Supplemental Material: Can bilayer graphene become a fractional metal?

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Below we show the study of the stability of the quarter-metal against the umklapp interaction term.

S-1. BASIC EQUATIONS

For reader’s convenience, let us recall several basic equations and facts from the main text.

A. Definitions

The interaction Hamiltonian is

\[ \hat{H}_{\text{int}} = \hat{H}^{(1)} + \hat{H}^{(2)} + \hat{H}^{(3)} + \hat{H}^{(4)}, \]

where

\[ \hat{H}^{(1)} = -\frac{1}{N_c} \sum_{\mathbf{k}, \mathbf{p}} V_{\mathbf{k}, \mathbf{p}}^{(1)} \left[ \gamma_{\mathbf{k}1\sigma} \gamma_{\mathbf{p}1\sigma} \right] \left( \gamma_{\mathbf{k}2\sigma} \gamma_{\mathbf{p}2\sigma} \right) \]

\[ \hat{H}^{(2)} = -\frac{1}{2N_c} \sum_{\mathbf{k}, \mathbf{p}} V_{\mathbf{k}, \mathbf{p}}^{(2)} \left[ \gamma_{\mathbf{k}1\sigma} \gamma_{\mathbf{p}1\sigma} \right] \left( \gamma_{\mathbf{k}2\sigma} \gamma_{\mathbf{p}2\sigma} \right) \]

\[ \hat{H}^{(3)} = -\frac{1}{N_c} \sum_{\mathbf{k}, \mathbf{p}} V_{\mathbf{k}, \mathbf{p}}^{(3)} \left[ \gamma_{\mathbf{k}1\sigma} \gamma_{\mathbf{p}1\sigma} \right] \left( \gamma_{\mathbf{k}2\sigma} \gamma_{\mathbf{p}2\sigma} \right) \]

\[ \hat{H}^{(4)} = -\frac{1}{2N_c} \sum_{\mathbf{k}, \mathbf{p}} V_{\mathbf{k}, \mathbf{p}}^{(4)} \left[ \gamma_{\mathbf{k}1\sigma} \gamma_{\mathbf{p}1\sigma} \right] \left( \gamma_{\mathbf{k}2\sigma} \gamma_{\mathbf{p}2\sigma} \right) \]

with the coupling constants \( V_{\mathbf{k}, \mathbf{p}}^{(1,2,3,4)} \) defined as

\[ V_{\mathbf{k}, \mathbf{p}}^{(1,3)} = \frac{1}{8} \left[ V_{AA}^{00} + V_{AA}^{10} \pm (V_{AB}^{00} + V_{AB}^{10}) e^{-i\Delta \varphi} + C.c. \right] \]

\[ V_{\mathbf{k}, \mathbf{p}}^{(2,4)} = \frac{1}{8} \left[ V_{AA}^{00} - V_{AA}^{10} \mp (V_{AB}^{00} - V_{AB}^{10}) e^{-i\Delta \varphi} + C.c. \right] \]

Our first-step approximation is

\[ V_{\mathbf{k}, \mathbf{p}}^{(1,3)} \approx \frac{1}{2} V_{C} (\mathbf{k} - \mathbf{p}) \left[ 1 \pm \cos(\Delta \varphi) \right], \quad V_{\mathbf{k}, \mathbf{p}}^{(2,4)} \approx 0. \]

The interaction can be approximated as \( \hat{H}_{\text{int}} \approx \hat{H}^{(1)} + \hat{H}^{(3)} \).

B. Mean field approximation

The mean field version of \( \hat{H}_{\text{int}} \) is

\[ \hat{H}_{\text{int}}^{\text{MF}} = \frac{1}{N_c} \left( B_{+} + B_{-} \right) - \sum_{\mathbf{p} \sigma} \left( \Delta_{\mathbf{p} \sigma} \gamma_{\mathbf{p}1\sigma}^\dagger \gamma_{\mathbf{p}2\sigma} + \Delta_{\mathbf{p} \sigma} \gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}1\sigma} + H.c. \right), \]
where

\[ \Delta_{k\sigma} = \frac{1}{N_c} \sum_{p} \left[ V_{p,k}^{(1)} \langle \gamma_{p2\sigma}^{\dagger} \gamma_{p3\sigma} \rangle + V_{p,k}^{(3)} \langle \gamma_{p1\sigma}^{\dagger} \gamma_{p4\sigma} \rangle \right], \]  

