Supplemental Material for "Chiral-Extended Photon-Emitter Dressed States in Non-Hermitian Topological Baths"

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I. EFFECTIVE NON-HERMITIAN BATH IN SINGLE-EXCITATION SUBSPACE

As shown in the main text, we consider a set of N identical atoms, as quantum emitters, coupled to a 1D Su-Schrieffer–Heeger (SSH) photonic chain with L unit cells. The photonic chain consists of coupled cavities subjected to engineered nonlocal dissipation, as shown in Fig. 1 in the main text. Each two-level atom, with ground state $|g\rangle$ and excited state $|e\rangle$, is coupled to each cavity in the lattice. Under the Markovian and rotating-wave approximations, the dissipative dynamics of the system (in the rotating frame) is governed by the Lindblad master equation[S1–S4]:

$$\frac{d\hat{\rho}}{dt} = -i\left[\hat{\mathcal{H}}_{\rm e} + \hat{\mathcal{H}}_{\rm p} + \hat{\mathcal{H}}_{\rm int},\ \hat{\rho}\right] + \gamma\sum_{n=1}^{N}\mathcal{D}[\hat{\sigma}_n^-]\hat{\rho} + \kappa\sum_{j}\mathcal{D}[\hat{L}_j]\hat{\rho}, \tag{S1}$$

where the Hamiltonians of atoms $\hat{\mathcal{H}}_{\rm e}$, photonic SSH bath $\hat{\mathcal{H}}_{\rm p}$ and photon-emitter interaction $\hat{\mathcal{H}}_{\rm int}$ are written as

$$\hat{\mathcal{H}}_{\rm e} = \sum_{n=1}^{N}\Delta_0\hat{\sigma}_n^+\hat{\sigma}_n^-, \tag{S2}$$

$$\hat{\mathcal{H}}_{\rm p} = \sum_{j=1}^{L}\left(J_1\hat{b}_j^\dagger\hat{a}_j + J_2\hat{a}_{j+1}^\dagger\hat{b}_j + \text{H.c.}\right), \tag{S3}$$

$$\hat{\mathcal{H}}_{\rm int} = \sum_{n=1}^{N}\sum_{\alpha\in\{a,b\}} g\left(\hat{\alpha}_{j_n}^\dagger\hat{\sigma}_n^- + \text{H.c.}\right). \tag{S4}$$

Here, $\hat{\sigma}_n^- = (\hat{\sigma}_n^+)^\dagger = |g_n\rangle\langle e_n|$ is the pseudospin ladder operator of the nth atom, Δ_0 is frequency detuning of the atom with respect to the cavity frequency, $\hat{a}_j$ and $\hat{b}_j$ annihilate photons at sublattices a and b of the jth unit cell (see Fig. 1 in the main text), g is the photon-emitter interacting strength, and j_n labels the unit cell at which the nth atom is located. Moreover, $\hat{\rho}$ is the system density matrix, the Lindblad superoperator $\mathcal{D}[\mathcal{L}]\hat{\rho} = \mathcal{L}\hat{\rho}\mathcal{L}^\dagger - \{\mathcal{L}^\dagger\mathcal{L},\ \hat{\rho}\}/2$ represents atomic and photonic dissipation, γ is the atomic decay rate, and κ denotes the photonic loss. In this work, we consider the nonlocal photon decay between two sublattices a and b in each unit cell with $\hat{L}_j = \hat{a}_j - i\hat{b}_j$. This type of nonlocal dissipation has been extensively studied in theoretical frameworks[S5–S7], and demonstrated in experimental settings[S8]. The nonlocal dissipation of the photonic waveguide can be realized by coupling it to an auxiliary bath, When the auxiliary bath operates under conditions of large detuning or strong dissipation, it can be adiabatically eliminated, effectively implementing the desired nonlocal dissipation $\hat{L}_j = \hat{a}_j - i\hat{b}_j$[S5].

We consider the single-excitation subspace with an initial state $|\psi_0\rangle = \hat{\sigma}_n^+|g\rangle\otimes|\text{vac}\rangle$, here $|g\rangle \equiv |g_1g_2\ldots g_N\rangle$ and $|\text{vac}\rangle$ is the photon vacuum state, and so as the initial density matrix $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$. Then, the master equation in Eq. (S1) can be solved as[S9–S12]

$$\hat{\rho}_t = e^{-i\hat{\mathcal{H}}_{\rm eff}t}\hat{\rho}_0 e^{i\hat{\mathcal{H}}_{\rm eff}^\dagger t} + p_t|g\rangle\langle g|\otimes|\text{vac}\rangle\langle\text{vac}|, \tag{S5}$$

with

$$p_t = 1 - \text{Tr}[e^{-i\hat{\mathcal{H}}_{\rm eff}t}\hat{\rho}_0 e^{i\hat{\mathcal{H}}_{\rm eff}^\dagger t}]. \tag{S6}$$

Therefore, when focusing only on the single-excitation subspace, the system's dynamics is governed by the effective non-Hermitian Hamiltonian

$$\hat{\mathcal{H}}_{\rm eff} = \hat{\mathcal{H}}_{\rm e} + \hat{\mathcal{H}}_{\rm p} + \hat{\mathcal{H}}_{\rm int} - i\gamma/2 \sum_n \hat{\sigma}_n^+ \hat{\sigma}_n^- - i\kappa/2 \sum_j \hat{L}_j^\dagger \hat{L}_j. \tag{S7}$$

Under periodic boundary conditions (PBCs), and using $\hat{\alpha}_k = L^{-1/2}\sum_j e^{-ikj}\hat{\alpha}_j$ $(\alpha = a, b)$, the momentum-space Hamiltonian becomes

$$\hat{\mathcal{H}}_{\rm eff}(k) = \Delta \sum_{n=1}^{N} \hat{\sigma}_n^+ \hat{\sigma}_n^- + \sum_k \hat{\rm a}_k^\dagger H_k \hat{\rm a}_k + \frac{1}{\sqrt{L}} \sum_{n=1}^{N} \sum_k \left(\hat{\sigma}_n^- \hat{\rm a}_k^\dagger {\rm g}_{kn} + {\rm H.c.} \right), \tag{S8}$$

where $\Delta = \Delta_0 - i\gamma/2$, $\hat{\rm a}_k \equiv [\hat{a}_k,\ \hat{b}_k]^T$, ${\rm g}_{kn} = [g_a e^{-ikj_n},\ g_b e^{-ikj_n}]^T$ with $g_a, g_b \in \{0, g\}$, and the Bloch Hamiltonian of the non-Hermitian SSH bath $H_p(k)$ is

$$H_k = -i\frac{\kappa}{2}\tau_0 + (J_1 + J_2 \cos k)\,\tau_x + (J_2 \sin k - i\kappa/2)\,\tau_y, \tag{S9}$$

with Pauli matrices τ_i $(i = x, y, z)$ and identity matrix τ_0.

II. BULK-BOUNDARY CORRESPONDENCE OF A NON-HERMITIAN SSH BATH

As shown in Eq. (S9), the Bloch Hamiltonian of the non-Hermitian SSH bath is rewritten as

$$H_k = \mathcal{H}_k - i(\kappa/2)\tau_0, \quad \text{with} \quad \mathcal{H}_k = (J_1 + J_2 \cos k)\,\tau_x + (J_2 \sin k - i\kappa/2)\,\tau_y, \tag{S10}$$

where the Hamiltonians H_k and $\mathcal{H}_k$ are topologically equivalent. The non-Hermitian skin effect leads to the breakdown of the conventional bulk-boundary correspondence. The topological-phase boundary can be recovered by the non-Bloch theory[S13], where the non-Bloch Hamiltonian for $\mathcal{H}_k$ reads

$$\mathcal{H}_\beta = \begin{pmatrix} 0 & J_1 - \frac{\kappa}{2} + J_2\beta^{-1} \\ J_1 + \frac{\kappa}{2} + J_2\beta & 0 \end{pmatrix}. \tag{S11}$$

One can obtain the eigenvalue equation for β as $\det[\mathcal{H}_\beta - E] = 0$. Therefore, we have

$$\left[\left(J_1 - \frac{\kappa}{2}\right) + J_2\beta^{-1}\right]\left[\left(J_1 + \frac{\kappa}{2}\right) + J_2\beta\right] = E^2. \tag{S12}$$

