

Supplemental Material: Can bilayer graphene become a fractional metal?

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Below we show the study of the stability of the quarter-metal against the umklapp interaction term.

S-1. BASIC EQUATIONS

For reader's convenience, let us recall several basic equations and facts from the main text.

A. Definitions

The interaction Hamiltonian is

$$\hat{H}_{\text{int}} = \hat{H}^{(1)} + \hat{H}^{(2)} + \hat{H}^{(3)} + \hat{H}^{(4)}, \quad \text{where} \quad (\text{S1})$$

$$\hat{H}^{(1)} = -\frac{1}{N_c} \sum_{\mathbf{k}, \mathbf{p}, \sigma} V_{\mathbf{k}, \mathbf{p}}^{(1)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}}) (\gamma_{\mathbf{p}4\bar{\sigma}}^\dagger \gamma_{\mathbf{p}1\sigma}) + (\gamma_{\mathbf{k}3\bar{\sigma}}^\dagger \gamma_{\mathbf{k}2\sigma}) (\gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}3\bar{\sigma}}) \right], \quad (\text{S2})$$

$$\hat{H}^{(2)} = -\frac{1}{2N_c} \sum_{\mathbf{k}, \mathbf{p}, \sigma} V_{\mathbf{k}, \mathbf{p}}^{(2)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}}) (\gamma_{\mathbf{p}1\bar{\sigma}}^\dagger \gamma_{\mathbf{p}4\sigma}) + (\gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}}) (\gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma}) + \text{H.c.} \right], \quad (\text{S3})$$

$$\hat{H}^{(3)} = -\frac{1}{N_c} \sum_{\mathbf{k}, \mathbf{p}, \sigma} V_{\mathbf{k}, \mathbf{p}}^{(3)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}}) (\gamma_{\mathbf{p}3\bar{\sigma}}^\dagger \gamma_{\mathbf{p}2\sigma}) + (\gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}}) (\gamma_{\mathbf{p}4\bar{\sigma}}^\dagger \gamma_{\mathbf{p}1\sigma}) \right], \quad (\text{S4})$$

$$\hat{H}^{(4)} = -\frac{1}{2N_c} \sum_{\mathbf{k}, \mathbf{p}, \sigma} V_{\mathbf{k}, \mathbf{p}}^{(4)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}}) (\gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma}) + (\gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}}) (\gamma_{\mathbf{p}1\bar{\sigma}}^\dagger \gamma_{\mathbf{p}4\sigma}) + \text{H.c.} \right], \quad (\text{S5})$$

with the coupling constants $V_{\mathbf{k}, \mathbf{p}}^{(1,2,3,4)}$ defined as

$$V_{\mathbf{k}, \mathbf{p}}^{(1,3)} = \frac{1}{8} [V_{AA}^{00} + V_{AA}^{10} \pm (V_{AB}^{00} + V_{AB}^{10}) e^{-i\Delta\varphi} + \text{C.c.}], \quad (\text{S6})$$

$$V_{\mathbf{k}, \mathbf{p}}^{(2,4)} = \frac{1}{8} [V_{AA}^{00} - V_{AA}^{10} \mp (V_{AB}^{00} - V_{AB}^{10}) e^{-i\Delta\varphi} + \text{C.c.}]. \quad (\text{S7})$$

Our first-step approximation is

$$\bar{V}_{\mathbf{k}, \mathbf{p}}^{(1,3)} \approx \frac{1}{2} V_C(\mathbf{k} - \mathbf{p}) [1 \pm \cos(\Delta\varphi_{\mathbf{k}, \mathbf{p}})], \quad \bar{V}_{\mathbf{k}, \mathbf{p}}^{(2,4)} \approx 0. \quad (\text{S8})$$

The interaction can be approximated as $\hat{H}_{\text{int}} \approx \hat{H}^{(1)} + \hat{H}^{(3)}$.

B. Mean field approximation

The mean field version of \hat{H}_{int} is

$$\hat{H}_{\text{int}}^{\text{MF}} = \frac{1}{N_c} (B_\uparrow + B_\downarrow) - \sum_{\mathbf{p}, \sigma} \left(\tilde{\Delta}_{\mathbf{p}, \sigma} \gamma_{\mathbf{p}4\bar{\sigma}}^\dagger \gamma_{\mathbf{p}1\sigma} + \Delta_{\mathbf{p}, \sigma} \gamma_{\mathbf{p}3\bar{\sigma}}^\dagger \gamma_{\mathbf{p}2\sigma} + \text{H.c.} \right), \quad (\text{S9})$$