(10)

\[ \tilde{\Delta}_{k\sigma} = \frac{1}{N_c} \sum_{p} \left[ V_{p,k}^{(1)} \langle \gamma_{p1\sigma}^{\dagger} \gamma_{p4\sigma} \rangle + V_{p,k}^{(3)} \langle \gamma_{p2\sigma}^{\dagger} \gamma_{p3\sigma} \rangle \right], \]  

(11)

\[ B_{\sigma} = \frac{1}{N_c} \sum_{k} \left[ \Delta_{k\sigma} \langle \gamma_{k1\sigma} \gamma_{k2\sigma} \rangle + \tilde{\Delta}_{k\sigma} \langle \gamma_{k4\sigma} \gamma_{k1\sigma} \rangle \right]. \]  

(12)

The spectrum of the mean-field Hamiltonian can be easily derived

\[ E_{k\sigma}^{(2,3)} = \mp E_{k\sigma}^{m}, \quad E_{k\sigma}^{(1,4)} = \mp E_{k\sigma}^{h}, \]  

(13)

where

\[ E_{k\sigma}^{m} = \sqrt{\Delta_{k\sigma}^2 + (t_0 - t_{0k})^2}, \quad E_{k\sigma}^{h} = \sqrt{\tilde{\Delta}_{k\sigma}^2 + (t_0 + t_{0k})^2}. \]  

The total energy of the system is

\[ E = \sum_{\nu=1}^{4} \sum_{k\sigma} (E_{k\sigma}^{(\nu)} - \mu) \Theta(\mu - E_{k\sigma}^{(\nu)}), \]  

(14)

where \( \Theta(E) \) is the step-function. Using the Hellmann-Feynman theorem, we obtain

\[ \langle \gamma_{k1\sigma}^{\dagger} \gamma_{k2\sigma} \rangle = \frac{\Delta_{k\sigma}}{2E_{k\sigma}^{m}} \left[ \Theta(\mu + E_{k\sigma}^{m}) - \Theta(\mu - E_{k\sigma}^{m}) \right], \]

\[ \langle \gamma_{k4\sigma}^{\dagger} \gamma_{k1\sigma} \rangle = \frac{\tilde{\Delta}_{k\sigma}}{2E_{k\sigma}^{h}} \left[ \Theta(\mu + E_{k\sigma}^{h}) - \Theta(\mu - E_{k\sigma}^{h}) \right]. \]  

(15)

Formally, the summation in Eq. (10) covers the whole Brillouin zone. However, the interaction \( V_{p,k}^{(1,3)} \) is the strongest when \( p \approx k \), and decays for larger \( |p - k| \). In the limit of vanishing backscattering

\[ V_{bs}^{(1,3)} = V_{K_{1}, K_{2}}^{(1,3)} \approx 0, \]  

(16)

it is possible to define order parameters localized near a specific Dirac point \( K_{\xi} \): \( \Delta_{k\sigma \xi} = \Delta_{k\sigma} \), when \( k \approx K_{\xi} \). Combining Eqs. (10) and (15), we obtain the self-consistent equations in the form

\[ \Delta_{k\sigma \xi} = \frac{1}{N_c} \sum_{p \in K_{\xi}} \left\{ V_{p,k}^{(1)} \frac{\Delta_{p\sigma \xi}}{2E_{p\sigma}^{m}} \left[ \Theta(\mu + E_{p\sigma}^{m}) - \Theta(\mu - E_{p\sigma}^{m}) \right] + V_{p,k}^{(3)} \frac{\Delta_{p\sigma \xi}}{2E_{p\sigma}^{h}} \left[ \Theta(\mu + E_{p\sigma}^{h}) - \Theta(\mu - E_{p\sigma}^{h}) \right] \right\}, \]  

(17)

\[ \tilde{\Delta}_{k\sigma \xi} = \frac{1}{N_c} \sum_{p \in K_{\xi}} \left\{ V_{p,k}^{(1)} \frac{\tilde{\Delta}_{p\sigma \xi}}{2E_{p\sigma}^{m}} \left[ \Theta(\mu + E_{p\sigma}^{m}) - \Theta(\mu - E_{p\sigma}^{m}) \right] + V_{p,k}^{(3)} \frac{\tilde{\Delta}_{p\sigma \xi}}{2E_{p\sigma}^{h}} \left[ \Theta(\mu + E_{p\sigma}^{h}) - \Theta(\mu - E_{p\sigma}^{h}) \right] \right\}. \]  

(18)