This leads to two solutions

$$\beta_{1,2}(E) = \frac{[E^2 + \kappa^2/4 - J_1^2 - J_2^2] \pm \sqrt{[E^2 + \kappa^2/4 - J_1^2 - J_2^2]^2 - 4J_2^2(J_1^2 - \kappa^2/4)}}{2J_2(J_1 - \kappa/2)}, \tag{S13}$$

where $+(-)$ corresponds to β_1 (β_2). Then, we obtain

$$\beta_1\beta_2 = \frac{J_1 + \kappa/2}{J_1 - \kappa/2}. \tag{S14}$$

According to the generalized Bloch band theory[S14], we require $|\beta_1| = |\beta_2|$. This leads to

$$|\beta_1| = |\beta_2| = \sqrt{\left|\frac{J_1 + \kappa/2}{J_1 - \kappa/2}\right|}. \tag{S15}$$

Due to chiral symmetry, we can obtain a generalized Q matrix[S13], defined as

$$Q(\beta) = |\tilde{\psi}_R(\beta)\rangle\langle\tilde{\psi}_L(\beta)| - |\psi_R(\beta)\rangle\langle\psi_L(\beta)| = \begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}, \tag{S16}$$

where $|\tilde{\psi}_R(\beta)\rangle = \sigma_z|\psi_R(\beta)\rangle$ and $|\tilde{\psi}_L(\beta)\rangle = \sigma_z|\psi_L(\beta)\rangle$, with the right and left eigenvectors given by the following eigenequations

$$\mathcal{H}_\beta|\psi_R(\beta)\rangle = E(\beta)|\psi_R(\beta)\rangle, \quad \mathcal{H}_\beta^\dagger|\psi_L(\beta)\rangle = E^*(\beta)|\psi_L(\beta)\rangle. \tag{S17}$$

The non-Bloch winding number $\mathcal{W}$ is given by

$$\mathcal{W} = \frac{i}{2\pi}\int_{\mathrm{GBZ}} \frac{dq}{q}, \tag{S18}$$

where GBZ denotes the generalized Brillouin zone.

According to Eqs. (S16-S18), the true topological-phase transition points in the presence of non-Hermitian skin effects are given by

$$J_1 = \pm\sqrt{J_2^2 + \kappa^2/4}, \tag{S19}$$

where the SSH bath is topologically nontrivial when $J_1 \in (-\sqrt{J_2^2+\kappa^2/4},\ \sqrt{J_2^2+\kappa^2/4})$.

This indicates that the bath is topologically trivial for $J_2 = J_1 - \kappa/2$ with $J_1 > \kappa/2$ due to $\kappa \geq 0$, where we explore this regime for chiral-extended photon-emitter dressed states.

III. CHIRAL AND HIDDEN BOUND STATES

Due to the particle number conservation of the system Hamiltonian $\hat{\mathcal{H}}_{\mathrm{eff}}(k)$ in Eq. (S8), in the single-excitation subspace, the bound state can be written as

$$|\psi_{\mathrm{b}}\rangle = \left(\frac{1}{\sqrt{L}}\sum_k \mathrm{c}_k\hat{\mathrm{a}}_k^\dagger + \sum_{n=1}^{N} c_{e,n}\hat{\sigma}_n^+\right)|g\rangle\otimes|\mathrm{vac}\rangle\,, \tag{S20}$$

with $\mathrm{c}_k \equiv [c_{k,a},\ c_{k,b}]^T$, which satisfies $\hat{\mathcal{H}}_{\mathrm{eff}}(k)\,|\psi_{\mathrm{b}}\rangle = E_{\mathrm{b}}\,|\psi_{\mathrm{b}}\rangle$. Then, we obtain

$$\Delta\mathrm{c}_e + \frac{1}{L}\sum_k \mathrm{g}_k^\dagger\mathrm{c}_k = E_{\mathrm{b}}\mathrm{c}_e, \quad \text{and} \quad H_k\mathrm{c}_k + \mathrm{g}_k\mathrm{c}_e = E_{\mathrm{b}}\mathrm{c}_k, \tag{S21}$$

for $\forall\ k$, where $\mathrm{c}_e \equiv [c_{e,1},\ c_{e,2},\ldots,\ c_{e,N}]^T$, and $\mathrm{g}_k \equiv [\mathrm{g}_{k1},\ \mathrm{g}_{k2},\ldots,\ \mathrm{g}_{kN}]$.

According to Eqs. (S21), we have

$$[E_{\mathrm{b}} - \Delta - \Sigma(E_{\mathrm{b}})]\,\mathrm{c}_e = 0, \tag{S22}$$

and for photon-emitter bound states, we require $\mathrm{c}_e \neq 0$. This yields

$$\det[E_{\mathrm{b}} - \Delta - \Sigma(E_{\mathrm{b}})] = 0, \tag{S23}$$

where $\Sigma(z)$ is the self-energy of the emitters, given by

$$\Sigma(z) = \frac{1}{L}\sum_k \mathrm{g}_k^\dagger\,(z - H_k)^{-1}\,\mathrm{g}_k. \tag{S24}$$

We then can determine the atomic and photonic weights as

$$|\mathrm{c}_e|^2 = \frac{1}{1 + \frac{1}{L}\sum_k \mathrm{g}_k^\dagger\left[(E_{\mathrm{b}} - H_k)(E_{\mathrm{b}}^* - H_k^\dagger)\right]^{-1}\mathrm{g}_k}, \quad \text{and} \quad \mathrm{c}_k = \frac{\mathrm{g}_k\mathrm{c}_e}{E_{\mathrm{b}} - H_k}. \tag{S25}$$

In this section, we are interest in a single emitter coupled to the sublattice a or b within the unit cell j_0 of the bath, and study the bound states with its eigenenergy lying within the regimes of both line and point gaps of the non-Hermitian SSH bath. In this work, unless otherwise specified, we assume $\gamma = \kappa$.

A. Line gap and chiral bound state

In the presence of the line gap for the bath Hamiltonian H_k, to have a simple form of the analytical solution for the bound-state wavefunction in Eq. (S25), we solve the bound state for $E_{\mathrm{b}} = -i\kappa/2$.

According to Eqs. (S23) and (S24), for $E_{\mathrm{b}} = -i\kappa/2$, we immediately have

$$\Sigma\left(E_{\mathrm{b}}\right) = 0, \quad \text{and} \quad \Delta = -i\kappa/2. \tag{S26}$$

Then, according to Eq. (S25), the atomic weight $|c_e|^2$ for the emitter coupled to the sublattice a or b is, respectively, derived as

$$|c_{e,a}|^2 = \frac{1}{1 + \frac{g^2}{L}\sum_k |J_1 + J_2 e^{-ik} - \kappa/2|^{-2}}, \tag{S27}$$

or

$$|c_{e,b}|^2 = \frac{1}{1 + \frac{g^2}{L}\sum_k |J_1 + J_2 e^{ik} + \kappa/2|^{-2}}. \tag{S28}$$

(i) When the emitter is coupled to the sublattice a, the photonic weight $c_{k,\alpha}$ ($\alpha = a, b$) for $\Delta = -i\kappa/2$ in momentum space is obtained as

$$c_{k,a} = 0, \quad \text{and} \quad c_{k,b} = -\frac{g_k c_{e,a}}{J_1 + J_2 e^{-ik} - \kappa/2}, \tag{S29}$$

where $g_k = g e^{-ikj_0}$. The real-space photonic profile can be obtained by the inverse Fourier transformation of Eq. (S29). This leads to $c_{j,a} = 0$, and

$$c_{j,b} = -\frac{g c_{e,a}}{L}\sum_k \frac{e^{ik(j-j_0)}}{J_1 + J_2 e^{-ik} - \frac{\kappa}{2}} = -\frac{g c_{e,a}}{2\pi i}\oint_{|y|=1} dy \frac{y^{j-j_0}}{J_2 + (J_1 - \kappa/2) y}. \tag{S30}$$

According to Eq. (S30), for $|J_2| < |J_1 - \kappa/2|$, where the PBC spectrum of the SSH bath Hamiltonian H_k exhibits a line gap, we obtain

$$c_{j,b} = \begin{cases} -\frac{g c_{e,a}}{J_1 - \kappa/2}\left(-\frac{J_2}{J_1 - \kappa/2}\right)^{j-j_0}, & j \geq j_0, \\ 0, & j < j_0. \end{cases} \tag{S31}$$