where

$$\Delta_{\mathbf{k}\sigma} = \frac{1}{N_c} \sum_{\mathbf{p}} \left[V_{\mathbf{p},\mathbf{k}}^{(1)} \langle \gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}3\bar{\sigma}} \rangle + V_{\mathbf{p},\mathbf{k}}^{(3)} \langle \gamma_{\mathbf{p}1\sigma}^\dagger \gamma_{\mathbf{p}4\bar{\sigma}} \rangle \right], \quad (\text{S10})$$

$$\tilde{\Delta}_{\mathbf{k}\sigma} = \frac{1}{N_c} \sum_{\mathbf{p}} \left[V_{\mathbf{p},\mathbf{k}}^{(1)} \langle \gamma_{\mathbf{p}1\sigma}^\dagger \gamma_{\mathbf{p}4\bar{\sigma}} \rangle + V_{\mathbf{p},\mathbf{k}}^{(3)} \langle \gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}3\bar{\sigma}} \rangle \right], \quad (\text{S11})$$

$$B_\sigma = \frac{1}{N_c} \sum_{\mathbf{k}} \left[\Delta_{\mathbf{k}\sigma} \langle \gamma_{\mathbf{k}3\bar{\sigma}}^\dagger \gamma_{\mathbf{k}2\sigma} \rangle + \tilde{\Delta}_{\mathbf{k}\sigma} \langle \gamma_{\mathbf{k}4\bar{\sigma}}^\dagger \gamma_{\mathbf{k}1\sigma} \rangle \right]. \quad (\text{S12})$$

The spectrum of the mean-field Hamiltonian can be easily derived

$$E_{\mathbf{k}\sigma}^{(2,3)} = \mp E_{\mathbf{k}\sigma}^m, \quad E_{\mathbf{k}\sigma}^{(1,4)} = \mp E_{\mathbf{k}\sigma}^h, \quad (\text{S13})$$

where

$$E_{\mathbf{k}\sigma}^m = \sqrt{|\Delta_{\mathbf{k}\sigma}|^2 + (t_0 - t\zeta_{\mathbf{k}})^2}, \quad E_{\mathbf{k}\sigma}^h = \sqrt{|\tilde{\Delta}_{\mathbf{k}\sigma}|^2 + (t_0 + t\zeta_{\mathbf{k}})^2}.$$

The total energy of the system is

$$E = \sum_{\nu=1}^4 \sum_{\mathbf{k}\sigma} (E_{\mathbf{k}\sigma}^{(\nu)} - \mu) \Theta(\mu - E_{\mathbf{k}\sigma}^{(\nu)}), \quad (\text{S14})$$

where $\Theta(E)$ is the step-function. Using the Hellmann-Feynman theorem, we obtain

$$\begin{aligned} \langle \gamma_{\mathbf{k}3\bar{\sigma}}^\dagger \gamma_{\mathbf{k}2\sigma} \rangle &= \frac{\Delta_{\mathbf{k}\sigma}}{2E_{\mathbf{k}\sigma}^m} [\Theta(\mu + E_{\mathbf{k}\sigma}^m) - \Theta(\mu - E_{\mathbf{k}\sigma}^m)], \\ \langle \gamma_{\mathbf{k}4\bar{\sigma}}^\dagger \gamma_{\mathbf{k}1\sigma} \rangle &= \frac{\tilde{\Delta}_{\mathbf{k}\sigma}}{2E_{\mathbf{k}\sigma}^h} [\Theta(\mu + E_{\mathbf{k}\sigma}^h) - \Theta(\mu - E_{\mathbf{k}\sigma}^h)]. \end{aligned} \quad (\text{S15})$$

Formally, the summation in Eq. (S10) covers the whole Brillouin zone. However, the interaction $V_{\mathbf{p},\mathbf{k}}^{(1,3)}$ is the strongest when $\mathbf{p} \approx \mathbf{k}$, and decays for larger $|\mathbf{p} - \mathbf{k}|$. In the limit of vanishing backscattering

$$V_{\text{bs}}^{(1,3)} \equiv V_{\mathbf{K}_1, \mathbf{K}_2}^{(1,3)} \approx 0, \quad (\text{S16})$$

it is possible to define order parameters localized near a specific Dirac point \mathbf{K}_ξ : $\Delta_{\mathbf{k}\sigma\xi} = \Delta_{\mathbf{k}\sigma}$, when $\mathbf{k} \approx \mathbf{K}_\xi$. Combining Eqs. (S10) and (S15), we obtain the self-consistent equations in the form