Simplifying the latter equations in the regime \( \mu > 0 \), we derive

\[ \Delta_{k\sigma \xi} = \frac{1}{N_c} \sum_{p \in K_{\xi}} \left\{ V_{p,k}^{(1)} \frac{\Delta_{p\sigma \xi}}{2E_{p\sigma}^{m}} \Theta(E_{p\sigma}^{m} - \mu) + V_{p,k}^{(3)} \frac{\Delta_{p\sigma \xi}}{2E_{p\sigma}^{h}} \Theta(E_{p\sigma}^{h} - \mu) \right\}, \]  

(19)

\[ \tilde{\Delta}_{k\sigma \xi} = \frac{1}{N_c} \sum_{p \in K_{\xi}} \left\{ V_{p,k}^{(1)} \frac{\tilde{\Delta}_{p\sigma \xi}}{2E_{p\sigma}^{m}} \Theta(E_{p\sigma}^{m} - \mu) + V_{p,k}^{(3)} \frac{\tilde{\Delta}_{p\sigma \xi}}{2E_{p\sigma}^{h}} \Theta(E_{p\sigma}^{h} - \mu) \right\}. \]  

(20)

We see that, within our approximations, the electronic states and the order parameters can be split into four independent sectors, which can be labeled by the multi-index \( s = (\sigma, \xi) \). Our derivation implies that the sectors are not entirely independent: neglected contributions proportional to \( V_{bs} \) and \( V_{(2,4)} \) couple them. Due to the smallness of these couplings, they can be treated perturbatively.
We add and subtract Eqs. \[ \text{(S19)} \] and \[ \text{(S20)}, \] use Eqs. \[ \text{(S8)} \], and change the summation over momentum by an integration. We also assume that both $\Delta$ and $\tilde{\Delta}$ only depend on $|k|$. Finally, using the symmetry of our theory with respect to the sign of $\mu$, we derive for $0 < \mu < t_0$

$$
\Delta_{ks} + \Delta_{ks} = \int_p \tilde{V}_Q^{(0)} \left[ \frac{\Delta_{ps}}{2E_{ps}} \Theta(E_{ps} - \mu) + \frac{\tilde{\Delta}_{ps}}{2E_{ps}} \right],
$$

$$
\Delta_{ks} - \Delta_{ks} = \int_p \tilde{V}_Q^{(1)} \left[ \frac{\Delta_{ps}}{2E_{ps}} \Theta(E_{ps} - \mu) - \frac{\tilde{\Delta}_{ps}}{2E_{ps}} \right],
$$

(S21)

where the integration symbol stands for $\int_p \ldots = (2\pi p/v_{\text{BZ}}) \int dp \ldots$, and the volume (area) of the Brillouin zone is $v_{\text{BZ}} = 8\pi^2/(3\sqrt{3}a^2)$. In Eqs. \[ \text{(S21)}, \] the averaged coupling constants are

$$
\tilde{V}_Q^{(0)}(k, p) = \int_0^{2\pi} \frac{d\phi}{2\pi} V_C(\sqrt{k^2 + p^2 + 2kp\cos\phi}),
$$

(S22)

$$
\tilde{V}_Q^{(1)}(k, p) = \int_0^{2\pi} \frac{d\phi}{2\pi} V_C(\sqrt{k^2 + p^2 + 2kp\cos\phi}) \cos\phi,
$$

and the spectrum \[ \text{\[S13\]} \] in sector $s = (\sigma, \xi)$ can be approximated as

$$
E_{ps}^m \approx \sqrt{\Delta_s^2 + t_0^2(1 - p/kF_0)^2},
$$

$$
E_{ps}^h \approx \sqrt{\tilde{\Delta}_s^2 + t_0^2(1 + p/kF_0)^2} \approx t_0(1 + p/kF_0),
$$

(S23)

where $p = |p - K_\xi|$. To solve the integral equations \[ \text{(S21)}, \] we use the simple BCS-like ansatz

$$
\Delta_n(q) = \Delta_s \Theta(\Lambda - |q - kF_0|) \quad \text{and} \quad \tilde{\Delta}_n(q) = \tilde{\Delta}_s \Theta(\Lambda - |q - kF_0|)
$$

(S24)

for the order parameters (the cutoff momentum $\Lambda$ satisfies $\Lambda \ll Q_0$), and assume that $\tilde{V}_Q^{(0,1)}$ are constants independent of $k$ and $p$. This allows us to convert the integral equations into non-linear algebraic equations

$$
\Delta_s + \tilde{\Delta}_s = g\Delta_s \ln \left( \frac{E^*}{\mu + \sqrt{\mu^2 - \Delta_s^2}} \right) + \tilde{g}\tilde{\Delta}_s,
$$