(ii) When the emitter is coupled to the sublattice b, the photonic weight $c_{\alpha,k}$ ($\alpha = a, b$) for $\Delta = -i\kappa/2$ in momentum space is obtained as

$$c_{k,a} = -\frac{g_k c_{e,b}}{J_1 + J_2 e^{ik} + \kappa/2}, \quad \text{and} \quad c_{k,b} = 0. \tag{S32}$$

The real-space photonic profile is obtained by the inverse Fourier transformation of Eq. (S32). This leads to $c_{j,b} = 0$, and

$$c_{j,a} = -\frac{g c_{e,b}}{L}\sum_k \frac{e^{ik(j-j_0)}}{J_1 + J_2 e^{ik} + \frac{\kappa}{2}} = -\frac{g c_{e,b}}{2\pi i}\oint_{|y|=1} dy \frac{y^{j-j_0-1}}{J_2 y + J_1 + \kappa/2}. \tag{S33}$$

According to Eq. (S33), for $|J_2| < |J_1 + \kappa/2|$, where the PBC spectrum exhibits a line gap of the SSH bath Hamiltonian H_k, we obtain

$$c_{j,a} = \begin{cases} 0, & j > j_0, \\ \frac{g c_{e,b}}{J_2}\left(-\frac{J_1 + \kappa/2}{J_2}\right)^{j-j_0-1}, & j \leq j_0. \end{cases} \tag{S34}$$

The above analytical results show that the bound state, with its eigenenergy lying inside the line gap, has its eigenstate located on just the left or right side of the emitter, depending on the sublattice a or b to which the emitter is coupled, for $\Delta = -i\kappa/2$, as shown in Fig. S1(a,b1). As explained in the main text, such a chiral bound state has a topological origin, which can be interpreted as the effective domain-wall state between two semi-infinite chains with different topology.

In spite of the nonreciprocal hopping, the bound state inside the line gap behaves more like conventional Hermitian bound states for $\Delta = -i\kappa/2$[S15]. When the detuning Δ is deviated from the $-i\kappa/2$, the chirality of the bound states inside the line gap decreases due to coupling with the bulk bands of the bath, as shown in Fig. S1(b2). Outside the eigenenergy-spectrum range (i.e., above or below the two bands), the bound states do not exhibit chirality due to the absence of topological protection [see Fig. S1(b3)].

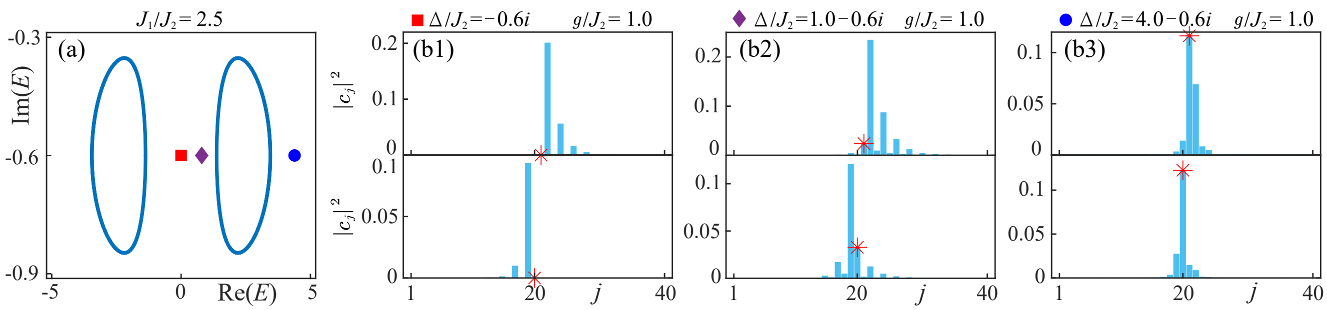

FIG. S1. Single-excitation line-gap spectrum (blue loops) of the bath H_k under PBCs for $J_1/J_2 = 2.5$, The markers denote the eigenenergies of the bound states of a single emitter coupled to the bath for different Δ. The corresponding site-resolved photon weights $|c_j|^2$ are shown in (b1-b3), where the emitter is coupled to the sublattice a (b), denoted by the red asterisk, for the top (bottom) plot. The other parameters used are $\kappa/J_2 = 1.2$.

B. Point gap and hidden bound state

We now consider the bound state with its eigenenergy enclosed by the point gap of the bath Hamiltonian H_k. To have a simple form of the analytical solution for the bound-state wavefunction in Eq. (S25), we solve the bound state for $J_1 = \kappa/2$. In this case, the eigenenergy of the bath Hamiltonian H_k in Eq. (S9) reads

$$E_0 = -i(\kappa/2) \pm \left[J_2(J_2 + \kappa e^{-ik})\right]^{-1/2}. \tag{S35}$$

According to Eq. (S24), the self energy of the bound state is written as

$$\Sigma(E_\mathrm{b}) = \frac{g^2}{2\pi i}\oint_{|y|=1} dy \frac{E_\mathrm{b} + \frac{i\kappa}{2}}{\left[(E_\mathrm{b} + i\kappa/2)^2 - J_2^2\right] y - J_2\kappa} = \begin{cases} -\frac{g^2(E_\mathrm{b}+\frac{i\kappa}{2})}{J_2^2-(E_\mathrm{b}+\frac{i\kappa}{2})^2}, & |\kappa J_2| < \left|J_2^2 - (E_\mathrm{b} + i\kappa/2)^2\right|, \\ 0, & |\kappa J_2| > \left|J_2^2 - (E_\mathrm{b} + i\kappa/2)^2\right|. \end{cases} \tag{S36}$$

It turns out that the self-energy vanishes when the E_b lies inside the loop of the point-gap spectrum. We will reveal that such a bound state shows a skin-mode-like photonic profile resulting from the nontrivial point-gap topology, dubbed hidden bound state [S11,S12].

(i) When the emitter is coupled to the sublattice a, the photonic weight $c_{k,\alpha}$ ($\alpha = a, b$) in momentum space is obtained as

$$c_{k,a} = c_{e,a} g_k \left[E_b + \frac{i\kappa}{2} - \frac{J_2^2 + \kappa J_2 e^{-ik}}{E_b + i\kappa/2}\right]^{-1}, \quad \text{and} \quad c_{k,b} = c_{e,a} g_k \left[\frac{(E_b + i\kappa/2)^2}{\kappa + J_2 e^{ik}} - J_2 e^{-ik}\right]^{-1}, \tag{S37}$$

where $g_k = g e^{-ikj_0}$. The real-space photonic profile can be obtained by the inverse Fourier transformation of Eq. (S37). This leads to

$$c_{j,a} = \frac{g c_{e,a}(E_b + \frac{i\kappa}{2})}{L} \sum_k \frac{e^{ik(j-j_0)}}{[(E_b + \frac{i\kappa}{2})^2 - J_2^2] - \kappa J_2 e^{-ik}} = \frac{g c_{e,a}(E_b + \frac{i\kappa}{2}) y_A}{2\pi i \kappa J_2} \oint_{|y|=1} dy \frac{y^{j-j_0}}{y - \eta}, \tag{S38}$$

and

$$c_{j,b} = \frac{g c_{e,a}}{L} \sum_k \frac{(\kappa + J_2 e^{ik}) e^{ik(j-j_0)}}{[(E_b + \frac{i\kappa}{2})^2 - J_2^2] - \kappa J_2 e^{-ik}} = \frac{g c_{e,a}\eta}{2\pi i \kappa J_2} \oint_{|y|=1} dy \frac{\kappa y^{j-j_0} + J_2 y^{j-j_0+1}}{y - \eta}, \tag{S39}$$

where $\eta = (\kappa J_2)/[(E_b + i\kappa/2)^2 - J_2^2]$.