$$\Delta_{\mathbf{k}\sigma\xi} = \frac{1}{N_c} \sum_{\mathbf{p} \in \mathbf{K}_\xi} \left\{ \frac{V_{\mathbf{p},\mathbf{k}}^{(1)} \Delta_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^m} [\Theta(\mu + E_{\mathbf{p}\sigma}^m) - \Theta(\mu - E_{\mathbf{p}\sigma}^m)] + \frac{V_{\mathbf{p},\mathbf{k}}^{(3)} \tilde{\Delta}_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^h} [\Theta(\mu + E_{\mathbf{p}\sigma}^h) - \Theta(\mu - E_{\mathbf{p}\sigma}^h)] \right\}, \quad (\text{S17})$$

$$\tilde{\Delta}_{\mathbf{k}\sigma\xi} = \frac{1}{N_c} \sum_{\mathbf{p} \in \mathbf{K}_\xi} \left\{ \frac{V_{\mathbf{p},\mathbf{k}}^{(1)} \tilde{\Delta}_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^h} [\Theta(\mu + E_{\mathbf{p}\sigma}^h) - \Theta(\mu - E_{\mathbf{p}\sigma}^h)] + \frac{V_{\mathbf{p},\mathbf{k}}^{(3)} \Delta_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^m} [\Theta(\mu + E_{\mathbf{p}\sigma}^m) - \Theta(\mu - E_{\mathbf{p}\sigma}^m)] \right\}. \quad (\text{S18})$$

Simplifying the latter equations in the regime $\mu > 0$, we derive

$$\Delta_{\mathbf{k}\sigma\xi} = \frac{1}{N_c} \sum_{\mathbf{p} \in \mathbf{K}_\xi} \left\{ \frac{V_{\mathbf{p},\mathbf{k}}^{(1)} \Delta_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^m} \Theta(E_{\mathbf{p}\sigma}^m - \mu) + \frac{V_{\mathbf{p},\mathbf{k}}^{(3)} \tilde{\Delta}_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^h} \Theta(E_{\mathbf{p}\sigma}^h - \mu) \right\}, \quad (\text{S19})$$

$$\tilde{\Delta}_{\mathbf{k}\sigma\xi} = \frac{1}{N_c} \sum_{\mathbf{p} \in \mathbf{K}_\xi} \left\{ \frac{V_{\mathbf{p},\mathbf{k}}^{(1)} \tilde{\Delta}_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^h} \Theta(E_{\mathbf{p}\sigma}^h - \mu) + \frac{V_{\mathbf{p},\mathbf{k}}^{(3)} \Delta_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^m} \Theta(E_{\mathbf{p}\sigma}^m - \mu) \right\}. \quad (\text{S20})$$

We see that, within our approximations, the electronic states and the order parameters can be split into four independent sectors, which can be labeled by the multi-index $s = (\sigma, \xi)$. Our derivation implies that the sectors are not entirely independent: neglected contributions proportional to V_{bs} and $V^{(2,4)}$ couple them. Due to the smallness of these couplings, they can be treated perturbatively.

We add and subtract Eqs. (S19) and (S20), use Eqs. (S8), and change the summation over momentum by an integration. We also assume that both Δ and $\tilde{\Delta}$ only depend on $|\mathbf{k}|$. Finally, using the symmetry of our theory with respect to the sign of μ , we derive for $0 < \mu < t_0$

$$\begin{aligned}\Delta_{ks} + \tilde{\Delta}_{ks} &= \int_p \bar{V}_Q^{(0)} \left[\frac{\Delta_{ps}}{2E_{ps}^m} \Theta(E_{ps}^m - \mu) + \frac{\tilde{\Delta}_{ps}}{2E_{ps}^h} \right], \\ \Delta_{ks} - \tilde{\Delta}_{ks} &= \int_p \bar{V}_Q^{(1)} \left[\frac{\Delta_{ps}}{2E_{ps}^m} \Theta(E_{ps}^m - \mu) - \frac{\tilde{\Delta}_{ps}}{2E_{ps}^h} \right],\end{aligned}\quad (\text{S21})$$

where the integration symbol stands for $\int_p \dots = (2\pi p/v_{\text{BZ}}) \int dp \dots$, and the volume (area) of the Brillouin zone is $v_{\text{BZ}} = 8\pi^2/(3\sqrt{3}a^2)$. In Eqs. (S21), the averaged coupling constants are

$$\begin{aligned}\bar{V}_Q^{(0)}(k, p) &= \int \frac{d\phi}{2\pi} V_C(\sqrt{k^2 + p^2 + 2kp \cos \phi}), \\ \bar{V}_Q^{(1)}(k, p) &= \int \frac{d\phi}{2\pi} V_C(\sqrt{k^2 + p^2 + 2kp \cos \phi}) \cos \phi,\end{aligned}\quad (\text{S22})$$

and the spectrum (S13) in sector $s = (\sigma, \xi)$ can be approximated as

$$\begin{aligned}E_{ps}^m &\approx \sqrt{|\Delta_s|^2 + t_0^2(1 - p/k_{F0})^2}, \\ E_{ps}^h &\approx \sqrt{|\tilde{\Delta}_s|^2 + t_0^2(1 + p/k_{F0})^2} \approx t_0(1 + p/k_{F0}),\end{aligned}\quad (\text{S23})$$

where $p = |\mathbf{p} - \mathbf{K}_\xi|$.