(S25)

$$
\Delta_s - \tilde{\Delta}_s = \frac{g}{\alpha} \Delta_s \ln \left( \frac{E^*}{\mu + \sqrt{\mu^2 - \tilde{\Delta}_s^2}} \right) - \frac{\tilde{g}}{\alpha} \tilde{\Delta}_s,
$$

where the energy scale is $E^* = 2t_0\Lambda/kF_0$, and the coupling constants are

$$
g = \frac{t_0}{\sqrt{3\pi^2}} \tilde{V}_Q^{(0)}, \quad \tilde{g} = \frac{\Lambda}{2kF_0} g, \quad \alpha = \frac{\tilde{V}_Q^{(0)}}{V_Q^{(1)}} > 1.
$$

(S26)

C. Solutions of the mean field equations

At zero doping, which corresponds to the case $\mu = \Delta_s$, the order parameters are

$$
\Delta_s = \Delta_0 = E^* \exp \left[ -\frac{1}{g} \frac{2\alpha - \tilde{g}(1 + \alpha)}{1 + \alpha - 2\tilde{g}} \right], \quad \tilde{\Delta}_s = \frac{\alpha - 1}{\alpha + 1 - 2\tilde{g}} \Delta_s.
$$

(S27)

This mean-field solution is valid in the weak-coupling limit, that is, when $g$ is small, and, consequently, $\Delta_0$ and $\tilde{\Delta}_0$ are much less than $t_0$. The doped state is characterized by $\mu > \Delta_s$. To describe the solution of Eq. \[ \text{(S25)}, \] in such a regime, let us define the partial doping $x_s$ for the concentration of electrons residing in sector $s$. It is known that a finite $x_s$ decreases the order parameter $\Delta_s$:

$$
\Delta_s(x_s) = \Delta_0 \sqrt{1 - \frac{4x_s}{x_0}}, \quad \mu = \Delta_0 \left( 1 - \frac{2x_s}{x_0} \right),
$$

(S28)
where \( x_0 = \Delta_0 t_0/(\pi \sqrt{3} \ell^2) \). It is easy to check that Eqs. \([S28]\) indeed guarantee that \( \mu \) exceeds \( \Delta_s \), making it possible to dope sector \( s \). At zero temperature, the partial free energy (per unit cell) associated with doping becomes

\[
\Delta F_s(x_s) = 4 \int_0^{x_s} \mu(x) \, dx = 4 \Delta_0 \left( x_s - \frac{x_s^2}{x_0} \right).
\]  

(S29)

As in the main text, the factor 4 accounts for the four carbon atoms in a single unit cell.

**S-2. STABILITY AGAINST THE UMKLAPP INTERACTION**

**A. Self-consistent equations**

The next step is to add the inter-sector interaction. We will use \( H^{(2)} \) as an example of the inter-sector interaction. The other example is the backscattering \( V_{bs}^{(1,3)} \). The term \( H^{(2)} \) is a type of umklapp scattering: such term is non-zero only when the nesting vector is either zero or half of the elementary reciprocal lattice vector. If we average \( H^{(2)} \) we obtain

\[
\langle \hat{H}^{(2)} \rangle = -\frac{1}{2 \mathcal{N}_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k},\mathbf{p}}^{(2)} \left[ \langle \gamma^{\uparrow}_{\mathbf{k}1\sigma} \gamma^{\downarrow}_{\mathbf{k1}\sigma} \gamma^{\uparrow}_{\mathbf{p}1\sigma} \gamma^{\downarrow}_{\mathbf{p1}\sigma} \rangle + \langle \gamma^{\uparrow}_{\mathbf{k}2\sigma} \gamma^{\downarrow}_{\mathbf{k2}\sigma} \gamma^{\uparrow}_{\mathbf{p}3\sigma} \gamma^{\downarrow}_{\mathbf{p3}\sigma} \rangle \right] + C.c.
\]  