We consider the E_b lies inside the loop of the point-gap spectrum with $|\kappa J_2| > \left|J_2^2 - (E_b + i\kappa/2)^2\right|$. According to Eqs. (S38) and (S39), we obtain the real-space photonic profile as

$$c_{j,a} = \begin{cases} 0, & j \geq j_0, \\ \frac{-g c_{e,a}(E_b + i\kappa/2)\eta^{j-j_0+1}}{\kappa J_2}, & j < j_0, \end{cases} \tag{S40}$$

and

$$c_{j,b} = \begin{cases} 0, & j \geq j_0, \\ -\frac{g c_{e,a}}{J_2}, & j = j_0 - 1, \\ -\frac{g c_{e,a} \eta^{j-j_0+1}}{J_2} - \frac{g c_{c,a} \eta^{j-j_0+2}}{\kappa}, & j < j_0 - 1. \end{cases} \tag{S41}$$

(ii) When the emitter is coupled to the sublattice b, the photonic weight $c_{k,\alpha}$ ($\alpha = a, b$) in momentum space is obtained as

$$c_{k,a} = \frac{g c_{e,b} J_2 e^{-ik}}{(E_b + \frac{i\kappa}{2})^2 - J_2^2 - \kappa J_2}, \quad \text{and} \;\; c_{k,b} = \frac{g c_{e,b}(E_b + \frac{i\kappa}{2})}{(E_b + \frac{i\kappa}{2})^2 - J_2^2 - \kappa J_2}. \tag{S42}$$

The real-space photonic profile is obtained by the inverse Fourier transformation of Eq. (S42). This leads to

$$c_{j,a} = \frac{g c_{e,b} J_2}{L} \sum_k \frac{e^{ik(j-j_0-1)}}{(E_b + \frac{i\kappa}{2})^2 - J_2^2 - \kappa J_2 e^{-ik}} = \frac{g c_{e,b} \eta}{2\pi i \kappa} \oint_{|y|=1} dy \frac{y^{j-j_0-1}}{y - \eta}, \tag{S43}$$

and

$$c_{j,b} = \frac{g c_{e,b}(E_b + \frac{i\kappa}{2})}{L} \sum_k \frac{e^{ik(j-j_0)}}{(E_b + \frac{i\kappa}{2})^2 - J_2^2 - \kappa J_2 e^{-ik}} = \frac{g c_{e,b}(E_b + \frac{i\kappa}{2})\eta}{2\pi i \kappa J_2} \oint_{|y|=1} dy \frac{y^{j-j_0}}{y - \eta}. \tag{S44}$$

E_{b} lies inside the loop of the point-gap spectrum for $|\kappa J_2| > \left|J_2^2 - (E_b + i\kappa/2)^2\right|$. According to Eqs. (S43) and (S44), we obtain the real-space photonic profile as

$$c_{j,a} = \begin{cases} 0, & j > j_0, \\ -\frac{g c_{e,b} \eta^{(j-j_0)}}{\kappa}, & j \leq j_0, \end{cases} \tag{S45}$$

and

$$c_{j,b} = \begin{cases} 0, & j \geq j_0, \\ -\frac{g c_{e,b}(E_b + \frac{i\kappa}{2})\eta^{(j-j_0+1)}}{\kappa J_2}, & j < j_0. \end{cases} \tag{S46}$$

The above analytical results show that the bound state, with its eigenenergy lying inside the point gap, has its eigenstate located on only the left side of the emitter, no matter if the emitter is coupled to the sublattice a or b. Such a bound state behaves like the skin modes. Figure S2(a,b1,b2) plot the bound states and site-resolved photon weight for $J_1 = \kappa/2$. In spite of the detuning Δ, the eigenstates are located on only the left side of the emitter due to the NHSE. While, outside the loop of the point gap, the bound state behaves like the conventional Hermitian bound states [see Figure S2(b3)]. In addition, in spite of the coexistence of point gap and line gap, bound states inside the point gap behave like skin modes, as shown in Fig. S2(c,d)

C. Atomic weight

The atomic weight can be directly solved out using Eq. (S25). To have a simple form of the analytical solution for the bound-state wavefunction in Eq. (S25), we solve out the bound state for $J_1 = \kappa/2$. When the emitter is coupled to the sublattice a, the photonic weight reads

$$|c_{e,a}|^2 = \left[1 + \frac{4g^2}{w} + \frac{g^2(4J_2\kappa z_+^2 + u_a z_+ + 4J_2\kappa)}{4J_2\kappa v z_+(z_+ - z_-)}\theta(1 - |z_+|) + \frac{g^2(4J_2^2\kappa z_-^2 + u_a z_- + 4J_2\kappa)}{4J_2\kappa v z_-(z_- - z_+)}\theta(1 - |z_-|)\right]^{-1}, \tag{S47}$$

where

$$u_a = 4J_2^2 - 2iE_b\kappa + 5\kappa^2 + 4|E_{\mathrm{b}}|^2 + 2i\kappa E_{\mathrm{b}}^*, \tag{S48}$$

$$w = 4J_2^2 + \kappa^2 + 4i\kappa E_{\mathrm{b}}^* - 4(E_{\mathrm{b}}^*)^2, \quad \text{and} \quad v = J_2^2 - \left(E_{\mathrm{b}} + \frac{i\kappa}{2}\right)^2, \tag{S49}$$

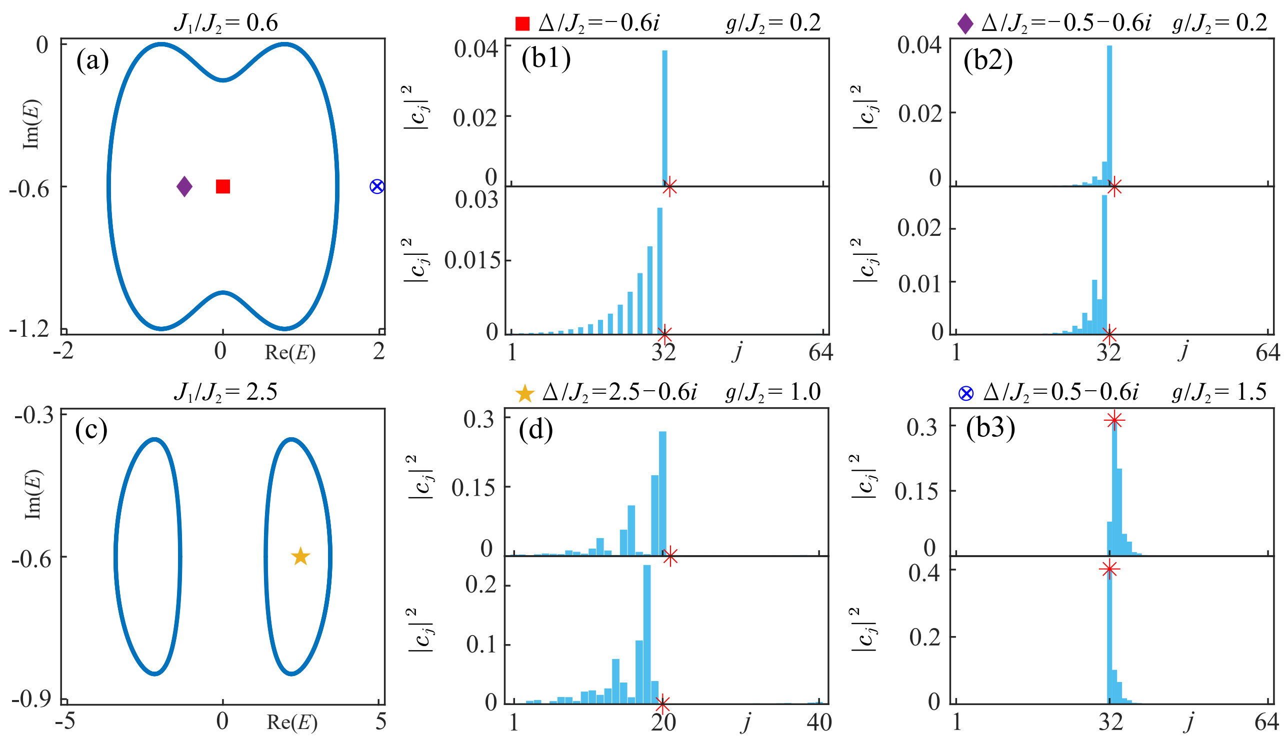

FIG. S2. Single-excitation point-gap spectrum (blue loops) of the bath under PBCs (a) for $J_1/J_2 = 2.5$, and (c) for $J_1/J_2 = 0.6$. The markers denote the eigenenergies of the bound states of a single emitter coupled to the bath for different Δ. The corresponding site-resolved photon weights $|c_j|^2$ are shown in (b1-b3) and (d), where the emitter is coupled to the sublattice a (b), denoted by the red asterisk, for the top (bottom) plot. The other parameters used are $\kappa/J_2 = 1.2$.