To solve the integral equations (S21) we use the simple BCS-like ansatz

$$\Delta_a(q) = \Delta_s \Theta(\Lambda - |q - k_{F0}|) \quad \text{and} \quad \tilde{\Delta}_{\xi\sigma}(q) = \tilde{\Delta}_s \Theta(\Lambda - |q - k_{F0}|) \quad (\text{S24})$$

for the order parameters (the cutoff momentum Λ satisfies $\Lambda \ll Q_0$), and assume that $\bar{V}_Q^{(0,1)}$ are constants independent of k and p . This allows us to convert the integral equations into non-linear algebraic equations

$$\begin{aligned}\Delta_s + \tilde{\Delta}_s &= g \Delta_s \ln \left(\frac{E^*}{\mu + \sqrt{\mu^2 - \Delta_s^2}} \right) + \tilde{g} \tilde{\Delta}_s, \\ \Delta_s - \tilde{\Delta}_s &= \frac{g}{\alpha} \Delta_s \ln \left(\frac{E^*}{\mu + \sqrt{\mu^2 - \Delta_s^2}} \right) - \frac{\tilde{g}}{\alpha} \tilde{\Delta}_s,\end{aligned}\quad (\text{S25})$$

where the energy scale is $E^* = 2t_0\Lambda/k_{F0}$, and the coupling constants are

$$g = \frac{t_0}{\sqrt{3}\pi t^2} \bar{V}^{(0)}, \quad \tilde{g} = \frac{\Lambda}{2k_{F0}} g, \quad \alpha = \frac{\bar{V}^{(0)}}{\bar{V}^{(1)}} > 1. \quad (\text{S26})$$

C. Solutions of the mean field equations

At zero doping, which corresponds to the case $\mu = \Delta_s$, the order parameters are

$$\Delta_s = \Delta_0 = E^* \exp \left[-\frac{1}{g} \frac{2\alpha - \tilde{g}(1 + \alpha)}{1 + \alpha - 2\tilde{g}} \right], \quad \tilde{\Delta}_s = \frac{\alpha - 1}{\alpha + 1 - 2\tilde{g}} \Delta_s. \quad (\text{S27})$$

This mean-field solution is valid in the weak-coupling limit, that is, when g is small, and, consequently, Δ_0 and $\tilde{\Delta}_0$ are much less than t_0 . The doped state is characterized by $\mu > \Delta_s$. To describe the solution of Eq. (S25) in such a regime, let us define the partial doping x_s for the concentration of electrons residing in sector s . It is known that a finite x_s decreases the order parameter Δ_s :

$$\Delta_s(x_s) = \Delta_0 \sqrt{1 - \frac{4x_s}{x_0}}, \quad \mu = \Delta_0 \left(1 - \frac{2x_s}{x_0} \right), \quad (\text{S28})$$

where $x_0 = \Delta_0 t_0 / (\pi \sqrt{3} t^2)$. It is easy to check that Eqs. (S28) indeed guarantee that μ exceeds Δ_s , making it possible to dope sector s . At zero temperature, the partial free energy (per unit cell) associated with doping becomes

$$\Delta F_s(x_s) = 4 \int_0^{x_s} \mu(x) dx = 4 \Delta_0 \left(x_s - \frac{x_s^2}{x_0} \right). \quad (\text{S29})$$

As in the main text, the factor 4 accounts for the four carbon atoms in a single unit cell.

S-2. STABILITY AGAINST THE UMKLAPP INTERACTION

A. Self-consistent equations

The next step is to add the inter-sector interaction. We will use $H^{(2)}$ as an example of the inter-sector interaction. The other example is the backscattering $V_{\text{bs}}^{(1,3)}$. The term $H^{(2)}$ is a type of umklapp scattering: such term is non-zero only when the nesting vector is either zero or half of the elementary reciprocal lattice vector. If we average $H^{(2)}$ we obtain