(S30)

\[
\approx -\frac{1}{2 \mathcal{N}_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k},\mathbf{p}}^{(2)} \left[ \langle \gamma^{\uparrow}_{\mathbf{k}2\sigma} \gamma^{\downarrow}_{\mathbf{k3}\sigma} \gamma^{\uparrow}_{\mathbf{p}3\sigma} \gamma^{\downarrow}_{\mathbf{p3}\sigma} \rangle + C.c. \right] \approx -\frac{V_{um}}{2 \mathcal{N}_c} \sum_{\mathbf{k}\mathbf{p}\sigma} \left[ \langle \gamma^{\uparrow}_{\mathbf{k}2\sigma} \gamma^{\downarrow}_{\mathbf{k3}\sigma} \gamma^{\uparrow}_{\mathbf{p}3\sigma} \gamma^{\downarrow}_{\mathbf{p3}\sigma} \rangle + C.c. \right],
\]  

where \( V_{um} \) is the averaged value of \( V_{\mathbf{k},\mathbf{p}}^{(2)} \). Using the definition of the order parameter in terms of the anomalous operator averages, Eq. \([S10]\), we derive

\[
\langle \hat{H}^{(2)} \rangle \approx -\mathcal{N}_c \frac{V_{um}}{4} \Delta_\xi \Delta_\xi + C.c.
\]  

(S31)

This suggests that the self-consistent equations for \( s = (\uparrow, \xi) \) and \( s' = (\downarrow, \xi) \) become coupled. To account for this, we take the first of the two equations \([S10]\) and add a term \( V_{\mathbf{k},\mathbf{p}}^{(2)} \langle \gamma^{\uparrow}_{\mathbf{p}3\sigma} \gamma^{\downarrow}_{\mathbf{p3}\sigma} \rangle \) to its right-hand side

\[
\Delta_{k\sigma} = \frac{1}{\mathcal{N}_c} \sum_{\mathbf{p}} \left[ V_{\mathbf{k},\mathbf{p}}^{(1)} \langle \gamma^{\uparrow}_{\mathbf{p}2\sigma} \gamma^{\downarrow}_{\mathbf{p3}\sigma} \rangle + V_{\mathbf{k},\mathbf{p}}^{(2)} \langle \gamma^{\uparrow}_{\mathbf{p}2\sigma} \gamma^{\downarrow}_{\mathbf{p3}\sigma} \rangle \right],
\]  

(S32)

where we discarded the term with bands 1 and 4. Finally, using Eq. \([S15]\), we derive

\[
\Delta_\uparrow = g \Delta_\uparrow \ln \left[ \frac{E^*}{M(\mu, \Delta_\uparrow)} \right] + g_{um} \Delta_\downarrow \ln \left[ \frac{E^*}{M(\mu, \Delta_\downarrow)} \right],
\]  

(S33)

\[
\Delta_\downarrow = g \Delta_\downarrow \ln \left[ \frac{E^*}{M(\mu, \Delta_\downarrow)} \right] + g_{um} \Delta_\uparrow \ln \left[ \frac{E^*}{M(\mu, \Delta_\uparrow)} \right].
\]  

(S34)

To describe two remaining sectors, \( (\uparrow, \xi) \) and \( (\downarrow, \xi) \), the identical set of equations should be used. In Eqs. \([S33]\) and \([S34]\), the quantity \( M(\mu, \Delta) \) effectively functions as the low-energy cutoff: if in a given sector \( \Delta > \mu \), this sector remains undoped, and \( M(\mu, \Delta) = \Delta \); when a sector accommodates finite doping \( \mu > \Delta \), in such a situation \( M(\mu, \Delta) = \mu + \sqrt{\mu^2 - \Delta^2} \). Formally, this can be expressed as

\[
M(\mu, \Delta) = (\mu + \sqrt{\mu^2 - \Delta^2}) \Theta(\mu - \Delta) + \Delta \Theta(\Delta - \mu).
\]  

(S35)

Note also that in Eqs. \([S33]\) and \([S34]\) we used the simplified notation \( \Delta_\uparrow = \Delta_{\uparrow\xi} \) and \( \Delta_\downarrow = \Delta_{\downarrow\xi} \). The coupling constant is \( g_{um} = \beta V_{um} \nu(\varepsilon_F) \), where \( \nu(\varepsilon_F) \) is the density of states, and \( \beta \) is a numerical coefficient of order unity.