$$p = 4J_2^2\kappa^2 + vw, \quad \text{and} \quad z_\pm = \frac{-p \pm \sqrt{p^2 - 16J_2^2\kappa^2 vw}}{8J_2\kappa v}. \tag{S50}$$

When the emitter is coupled to the sublattice b, the photonic weight reads

$$|c_{e,b}|^2 = \left[1 + \frac{g^2 u_b}{4J_2\kappa v(z_+ - z_-)}\theta(1-|z_+|) + \frac{g^2 u_b}{4J_2\kappa v(z_- - z_+)}\theta(1-|z_-|)\right]^{-1}, \tag{S51}$$

where

$$u_b = 4J_2^2 - 2iE_b\kappa + \kappa^2 + 4|E_\mathrm{b}|^2 + 2i\kappa E_\mathrm{b}^*. \tag{S52}$$

In Fig. S3, we calculate the dependence of the atomic weights $|c_{e,a}|$ and $|c_{e,b}|$ of the hidden (blue curves) and Hermitian-like (red curves) bound states on the coupling strength g under PBCs for $J_1 = \kappa/2$, where the emitter is coupled to the sublattice a or b. The emergence of hidden bound states with eigenenergies inside the point gap does not rely on the coupling strength g. In contrast, the conventional bound states with energies outside the point gap only appear for sufficiently large g [see also Fig. S2(a,b1-b3)].

IV. EFFECTS OF DISORDER ON CHIRAL AND EXTENDED PHOTON-EMITTER DRESSED STATES

The chiral-extended photon-emitter dressed state has the topological origin, and it is thus robust against the disorder. To illustrate this, we investigate the effect of two types of disorders: (a) the random cavity frequencies with the addition of the diagonal terms to the original Hamiltonian $\hat{\mathcal{H}}_\mathrm{eff} \to \hat{\mathcal{H}}_\mathrm{eff} + \hat{\mathcal{H}}_\mathrm{diag}$, and (b) the random hopping between cavities with the addition of the off-diagonal terms to the original Hamiltonian $\hat{\mathcal{H}}_\mathrm{eff} \to \hat{\mathcal{H}}_\mathrm{eff} + \hat{\mathcal{H}}_\mathrm{off}$. The Hamiltonians $\hat{\mathcal{H}}_\mathrm{diag}$ and $\hat{\mathcal{H}}_\mathrm{off}$ are written as

$$\hat{\mathcal{H}}_\mathrm{diag} = \sum_{j=1}^{L}\left(\varepsilon_{a,j}\hat{a}_j^\dagger\hat{a}_j + \varepsilon_{b,j}\hat{b}_j^\dagger\hat{b}_j\right), \tag{S53}$$

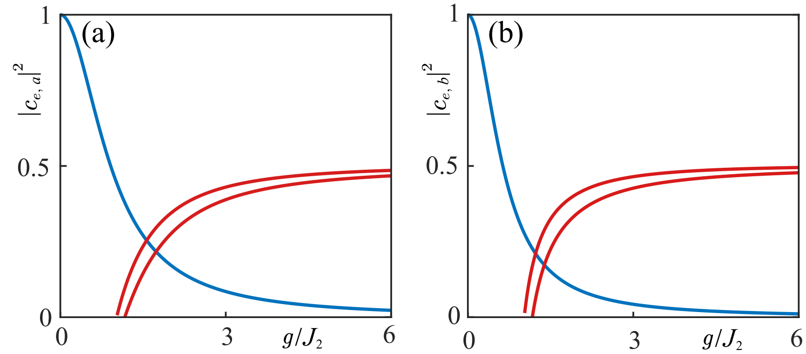

FIG. S3. Dependence of the atomic weights $|c_{e,a}|$ and $|c_{e,b}|$ of the hidden (blue curves) and Hermitian-like (red curves) bound states on the coupling strength g under PBCs for $J_1 = \kappa/2$, where the emitter is coupled to the sublattice a or b. The parameters used are $\Delta/J_2 = 0.2 - 0.4i$.

and

$$\hat{\mathcal{H}}_{\text{off}} = \sum_{j=1}^{L} \left(\varepsilon_{1,j} \hat{b}_j^\dagger \hat{a}_j + \text{H.c.} \right) + \sum_{j=1}^{L-1} \left(\varepsilon_{2,j} \hat{b}_j^\dagger \hat{a}_{j+1} + \text{H.c.} \right), \tag{S54}$$

where $\varepsilon_{a,j}$, $\varepsilon_{b,j}$, $\varepsilon_{1,j}$ and $\varepsilon_{2,j}$ are taken from a uniform distribution within the range $[-V/2,\ V/2]$ with the disorder strength V.

We consider a single emitter coupled to the sublattice a of the disordered SSH bath. Figure S4(a-d) plots the real part of the complex eigenspectrum E and the corresponding site-resolved photon weights $|c_j|^2$ for randomly disordered cavity frequencies (diagonal disorder). The chirality of the in-gap dressed photon-emitter state, along with its extended photon profile, remains remarkably robust even under strong disorder.

Figure S4(e-h) shows the real part of the complex eigenspectrum E and the corresponding site-resolved photon weights $|c_j|^2$ for the random hopping between cavities (off-diagonal disorder). The chirality and extended photon profile of the dressed state remain remarkably robust even in the presence of strong off-diagonal disorder caused by random hopping between cavities.

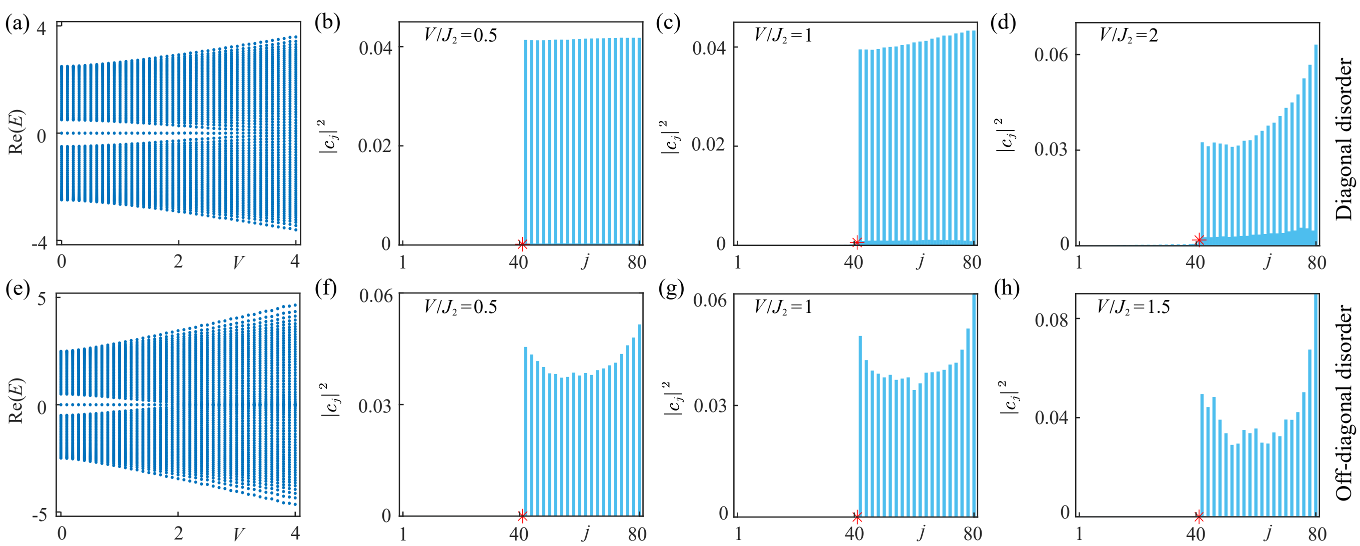

FIG. S4. Real part of the complex eigenspectrum E and the corresponding site-resolved photon weights $|c_j|^2$ at the transition point $J_2 = J_1 - \kappa/2$ under OBCs as a function of the disorder strength V, where a single emitter is coupled to the sublattice a of the disordered bath. The in-gap modes are the dressed photon-emitter state. (a-d) Disorder is applied to the cavity frequencies (diagonal disorder), and (e-h) the disorder is applied to the intercell couplings between cavities (off-diagonal disorder). The results are averaged over 1000 random realizations. The other parameters used are $\Delta/J_2 = -0.6i$, $g/J_2 = 0.5$, $\kappa/J_2 = 1.2$, $J_1/J_2 = 1.6$ and $L = 40$.