$$\begin{aligned} \langle \hat{H}^{(2)} \rangle &= -\frac{1}{2N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k},\mathbf{p}}^{(2)} \left[\langle \gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}} \rangle \langle \gamma_{\mathbf{p}1\bar{\sigma}}^\dagger \gamma_{\mathbf{p}4\sigma} \rangle + \langle \gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}} \rangle \langle \gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma} \rangle + \text{C.c.} \right] \\ &\approx -\frac{1}{2N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k},\mathbf{p}}^{(2)} \left[\langle \gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}} \rangle \langle \gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma} \rangle + \text{C.c.} \right] \approx -\frac{\bar{V}_{\text{um}}}{2N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} \left[\langle \gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}} \rangle \langle \gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma} \rangle + \text{C.c.} \right], \end{aligned} \quad (\text{S30})$$

where \bar{V}_{um} is the averaged value of $V_{\mathbf{k},\mathbf{p}}^{(2)}$. Using the definition of the order parameter in terms of the anomalous operator averages, Eq. (S10), we derive

$$\langle \hat{H}^{(2)} \rangle \approx -N_c \frac{\bar{V}_{\text{um}}}{V_1^2} \Delta_{\uparrow\xi} \Delta_{\downarrow\xi} + \text{C.c.} \quad (\text{S31})$$

This suggests that the self-consistent equations for $s = (\uparrow, \xi)$ and $s' = (\downarrow, \xi)$ become coupled. To account for this, we take the first of the two equations (S10) and add a term $V_{\mathbf{p},\mathbf{k}}^{(2)} \langle \gamma_{\bar{\sigma}}^\dagger \gamma_{\sigma} \rangle$ to its right-hand side

$$\Delta_{\mathbf{k}\sigma} = \frac{1}{N_c} \sum_{\mathbf{p}} \left[V_{\mathbf{p},\mathbf{k}}^{(1)} \langle \gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}3\bar{\sigma}} \rangle + V_{\mathbf{p},\mathbf{k}}^{(2)} \langle \gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma} \rangle \right], \quad (\text{S32})$$

where we discarded the term with bands 1 and 4. Finally, using Eq. (S15), we derive

$$\Delta_{\uparrow} = g \Delta_{\uparrow} \ln \left[\frac{E^*}{M(\mu, \Delta_{\uparrow})} \right] + g_{\text{um}} \Delta_{\downarrow} \ln \left[\frac{E^*}{M(\mu, \Delta_{\downarrow})} \right], \quad (\text{S33})$$

$$\Delta_{\downarrow} = g \Delta_{\downarrow} \ln \left[\frac{E^*}{M(\mu, \Delta_{\downarrow})} \right] + g_{\text{um}} \Delta_{\uparrow} \ln \left[\frac{E^*}{M(\mu, \Delta_{\uparrow})} \right]. \quad (\text{S34})$$

To describe two remaining sectors, $(\uparrow, \bar{\xi})$ and $(\downarrow, \bar{\xi})$, the identical set of equations should be used. In Eqs. (S33) and (S34), the quantity $M(\mu, \Delta)$ effectively functions as the low-energy cutoff: if in a given sector $\Delta > \mu$, this sector remains undoped, and $M(\mu, \Delta) = \Delta$; when a sector accommodates finite doping $\mu > \Delta$, in such a situation $M(\mu, \Delta) = \mu + \sqrt{\mu^2 - \Delta^2}$. Formally, this can be expressed as

$$M(\mu, \Delta) = (\mu + \sqrt{\mu^2 - \Delta^2}) \Theta(\mu - \Delta) + \Delta \Theta(\Delta - \mu). \quad (\text{S35})$$

Note also that in Eqs. (S33) and (S34) we used the simplified notation $\Delta_{\uparrow} \equiv \Delta_{\uparrow\xi}$ and $\Delta_{\downarrow} \equiv \Delta_{\downarrow\xi}$. The coupling constant is $g_{\text{um}} = \beta \bar{V}_{\text{um}} \nu(\varepsilon_{\text{F}})$, where $\nu(\varepsilon_{\text{F}})$ is the density of states, and β is a numerical coefficient of order unity.

When the system is undoped, we can introduce Δ_0 as follows $\mu = \Delta_{\uparrow} = \Delta_{\downarrow} \equiv \Delta_0$ [note that this is a redefinition of Δ_0 initially given by Eq. (S27)]. In such a limit, both equations become identical

$$\Delta_0 = g(1 + \gamma) \Delta_0 \ln \left(\frac{E^*}{\Delta_0} \right), \quad \text{where } \gamma = \frac{g_{\text{um}}}{g}. \quad (\text{S36})$$

This equation has one non-zero solution

$$\Delta_0 = E^* \exp \left(-\frac{1}{g(1 + \gamma)} \right). \quad (\text{S37})$$

We can see that the umklapp coupling increases Δ_0 .

