the

Landé factor and μ_B is the Bohr magneton). This energy scale is much smaller than the superconducting order parameter and can be neglected as we assumed above. Finally, in thin films the London penetration depth is large ($\lambda_L \sim 12\,000$ nm, see Ref. [42]). Consequently, the assumption that B is uniform is reasonable.

V. SUMMARY

We analyzed the generation of MFs in topological superconductors in the presence of vortices. Discussing both (1) the topological insulator/superconductor heterostructure and (2) a spinless p-wave superconductor, we established that the combined effect of the vortex and the transverse magnetic field B gives rise to the generation of two MFs. The first MF is well known: It localizes near the vortex core. The second, the exterior MF, is localized by the magnetic field. Its wave-function weight is centered along a circle of a radius of $r^* \propto 1/B$. Varying the magnetic field we can change the positions of the MFs and tune the splitting between the core and the exterior MFs of the same vortex or between the exterior MFs of different vortices in real time. The tunability of the pairwise couplings between MFs by a magnetic field may open a new route for future topological quantum operations.

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