V. ANALYTICAL SOLUTION OF CHIRAL-EXTENDED PHOTON-EMITTER DRESSED STATES

A. Single Emitter

We now consider a single emitter coupled to the sublattice $\alpha \in \{a, b\}$ of the unit cell j_0. In the single-excitation subspace, spanned by $\{|e\rangle|\text{vac}\rangle, \; |g\rangle|j,a\rangle, |g\rangle|j,b\rangle\}$ with $j \in [1, \; L]$, and under the open boundary condition (OBC), the Hamiltonian of the photon-emitter hybrid system is written as

$$\mathcal{H}_\alpha = \begin{pmatrix} \Delta & V_\alpha \\ V_\alpha^\dagger & H_p \end{pmatrix}, \tag{S55}$$

where $\alpha = a, b$, indicating the sublattice a or b to which the emitter is coupled, the coupling vector

$$V_\alpha = (0, \; 0, \; 0, \; 0, \; \cdots, \; g\delta_{\alpha,a}, \; g\delta_{\alpha,a}, \; 0, \; 0, \; \cdots, \; 0), \tag{S56}$$

and the Hamiltonian matrix of the SSH chain H_p becomes

$$H_p = \begin{pmatrix} -i\frac{\kappa}{2} & J_1 - \frac{\kappa}{2} & 0 & 0 & \cdots & 0 & 0 \\ J_1 + \frac{\kappa}{2} & -i\frac{\kappa}{2} & J_2 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & -i\frac{\kappa}{2} & J_1 - \frac{\kappa}{2} & \cdots & 0 & 0 \\ 0 & 0 & J_1 + \frac{\kappa}{2} & -i\frac{\kappa}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -i\frac{\kappa}{2} & J_1 - \frac{\kappa}{2} \\ 0 & 0 & 0 & 0 & \cdots & J_1 + \frac{\kappa}{2} & -i\frac{\kappa}{2} \end{pmatrix}. \tag{S57}$$

We first discuss the emitter coupled to the sublattice $\alpha = a$. We implement a similarity transformation to the Hamiltonian $\mathcal{H}_a$ in Eq. (S55) with

$$\bar{\mathcal{H}}_a = S_a^{-1} \mathcal{H}_a S_a, \tag{S58}$$

where S_a is a diagonal matrix whose diagonal elements are

$$\{1, \; r^{-(j_0-1)}, \; r^{1-(j_0-1)}, \; r^{1-(j_0-1)}, \; r^{2-(j_0-1)}, \; \cdots, \; r^{L-1-(j_0-1)}, \; r^{L-1-(j_0-1)}, \; r^{L-(j_0-1)}\}, \quad \text{with } r = \sqrt{\frac{J_1 + \frac{\kappa}{2}}{J_1 - \frac{\kappa}{2}}}. \tag{S59}$$

Then, the $\bar{\mathcal{H}}_a$ is written as

$$\bar{\mathcal{H}}_a = \begin{pmatrix} \Delta & V_a \\ V_a^\dagger & \bar{H}_p \end{pmatrix}, \tag{S60}$$

where $\bar{H}_p$ reads

$$\bar{H}_p = \begin{pmatrix} -i\frac{\kappa}{2} & \bar{J}_1 & 0 & 0 & \cdots & 0 & 0 \\ \bar{J}_1 & -i\frac{\kappa}{2} & J_2 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & -i\frac{\kappa}{2} & \bar{J}_1 & \cdots & 0 & 0 \\ 0 & 0 & \bar{J}_1 & -i\frac{\kappa}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -i\frac{\kappa}{2} & \bar{J}_1 \\ 0 & 0 & 0 & 0 & \cdots & \bar{J}_1 & -i\frac{\kappa}{2} \end{pmatrix}, \tag{S61}$$

with $\bar{J}_1 = \sqrt{(J_1 - \kappa/2)(J_1 + \kappa/2)}$. After the similarity transformation, the photon-emitter Hamiltonian $\bar{\mathcal{H}}_a$ describes a Hermitian system subject to the uniform local dissipation with rate $\kappa/2$.

When the emitter is coupled to the sublattice $\alpha = b$, we can also implement a similarity transformation to the Hamiltonian $\mathcal{H}_b$ in Eq. (S55) with

$$\bar{\mathcal{H}}_b = S_b^{-1}\mathcal{H}_a S_b, \tag{S62}$$

where S_b is a diagonal matrix whose diagonal elements are

$$\{1,\ r^{-j_0},\ r^{1-j_0},\ r^{1-j_0},\ r^{2-j_0},\ \cdots,\ r^{L-1-j_0},\ r^{L-1-j_0},\ r^{L-j_0}\}. \tag{S63}$$

Then, $\bar{\mathcal{H}}_b$ is written as

$$\bar{\mathcal{H}}_b = \begin{pmatrix} \Delta & V_b \\ V_b^\dagger & \bar{H}_p \end{pmatrix}, \tag{S64}$$

After the similarity transformation, the photon-emitter Hamiltonian $\bar{\mathcal{H}}_b$ also describes a Hermitian system subjected to the uniform local dissipation with rate $\kappa/2$.

According to the Hamiltonian $\bar{\mathcal{H}}_a$ in Eq. (S60) or $\bar{\mathcal{H}}_b$ in Eq. (S64), in the single-excitation subspace, spanned by $\{|e\rangle|\mathrm{vac}\rangle,\ \ |g\rangle|j,\bar{a}\rangle, |g\rangle|j,\bar{b}\rangle\}$ with the similarity-transformed basis $|g\rangle|j,\bar{\alpha}\rangle$ ($\bar{\alpha} = \bar{a},\bar{b}$) and $j \in [1,\ L]$, the eigenequation for the photon-emitter dressed states reads

$$\bar{\mathcal{H}}|\bar{\psi}_\alpha\rangle = \left(\bar{\mathcal{H}}_0 + \bar{\mathcal{H}}_e + \bar{\mathcal{V}}_\alpha\right)|\bar{\psi}_\alpha\rangle = E|\bar{\psi}_\alpha\rangle, \tag{S65}$$

with the eigenenergy of the dressed state being $E_\mathrm{d} = E - i\kappa/2$, and

$$\bar{\mathcal{H}}_0 = \sum_{j=1}^{L}\left(\bar{J}_1|j,\bar{b}\rangle\langle j,\bar{a}| + J_2|j,\bar{b}\rangle\langle j+1,\bar{a}| + \mathrm{H.c.}\right), \tag{S66}$$

$$\bar{\mathcal{H}}_e = \Delta_0|e\rangle\langle e|, \tag{S67}$$

$$\bar{\mathcal{V}}_\alpha = g\left(|e\rangle\langle j_0,\bar{\alpha}| + |j_0,\bar{\alpha}\rangle\langle e|\right). \tag{S68}$$

The Hermitian SSH Hamiltonian $\bar{\mathcal{H}}_0$ for $J_2 = J_1 - \kappa/2$ (topological trivial phase) under OBCs can be solved out as[S16]

$$\bar{\mathcal{H}}_0 = \sum_{m=1}^{2L}\varepsilon_m|\varphi_m\rangle\langle\varphi_m|, \tag{S69}$$

where the eigenvalue reads

$$\varepsilon_m = (-1)^m \sqrt{2\bar{J}_1 J_2 \cos\theta_m + \bar{J}_1^2 + J_2^2}, \tag{S70}$$

with θ_m being a real number satisfying

$$\bar{J}_1 \sin[(L+1)\theta_m] + J_2 \sin[L\theta_m] = 0, \tag{S71}$$

and the eigenvectors are

$$|\varphi_m\rangle = \frac{1}{\sqrt{\mathcal{N}_m}} \sum_{j=1}^{L} \left(\varphi_{m,a}(j) |j, \bar{a}\rangle + \varphi_{m,b}(j) |j, \bar{b}\rangle\right). \tag{S72}$$