B. Doped state

Now we discuss the doped system. Below we will consider two possibilities: (i) all four sectors are doped equally, and (ii) three sectors remain undoped, and all doping only enters a single sector. Let us start with (i). In such a situation $\mu > \Delta_s = \Delta(x)$ for all four s . Equations (S33) and (S34) become identical

$$\Delta = g(1 + \gamma)\Delta \ln \left(\frac{E^*}{\mu + \sqrt{\mu^2 - \Delta^2}} \right), \quad (\text{S38})$$

valid in all four sectors. The solution to this equation is similar to Eq. (S28)

$$\Delta(x) = \Delta_0 \sqrt{1 - \frac{x}{x_0}}, \quad \mu = \Delta_0 \left(1 - \frac{x}{2x_0} \right), \quad (\text{S39})$$

where we took into account that partial dopings equal to half of the total doping: $x_s = x/4$. The expression for $\mu(x)$ allows us to calculate $\Delta F(x)$

$$\Delta F(x) = 4 \int_0^x \mu(x) dx = 4\Delta_0 x - \Delta_0 \frac{x^2}{x_0}. \quad (\text{S40})$$

This free energy is denoted as ΔF_e in the main text.

For case (ii), the calculations are more complicated. We define $\delta_\sigma(x)$ as follows $\Delta_\sigma(x) = \Delta_0[1 - \delta_\sigma(x)]$. For definiteness, we assume that the sector $s = (\uparrow, \xi)$ is undoped, while $s = (\downarrow, \xi)$ is doped. This means that $\Delta_\uparrow > \mu > \Delta_\downarrow$. Two other sectors, $(\uparrow, \bar{\xi})$ and $(\downarrow, \bar{\xi})$, are undoped, and decoupled from s and s' . Therefore, they are characterized by the order parameter Δ_0 , given by Eq. (S37)

$$0 < \delta_\uparrow < m < \delta_\downarrow, \quad \text{where} \quad m = \frac{\Delta_0 - \mu}{\Delta_0}. \quad (\text{S41})$$

Let us introduce yet another quantity, δS , as follows

$$\mu + \sqrt{\mu^2 - \Delta_\downarrow^2} = \Delta_0 \left[1 - m + \sqrt{(1 - m)^2 - (1 - \delta_\downarrow)^2} \right] = \Delta_0(1 + \delta S), \quad (\text{S42})$$

$$\delta S = \sqrt{(1 - m)^2 - (1 - \delta_\downarrow)^2} - m. \quad (\text{S43})$$

The parameters δ_σ , δS , and m are small in the limit of small doping x . However, they have different degrees of smallness. Indeed, as we will see later

$$\delta_\sigma = O(m), \quad \delta S = O(m^{1/2}). \quad (\text{S44})$$

These relations become important when we solve the self-consistent equations in the limit of small doping.

Our goal is to solve the following equations

$$(1 - \delta_\uparrow) = g(1 - \delta_\uparrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 - \delta_\uparrow) \right] + \gamma g(1 - \delta_\downarrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 + \delta S) \right], \quad (\text{S45})$$

$$(1 - \delta_\downarrow) = g(1 - \delta_\downarrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 + \delta S) \right] + \gamma g(1 - \delta_\uparrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 - \delta_\uparrow) \right], \quad (\text{S46})$$

to find δ_σ as a function of m , and then determine m versus x . In the limit of small x , we expand the self-consistent equations and, keeping in mind Eq. (S44), we derive

$$(1 - \delta_\uparrow) = g(1 - \delta_\uparrow) \left[\frac{1}{g(1 + \gamma)} + \delta_\uparrow \right] + \gamma g(1 - \delta_\downarrow) \left[\frac{1}{g(1 + \gamma)} - \delta S + \frac{\delta S^2}{2} \right] + O(m^{3/2}), \quad (\text{S47})$$

$$(1 - \delta_\downarrow) = g(1 - \delta_\downarrow) \left[\frac{1}{g(1 + \gamma)} - \delta S + \frac{\delta S^2}{2} \right] + \gamma g(1 - \delta_\uparrow) \left[\frac{1}{g(1 + \gamma)} + \delta_\uparrow \right] + O(m^{3/2}). \quad (\text{S48})$$

Simplifying, we obtain

$$\delta_\uparrow \approx \left[\frac{\delta_\uparrow}{1 + \gamma} - g\delta_\uparrow \right] + \gamma \left[\frac{1}{1 + \gamma} \delta_\downarrow + g \left(\delta S - \frac{\delta S^2}{2} \right) \right], \quad (\text{S49})$$

$$\delta_\downarrow \approx \left[\frac{\delta_\downarrow}{1 + \gamma} + g \left(\delta S - \frac{\delta S^2}{2} \right) \right] + \gamma \left[\frac{1}{1 + \gamma} \delta_\uparrow - g\delta_\uparrow \right], \quad (\text{S50})$$