When the system is undoped, we can introduce \( \Delta_0 \) as follows \( \mu = \Delta_\uparrow = \Delta_\downarrow = \Delta_0 \) [note that this is a redefinition of \( \Delta_0 \) initially given by Eq. \([S27]\)]. In such a limit, both equations become identical

\[
\Delta_0 = g(1 + \gamma) \Delta_0 \ln \left( \frac{E^*}{\Delta_0} \right), \quad \text{where} \quad \gamma = \frac{g_{um}}{g}.
\]  

(S36)

This equation has one non-zero solution

\[
\Delta_0 = E^* \exp \left( -\frac{1}{g(1 + \gamma)} \right).
\]  

(S37)

We can see that the umklapp coupling increases \( \Delta_0 \).
B. Doped state

Now we discuss the doped system. Below we will consider two possibilities: (i) all four sectors are doped equally, and (ii) three sectors remain undoped, and all doping only enters a single sector. Let us start with (i). In such a situation $\mu > \Delta_s = \Delta(x)$ for all four $s$. Equations (S33) and (S44) become identical

$$\Delta = g(1 + \gamma)\Delta \ln \left(\frac{E^*}{\mu + \sqrt{\mu^2 - \Delta^2}}\right),$$  \hspace{1cm} (S38)

valid in all four sectors. The solution to this equation is similar to Eq. (S28)

$$\Delta(x) = \Delta_0 \sqrt{1 - \frac{x}{x_0}}, \hspace{1cm} \mu = \Delta_0 \left(1 - \frac{x}{2x_0}\right),$$  \hspace{1cm} (S39)

where we took into account that partial dopings equal to half of the total doping: $x_s = x/4$. The expression for $\mu(x)$ allows us to calculate $\Delta F(x)$.

$$\Delta F(x) = 4 \int_0^x \mu(x) \, dx = 4\Delta_0 x - \Delta_0 \frac{x^2}{x_0}.$$  \hspace{1cm} (S40)

This free energy is denoted as $\Delta F_\mu$ in the main text.

For case (ii), the calculations are more complicated. We define $\delta_\sigma(x)$ as follows $\Delta_\sigma(x) = \Delta_0[1 - \delta_\sigma(x)]$. For definiteness, we assume that the sector $s = (\uparrow, \xi)$ is undoped, while $s = (\downarrow, \xi)$ is doped. This means that $\Delta_\uparrow > \mu > \Delta_\downarrow$. Two other sectors, $(\uparrow, \xi)$ and $(\downarrow, \xi)$, are undoped, and decoupled from $s$ and $s'$. Therefore, they are characterized by the order parameter $\Delta_0$, given by Eq. (S37)

$$0 < \delta_\uparrow < m < \delta_\downarrow, \hspace{1cm} \text{where} \hspace{1cm} m = \frac{\Delta_0 - \mu}{\Delta_0}. \hspace{1cm} (S41)$$

Let us introduce yet another quantity, $\delta S$, as follows

$$\mu + \sqrt{\mu^2 - \Delta_0^2} = \Delta_0 \left[1 - m + \sqrt{(1 - m)^2 - (1 - \delta_\downarrow)^2}\right] = \Delta_0(1 + \delta S),$$  \hspace{1cm} (S42)

$$\delta S = \sqrt{(1 - m)^2 - (1 - \delta_\downarrow)^2 - m}.$$  \hspace{1cm} (S43)

The parameters $\delta_\sigma$, $\delta S$, and $m$ are small in the limit of small doping $x$. However, they have different degrees of smallness. Indeed, as we will see later

$$\delta_\sigma = O(m), \hspace{1cm} \delta S = O(m^{3/2}).$$  \hspace{1cm} (S44)

These relations become important when we solve the self-consistent equations in the limit of small doping.

Our goal is to solve the following equations

$$(1 - \delta_\uparrow) = g(1 - \delta_\uparrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 - \delta_\uparrow)\right] + \gamma g(1 - \delta_\downarrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 + \delta S)\right],$$  \hspace{1cm} (S45)

$$(1 - \delta_\downarrow) = g(1 - \delta_\downarrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 + \delta S)\right] + \gamma g(1 - \delta_\uparrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 - \delta_\uparrow)\right],$$  \hspace{1cm} (S46)

to find $\delta_\sigma$ as a function of $m$, and then determine $m$ versus $x$. In the limit of small $x$, we expand the self-consistent equations and, keeping in mind Eq. (S44), we derive

$$\frac{1}{g(1 + \gamma)} - \delta_\uparrow = \frac{1}{g(1 + \gamma)} + \delta_\downarrow + \gamma \left(\frac{1}{g(1 + \gamma)} - \delta S - \frac{\delta S^2}{2}\right) + O(m^{3/2}),$$  \hspace{1cm} (S47)

$$\frac{1}{g(1 + \gamma)} - \delta_\downarrow = \frac{1}{g(1 + \gamma)} - \delta S + \frac{\delta S^2}{2} + \gamma \left(\frac{1}{g(1 + \gamma)} + \delta_\uparrow\right) + O(m^{3/2}).$$  \hspace{1cm} (S48)