Here, $\mathcal{N}_m$ is the normalized constant for $|\varphi_m\rangle$, and the amplitudes $\varphi_{m,a}(j)$ and $\varphi_{m,b}(j)$ are given by[S16]

$$\varphi_{m,a}(j) = \sin[j\theta_m] + \frac{J_2}{\bar{J}_1} \sin[(j-1)\theta_m], \tag{S73}$$

$$\varphi_{m,b}(j) = \frac{\varepsilon_m}{\bar{J}_1} \sin[j\theta_m]. \tag{S74}$$

Utilizing the eigenvector $\{|\varphi_m\rangle\}$, we rewrite the Hamiltonian $\bar{\mathcal{V}}_\alpha$ in Eq. (S68) as

$$\begin{aligned}\bar{\mathcal{V}}_\alpha &= g\left(\sum_{m=1}^{2L} |e\rangle \langle j_0, \bar{\alpha}|\varphi_m\rangle \langle\varphi_m| + \text{H.c.}\right), \\ &= g\left(\sum_{m=1}^{2L} \frac{\varphi_{m,\alpha}(j_0)}{\sqrt{\mathcal{N}_m}} |e\rangle \langle\varphi_m| + \text{H.c.}\right).\end{aligned} \tag{S75}$$

Therefore, in the single-excitation subspace, spanned by $\{|e\rangle|\text{vac}\rangle,\ |g\rangle|\varphi_m\rangle\}$ with $m \in [1,\ 2L]$, we would like to consider the dressed state

$$|\bar{\psi}_\alpha\rangle = \sum_{m=1}^{2L} \bar{c}_m |g\rangle |\varphi_m\rangle + \bar{c}_e |e\rangle |\text{vac}\rangle, \tag{S76}$$

that satisfies the eigenequation in Eq. (S65) with total Hamiltonian $\bar{\mathcal{H}}$ in the new basis $\{|\varphi_m\rangle\}$. Then we obtain

$$\varepsilon_m \bar{c}_m + \frac{g\varphi^*_{m,\alpha}(j_0)\bar{c}_e}{\sqrt{\mathcal{N}_m}} = E\bar{c}_m, \quad \forall m, \tag{S77}$$

$$g\sum_{m=1}^{2L} \frac{\varphi_{m,\alpha}(j_0)\bar{c}_m}{\sqrt{\mathcal{N}_m}} + \Delta_0 \bar{c}_e = E\bar{c}_e. \tag{S78}$$

According to Eq. (S77), the photon profile of the dressed state can be given by

$$\bar{c}_m = \frac{g\varphi^*_{m,\alpha}(j_0)}{\sqrt{\mathcal{N}_m}(E-\varepsilon_m)} \bar{c}_e, \tag{S79}$$

where the atom profile $\bar{c}_e$ can be determined by the normalization of the dressed state as

$$|\bar{c}_e|^2 = \left[1 + \sum_{m=1}^{2L} \frac{g^2 |\varphi_{m,\alpha}(j_0)|^2}{(E-\varepsilon_m)(E^* - \varepsilon^*_m)\mathcal{N}_m}\right]^{-1}. \tag{S80}$$

Inserting Eq. (S79) into Eq. (S78) to eliminate $\bar{c}_m$ yields

$$\left[E - \Delta_0 - g^2 \sum_{m=1}^{2L} \frac{|\varphi_{m,\alpha}(j_0)|^2}{(E-\varepsilon_m)\mathcal{N}_m}\right] \bar{c}_e = 0. \tag{S81}$$

According to Eqs. (S79)-(S81), we can solve E, $\bar{c}_m$, and $\bar{c}_e$. Then, at the basis $\{|e\rangle|\text{vac}\rangle,\ |g\rangle|j,a\rangle, |g\rangle|j,b\rangle\}$, we obtain the wavefunction of the dressed state $\psi_\alpha = S_\alpha \bar{\psi}_\alpha$, where

$$\bar{\psi}_\alpha = [\bar{c}_e, \phi_1, \phi_2, \cdots, \phi_m, \cdots, \phi_{2L}]^T, \quad \text{with} \quad \phi_m = \langle g, \varphi_m|\bar{\psi}_\alpha\rangle. \tag{S82}$$

B. Two Emitters

We now consider two emitters ($|g_1\rangle, |e_1\rangle$) and ($|g_2\rangle, |e_2\rangle$) coupled to site ($\alpha_1,\ j_1$) and ($\alpha_2,\ j_2$) of the same SSH bath, respectively. In this subsection, we focus on the situation when the atom-photon interaction strength between both emitters and the SSH bath is set as $g_1 = g_2 = g$. In the single-excitation subspace, spanned by $\{|e_1\rangle|\text{vac}\rangle,\ \ |e_2\rangle|\text{vac}\rangle,\ \ |g\rangle|j,a\rangle, |g\rangle|j,b\rangle\}$ with $j \in [1,\ L]$, and under OBCs, the system Hamiltonian reads

$$\mathcal{H}_{\alpha_1\alpha_2} = \begin{pmatrix} \Delta & 0 & V_{\alpha_1} \\ 0 & \Delta & V_{\alpha_2} \\ V_{\alpha_1}^\dagger & V_{\alpha_2}^\dagger & H_p \end{pmatrix}. \tag{S83}$$

Then the eigenequation for the photon-emitter dressed states becomes

$$\mathcal{H}|\Psi_{\alpha_1\alpha_2}\rangle = (\mathcal{H}_p + \mathcal{H}_{e_1} + \mathcal{H}_{e_2} + \mathcal{V}_{\alpha_1\alpha_2})|\Psi_{\alpha_1\alpha_2}\rangle = E_{\text{d}}|\Psi_{\alpha_1\alpha_2}\rangle, \tag{S84}$$

with the eigenenergy of the dressed state being E_{d}, and

$$\begin{aligned}\mathcal{H}_p = &\sum_{j=1}^{L}\left[\left(J_1 + \frac{\kappa}{2}\right)|j,b\rangle\langle j,a| + \left(J_1 - \frac{\kappa}{2}\right)|j,a\rangle\langle j,b|\right] \\ &+ \sum_{j=1}^{L-1}\left(J_2|j,b\rangle\langle j+1,a| + J_2|j+1,a\rangle\langle j,b|\right) \\ &- \sum_{j=1}^{L}\frac{i\kappa}{2}\left(|j,a\rangle\langle j,a| + |j,b\rangle\langle j,b|\right),\end{aligned} \tag{S85}$$

$$\mathcal{H}_e = \mathcal{H}_{e_1} + \mathcal{H}_{e_2} = \Delta\left(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|\right), \tag{S86}$$

$$\mathcal{V}_{\alpha_1\alpha_2} = g(|e_1\rangle\langle j_1,\alpha_1| + |j_1,\alpha_1\rangle\langle e_1|) + g(|e_2\rangle\langle j_2,\alpha_2| + |j_2,\alpha_2\rangle\langle e_2|). \tag{S87}$$

Due to the non-Hermiticity of the bath Hamiltonian $\mathcal{H}_p$ in Eq. (S85), the right and left eigenstates of $\mathcal{H}_p$ can be defined as

$$\mathcal{H}_p|\varphi_m^R\rangle = E|\varphi_m^R\rangle, \quad \mathcal{H}_p^\dagger|\varphi_m^L\rangle = E^*|\varphi_m^L\rangle, \tag{S88}$$

whose biorthogonal conditions and completeness conditions are given by $\langle\varphi_m^R|\varphi_n^L\rangle = \langle\varphi_m^L|\varphi_n^R\rangle = \delta_{mn}$ and $\sum_m|\varphi_m^L\rangle\langle\varphi_m^R| = \sum_m|\varphi_m^R\rangle\langle\varphi_m^L| = 1$, respectively. Using these relations, the bath Hamiltonian $\mathcal{H}_p$ can be expressed in terms of quasi-particle energy bands as

$$\mathcal{H}_p = \sum_{m=1}^{2L}\left(\varepsilon_m - \frac{i\kappa}{2}\right)|\varphi_m^R\rangle\langle\varphi_m^L|, \tag{S89}$$

and the right and left eigenvectors are given by

$$|\varphi_m^R\rangle = \frac{1}{\sqrt{\bar{\mathcal{N}}_m}}\sum_{j=1}^{L}(\varphi_{m,a}^R(j)|j,a\rangle + \varphi_{m,b}^R(j)|j,b\rangle), \tag{S90}$$

$$|\varphi_m^L\rangle = \frac{1}{\sqrt{\bar{\mathcal{N}}_m}}\sum_{j=1}^{L}(\varphi_{m,a}^L(j)|j,a\rangle + \varphi_{m,b}^L(j)|j,b\rangle). \tag{S91}$$