Next step:

$$\left(\frac{\gamma}{1+\gamma} + g\right) \delta_{\uparrow} = \frac{\gamma}{1+\gamma} \delta_{\downarrow} + g\gamma \left(\delta S - \frac{\delta S^2}{2}\right), \quad (\text{S51})$$

$$\left(\frac{\gamma}{1+\gamma} - g\gamma\right) \delta_{\uparrow} = \frac{\gamma}{1+\gamma} \delta_{\downarrow} - g \left(\delta S - \frac{\delta S^2}{2}\right). \quad (\text{S52})$$

Subtracting these two equations we derive

$$g(1+\gamma)\delta_{\uparrow} = g(1+\gamma) \left(\delta S - \frac{\delta S^2}{2}\right) \Leftrightarrow \delta_{\uparrow} = \delta S - \frac{\delta S^2}{2}. \quad (\text{S53})$$

Now δ_{\uparrow} can be eliminated

$$\left[\frac{\gamma}{1+\gamma} + g(1-\gamma)\right] \left(\delta S - \frac{\delta S^2}{2}\right) = \frac{\gamma}{1+\gamma} \delta_{\downarrow}. \quad (\text{S54})$$

This relation is equivalent to

$$\delta S - \frac{\delta S^2}{2} = \alpha \delta_{\downarrow}, \quad \text{where} \quad \alpha = [1 + g(\gamma^{-1} - \gamma)]^{-1}. \quad (\text{S55})$$

Let us express δS in the limit of small doping

$$\delta S = \sqrt{(1-m)^2 - (1-\delta_{\downarrow})^2} - m = \sqrt{(2-m-\delta_{\downarrow})(\delta_{\downarrow}-m)} - m = \sqrt{2(\delta_{\downarrow}-m)} - m + O(m^{3/2}), \quad (\text{S56})$$

$$\delta S^2 = 2(\delta_{\downarrow}-m) + O(m^{3/2}). \quad (\text{S57})$$

Therefore

$$\delta S - \frac{\delta S^2}{2} = \sqrt{2(\delta_{\downarrow}-m)} - \delta_{\downarrow} + O(m^{3/2}). \quad (\text{S58})$$

The self-consistent equation becomes

$$\alpha \delta_{\downarrow} = \sqrt{2(\delta_{\downarrow}-m)} - \delta_{\downarrow} + O(m^{3/2}). \quad (\text{S59})$$

Its solution is

$$\delta_{\downarrow} \approx m + \frac{(1+\alpha)^2}{2} m^2, \quad \delta_{\uparrow} = \alpha \delta_{\downarrow} \approx \alpha m + \frac{\alpha(1+\alpha)^2}{2} m^2. \quad (\text{S60})$$

Let us check the consistency of these relations with known results in the $\alpha = 0$ limit. In this case

$$\delta_{\downarrow} \approx m + \frac{1}{2} m^2, \quad \delta_{\uparrow} = 0. \quad (\text{S61})$$

At the same time, Eqs. (S28) in the regime of small x can be written as

$$m = \frac{x}{2x_0}, \quad \delta_{\downarrow}(x) = \frac{\Delta_0 - \Delta(x)}{\Delta_0} \approx \frac{x}{2x_0} + \frac{x^2}{8x_0^2}. \quad (\text{S62})$$

We can now exclude x to obtain

$$\delta_{\downarrow}(x) \approx m + \frac{m^2}{2}, \quad (\text{S63})$$

which coincides with Eq. (S61).

The final step is to add doping into the formalism. To this end, we write

$$4x = 2\nu_F \int_{\Delta_{\downarrow}}^{\mu} d\varepsilon \frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta_{\downarrow}^2}}, \quad (\text{S64})$$

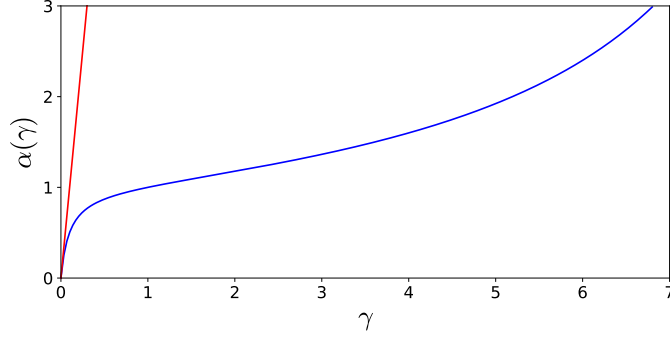


FIG. S1: The function $\alpha(\gamma)$ for $g = 0.1$ is shown by the blue curve. The straight (red) line is γ/g .