Simplifying, we obtain

$$\delta_\uparrow \approx \frac{\delta_\downarrow}{1 + \gamma} - g\delta_\uparrow + \gamma \left(\frac{1}{1 + \gamma} - \delta_\downarrow + g \left(\delta S - \frac{\delta S^2}{2}\right)\right),$$  \hspace{1cm} (S49)

$$\delta_\downarrow \approx \frac{\delta_\uparrow}{1 + \gamma} + g \left(\delta S - \frac{\delta S^2}{2}\right) + \gamma \left(\frac{1}{1 + \gamma} - \delta_\uparrow + g\delta_\downarrow\right),$$  \hspace{1cm} (S50)
Next step:

\[
\begin{align*}
\left(\frac{\gamma}{1+\gamma} + g\right)\delta_\uparrow &= \frac{\gamma}{1+\gamma}\delta_\downarrow + g\gamma \left(\delta S - \frac{\delta S^2}{2}\right), \\
\left(\frac{\gamma}{1+\gamma} - g\gamma\right)\delta_\uparrow &= \frac{\gamma}{1+\gamma}\delta_\downarrow - g \left(\delta S - \frac{\delta S^2}{2}\right).
\end{align*}
\]

(S51)  

(S52)

Subtracting these two equations we derive

\[
g(1+\gamma)\delta_\uparrow = g(1+\gamma) \left(\delta S - \frac{\delta S^2}{2}\right) \Leftrightarrow \delta_\uparrow = \delta S - \frac{\delta S^2}{2}.
\]

(S53)

Now \(\delta_\uparrow\) can be eliminated

\[
\left[\frac{\gamma}{1+\gamma} + g(1-\gamma)\right] \left(\delta S - \frac{\delta S^2}{2}\right) = \frac{\gamma}{1+\gamma}\delta_\downarrow.
\]

(S54)

This relation is equivalent to

\[
\delta S - \frac{\delta S^2}{2} = \alpha\delta_\downarrow, \quad \text{where} \quad \alpha = \left[1 + g(\gamma^{-1} - \gamma)\right]^{-1}.
\]

(S55)

Let us express \(\delta S\) in the limit of small doping

\[
\delta S = \sqrt{(1-m)^2 - (1-\delta_\downarrow)^2} - m = \sqrt{(2-m-\delta_\downarrow)(\delta_\downarrow-m)} - m = \sqrt{2(\delta_\downarrow-m)} - m + O(m^{3/2}),
\]

\[
\delta S^2 = 2(\delta_\downarrow-m) + O(m^{3/2}).
\]

Therefore

\[
\delta S - \frac{\delta S^2}{2} = \sqrt{2(\delta_\downarrow-m)} - \delta_\downarrow + O(m^{3/2}).
\]

(S58)

The self-consistent equation becomes

\[
\alpha\delta_\downarrow = \sqrt{2(\delta_\downarrow-m)} - \delta_\downarrow + O(m^{3/2}).
\]

(S59)

Its solution is

\[
\delta_\downarrow \approx m + \frac{(1+\alpha)^2}{2}m^2, \quad \delta_\uparrow = \alpha\delta_\downarrow \approx \alpha m + \frac{\alpha(1+\alpha)^2}{2}m^2.
\]

(S60)

Let us check the consistency of these relations with known results in the \(\alpha = 0\) limit. In this case

\[
\delta_\downarrow \approx m + \frac{1}{2}m^2, \quad \delta_\uparrow = 0.
\]

(S61)

At the same time, Eqs. (S28) in the regime of small \(x\) can be written as

\[
m = \frac{x}{2x_0}, \quad \delta_\downarrow(x) = \frac{\Delta_0 - \Delta(x)}{\Delta_0} \approx \frac{x}{2x_0} + \frac{x^2}{8x_0^2}.
\]

(S62)

We can now exclude \(x\) to obtain

\[
\delta_\downarrow(x) \approx m + \frac{m^2}{2},
\]

(S63)

which coincides with Eq. (S61).