where $4x$ is the doping per unit cell, $\nu_F = t_0/(\sqrt{3}\pi t^2)$ is the density of states per unit cell for each single Fermi surface sheet (there are four Fermi surface sheets),

$$x = \frac{\nu_F}{2} \sqrt{\mu^2 - \Delta_{\downarrow}^2} = \frac{\nu_F \Delta_0}{2} \sqrt{(1-m)^2 - (1-\delta_{\downarrow})^2} = \frac{x_0}{2} \sqrt{(1-m)^2 - (1-\delta_{\downarrow})^2}, \quad (\text{S65})$$

where $x_0 = \nu_F \Delta_0$. It is possible to show that

$$4x^2 = x_0^2 (\delta_{\downarrow} - m)(2 - m - \delta_{\downarrow}) \Rightarrow 4x^2 = (1 + \alpha)^2 x_0^2 m^2 + O(m^3). \quad (\text{S66})$$

Deriving the latter relation we used Eq. (S60), which, among other things, demonstrates that $\delta_{\downarrow} - m = O(m^2)$. Equation (S66) allows us to establish the following connection between doping and the chemical potential

$$m = \frac{2x}{(1 + \alpha)x_0} + O(x^2) \Leftrightarrow \mu = \Delta_0 \left(1 - \frac{2x}{(1 + \alpha)x_0} \right) + O(x^2). \quad (\text{S67})$$

Integrating $\mu(x)$, we obtain

$$\Delta F_{\text{qm}} = 4\Delta_0 x - \left(\frac{4\Delta_0}{1 + \alpha} \right) \frac{x^2}{x_0}. \quad (\text{S68})$$

In the limit $\alpha \rightarrow 0$ we recover the expression for ΔF_{qm} given in the main text [see after Eq. (30)]. The free energy (S68) must be compared against the free energy given by Eq. (S40). We see that the quarter-metal is stable if $(1 + \alpha)^{-1} > 1/4$. Equivalently,

$$\text{quarter-metal is stable when } \alpha(\gamma) < 3. \quad (\text{S69})$$

To understand what the latter requirement entails, let us examine Fig. S1, which shows $\alpha(\gamma)$ for $g = 0.1$. We see that $\alpha < 3$ as long as $\gamma = g_{\text{um}}/g < 6.8$. That is, for $g = 0.1$, the umklapp satisfying

$$g_{\text{um}} < 0.68, \quad (\text{S70})$$

does not violate the stability of the quarter-metal.

We note that Eq. (S69) is not the absolute stability criterion, rather it describes the stability of the quarter-metal against the transition into an ordinary metal, when all four sectors are doped equally. A comprehensive investigation of the stability goes well beyond the present study, and, most likely, requires input from experiments.

It is interesting to note that perturbation theory in powers of small γ strongly underestimates the stability range of the quarter-metal. To demonstrate this, we expand the expression (S68) for ΔF_{qm} in powers of α

$$\Delta F_{\text{qm}} \approx 4\Delta_0 x - 4\Delta_0 \frac{x^2}{x_0} + 4\Delta_0 \frac{\alpha x^2}{x_0}. \quad (\text{S71})$$

Since at $\gamma \rightarrow 0$, the following holds $\alpha \approx \gamma/g = g_{\text{um}}/g^2$, the expression for ΔF_{qm} can be approximated as

$$\Delta F_{\text{qm}} \approx 4\Delta_0 x - 4\Delta_0 \frac{x^2}{x_0} \left(1 - \frac{g_{\text{um}}}{g^2} \right), \quad (\text{S72})$$

If we use this expression, instead of the more accurate Eq. (S68), we could (erroneously) conclude that the quarter-metal is stable when $(1 - g_{\text{um}}/g^2) > 1/4$. This inequality can be transformed to

$$\frac{\gamma}{g} = \frac{g_{\text{um}}}{g^2} < \frac{3}{4} \quad \Leftrightarrow \quad g_{\text{um}} < \frac{3g^2}{4}. \quad (\text{S73})$$

In Fig. S1 we can see the low- γ approximation $\alpha(\gamma) \approx \gamma/g$ as a (red) straight line. We see that, at low g , this approximation works only at very small γ ; while for larger γ (larger g_{um}) it is completely useless. Thus, we conclude that the replacement $[1 + \alpha(\gamma)]^{-1} \rightarrow (1 - \gamma/g)$ artificially shrinks the stability range of the quarter-metal. Indeed, the requirement (S73) is very strict: at $g = 0.1$, as in Fig. S1, Eq. (S73) demand that $g_{\text{um}} < 0.0075$, cf. Eq. (S70). This is the origin of the serious disparity between the stability condition derived in the main text using simple perturbation theory and more the sophisticated criterion (S69).