The final step is to add doping into the formalism. To this end, we write

\[
4x = 2\nu_F \int_{\Delta_\downarrow}^\mu d\varepsilon \frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta_\downarrow^2}},
\]

(S64)
where $4x$ is the doping per unit cell, $
u_F = t_0/(\sqrt{3}\pi t^2)$ is the density of states per unit cell for each single Fermi surface sheet (there are four Fermi surface sheets),

$$x = \frac{\nu_F}{2} \sqrt{\mu^2 - \Delta_0^2} = \frac{\nu_F \Delta_0}{2} \sqrt{(1-m)^2 - (1-\delta_\uparrow)^2} = \frac{x_0}{2} \sqrt{(1-m)^2 - (1-\delta_\uparrow)^2},$$

(S65)

where $x_0 = \nu_F \Delta_0$. It is possible to show that $4x^2 = x_0^2(\delta_\uparrow - m)(2 - m - \delta_\downarrow)$ \Rightarrow $4x^2 = (1 + \alpha)^2 x_0^2 m^2 + O(m^3)$.

Deriving the latter relation we used Eq. (S60), which, among other things, demonstrates that $\delta_\downarrow - m = O(m^2)$.

Equation (S66) allows us to establish the following connection between doping and the chemical potential

$$m = \frac{2x}{(1 + \alpha)x_0} + O(x^2) \Leftrightarrow \mu = \Delta_0 \left(1 - \frac{2x}{(1 + \alpha)x_0}\right) + O(x^2).$$

(S67)

Integrating $\mu(x)$, we obtain

$$\Delta F_{qm} = 4\Delta_0 x - \frac{4\Delta_0}{1 + \alpha} \frac{x^2}{x_0}.$$  

(S68)

In the limit $\alpha \to 0$ we recover the expression for $\Delta F_{qm}$ given in the main text [see after Eq. (30)]. The free energy must be compared against the free energy given by Eq. (S40). We see that the quarter-metal is stable if $(1 + \alpha)^{-1} > 1/4$.

Equivalently,

quarter-metal is stable when $\alpha(\gamma) < 3$.

(S69)

To understand what the latter requirement entails, let us examine Fig. S1 which shows $\alpha(\gamma)$ for $g = 0.1$. We see that $\alpha < 3$ as long as $\gamma = g_{um}/g < 6.8$. That is, for $g = 0.1$, the umklapp satisfying

$$g_{um} < 0.68,$$

(S70)

does not violate the stability of the quarter-metal.

We note that Eq. (S69) is not the absolute stability criterion, rather it describes the stability of the quarter-metal against the transition into an ordinary metal, when all four sectors are doped equally. A comprehensive investigation of the stability goes well beyond the present study, and, most likely, requires input from experiments.

It is interesting to note that perturbation theory in powers of small $\gamma$ strongly underestimates the stability range of the quarter-metal. To demonstrate this, we expand the expression (S68) for $\Delta F_{qm}$ in powers of $\alpha$

$$\Delta F_{qm} \approx 4\Delta_0 x - 4\Delta_0 \frac{x^2}{x_0} + 4\Delta_0 \frac{\alpha x^2}{x_0}.$$  

(S71)

Since at $\gamma \to 0$, the following holds $\alpha \approx \gamma/g = g_{um}/g^2$, the expression for $\Delta F_{qm}$ can be approximated as

$$\Delta F_{qm} \approx 4\Delta_0 x - 4\Delta_0 \frac{x^2}{x_0} \left(1 - \frac{g_{um}}{g^2}\right),$$

(S72)
If we use this expression, instead of the more accurate Eq. (S68), we could (erroneously) conclude that the quarter-metal is stable when \((1 - g_{um}/g^2) > 1/4\). This inequality can be transformed to

\[
\frac{\gamma}{g} = \frac{g_{um}}{g^2} < \frac{3}{4} \quad \Leftrightarrow \quad g_{um} < \frac{3g^2}{4}.
\]  

(S73)

In Fig. [S1] we can see the low-\(\gamma\) approximation \(\alpha(\gamma) \approx \gamma/g\) as a (red) straight line. We see that, at low \(g\), this approximation works only at very small \(\gamma\); while for larger \(\gamma\) (larger \(g_{um}\)) it is completely useless. Thus, we conclude that the replacement \([1 + \alpha(\gamma)]^{-1} \rightarrow (1 - \gamma/g)\) artificially shrinks the stability range of the quarter-metal. Indeed, the requirement (S73) is very strict: at \(g = 0.1\), as in Fig. [S1], Eq. (S73) demand that \(g_{um} < 0.0075\), cf. Eq. (S70). This is the origin of the serious disparity between the stability condition derived in the main text using simple perturbation theory and more the sophisticated criterion (S69).