Here, $\bar{\mathcal{N}}_m$ is the normalized constant in the biorthogonal condition. By utilizing the inverse of the similarity transformation to Eqs. (S73) and (S74), the amplitudes of $\varphi_{m,a}^{R/L}(j)$ and $\varphi_{m,b}^{R/L}(j)$ are given by

$$\varphi_{m,a}^R(j) = \left(\frac{J_1 + \kappa/2}{J_1 - \kappa/2}\right)^{\frac{j}{2}}(\sin[j\theta_m] + \frac{J_2}{\bar{J}_1}\sin[(j-1)\theta_m]), \tag{S92}$$

$$\varphi^R_{m,b}(j) = \frac{\varepsilon_m}{J_1-\kappa/2}\left(\frac{J_1+\kappa/2}{J_1-\kappa/2}\right)^{\frac{j}{2}}\sin[j\theta_m], \tag{S93}$$

$$\varphi^L_{m,a}(j) = \left(\frac{J_1-\kappa/2}{J_1+\kappa/2}\right)^{\frac{j}{2}}(\sin[j\theta_m]+\frac{J_2}{\bar{J}_1}\sin[(j-1)\theta_m]), \tag{S94}$$

$$\varphi^L_{m,b}(j) = \frac{\varepsilon_m}{J_1+\kappa/2}\left(\frac{J_1-\kappa/2}{J_1+\kappa/2}\right)^{\frac{j}{2}}\sin[j\theta_m]. \tag{S95}$$

Utilizing the completeness condition $\sum_{m=1}^{2L}\left|\varphi^R_m\right\rangle\left\langle\varphi^L_m\right| = 1$, we rewrite the atom-photon interaction Hamiltonian $\mathcal{V}_{\alpha_1\alpha_2}$ in Eq. (S87) as

$$\begin{aligned}\mathcal{V}_{\alpha_1\alpha_2} = {} & g\left(\sum_{m=1}^{2L}\frac{\varphi^R_{m,\alpha_1}(j_1)}{\sqrt{\bar{\mathcal{N}}_m}}\left|e_1\right\rangle\left\langle\varphi^L_m\right| + \sum_{m=1}^{2L}\frac{[\varphi^L_{m,\alpha_1}(j_1)]^*}{\sqrt{\bar{\mathcal{N}}_m}}\left|\varphi^R_m\right\rangle\left\langle e_1\right|\right) \\ & + g\left(\sum_{m=1}^{2L}\frac{\varphi^R_{m,\alpha_2}(j_2)}{\sqrt{\bar{\mathcal{N}}_m}}\left|e_2\right\rangle\left\langle\varphi^L_m\right| + \sum_{m=1}^{2L}\frac{[\varphi^L_{m,\alpha_2}(j_2)]^*}{\sqrt{\bar{\mathcal{N}}_m}}\left|\varphi^R_m\right\rangle\left\langle e_2\right|\right).\end{aligned} \tag{S96}$$

We employ the resolvent method to solve the evolution dynamics of two emitters coupled to the topological bath[S17,S18]. Using the Hamiltonian $\mathcal{H} = \mathcal{H}_p + \mathcal{H}_{e_1} + \mathcal{H}_{e_2} + \mathcal{V}_{\alpha_1\alpha_2}$ in Eqs. (S86), (S89) and (S96), the resolvent operator of the whole system is defined as

$$\mathcal{G}(z) = \frac{1}{z-\mathcal{H}} = \frac{1}{z-\mathcal{H}_{pe}-\mathcal{V}_{\alpha_1\alpha_2}}, \tag{S97}$$

where

$$\mathcal{H}_{pe} = \mathcal{H}_p + \mathcal{H}_{e_1} + \mathcal{H}_{e_2}. \tag{S98}$$

We now consider the single-excitation spanned by the emitter and bath Hamiltonian $\mathcal{H}_{pe}$, which consists of the atomic excitation $\{\left|e_1\right\rangle\left|\text{vac}\right\rangle,\ \left|e_2\right\rangle\left|\text{vac}\right\rangle\}$, and the quasi-particle excitation $\{\left|g\right\rangle\left|\varphi^R_m\right\rangle\}$ with $m\in[1,\ 2L]$. The photon-emitter interaction term $\mathcal{V}_{\alpha_1\alpha_2}$ describes the coupling between the subspaces $\{\left|e_1\right\rangle\left|\text{vac}\right\rangle,\ \left|e_2\right\rangle\left|\text{vac}\right\rangle\}$ and $\{\left|g\right\rangle\left|\varphi^R_m\right\rangle\}$. In the following, we use the following notations $\left|e_1\right\rangle := \left|e_1\right\rangle\left|\text{vac}\right\rangle$, $\left|e_2\right\rangle := \left|e_2\right\rangle\left|\text{vac}\right\rangle$ and $\left|\varphi^R_m\right\rangle := \left|g\right\rangle\left|\varphi^R_m\right\rangle$ for convenience. Then, we define the projector operator

$$\mathcal{P} = \left|e_1\right\rangle\left\langle e_1\right| + \left|e_2\right\rangle\left\langle e_2\right|, \tag{S99}$$

and its complementary

$$\mathcal{Q} = \sum_{m=1}^{2L}\left|\varphi^R_m\right\rangle\left\langle\varphi^L_m\right|. \tag{S100}$$

Therefore, the constrained propagator $\mathcal{G}_p(z)$ is written as

$$\mathcal{G}_p(z) \equiv \mathcal{P}\mathcal{G}(z)\mathcal{P}. \tag{S101}$$

Starting from $(z-\mathcal{H})\mathcal{G}(z) = \mathbf{1}$, and manipulating it on the right by $\mathcal{P}$ and on the left by $\mathcal{P}$ or $\mathcal{Q}$, the constrained propagator can be derived as

$$\mathcal{G}_p(z) = \frac{\mathcal{P}}{z-\mathcal{P}\mathcal{H}_{pe}\mathcal{P}-\mathcal{P}\Sigma(z)\mathcal{P}}, \tag{S102}$$

where $\Sigma(z)$ is called the level-shift operator[S17], defined as

$$\begin{aligned}\Sigma_(z) &= \mathcal{V}_{\alpha_1\alpha_2} + \mathcal{V}_{\alpha_1\alpha_2}\frac{\mathcal{Q}}{z-\mathcal{Q}\mathcal{H}_{pe}\mathcal{Q}-\mathcal{Q}\mathcal{V}_{\alpha_1\alpha_2}\mathcal{Q}}\mathcal{V}_{\alpha_1\alpha_2}, \\ &= \mathcal{V}_{\alpha_1\alpha_2} + \mathcal{V}_{\alpha_1\alpha_2}\frac{\mathcal{Q}}{z-\mathcal{H}_p}\mathcal{V}_{\alpha_1\alpha_2},\end{aligned} \tag{S103}$$

with

$$\mathcal{V}_{\alpha_1\alpha_2}\frac{\mathcal{Q}}{z-\mathcal{H}_p}\mathcal{V}_{\alpha_1\alpha_2} = \sum_{m=1}^{2L}\frac{g^2\varphi^R_{m,\alpha_1}(j_1)[\varphi^L_{m,\alpha_1}(j_1)]^*/\bar{\mathcal{N}}_m}{z-\varepsilon_m+i\kappa/2}\left|e_1\right\rangle\left\langle e_1\right| + \sum_{m=1}^{2L}\frac{g^2\varphi^R_{m,\alpha_1}(j_1)[\varphi^L_{m,\alpha_2}(j_2)]^*/\bar{\mathcal{N}}_m}{z-\varepsilon_m+i\kappa/2}\left|e_1\right\rangle\left\langle e_2\right| + \sum_{m=1}^{2L}\frac{g^2\varphi^R_{m,\alpha_2}(j_2)[\varphi^L_{m,\alpha_1}(j_1)]^*/\bar{\mathcal{N}}_m}{z-\varepsilon_m+i\kappa/2}\left|e_2\right\rangle\left\langle e_1\right| + \sum_{m=1}^{2L}\frac{g^2\varphi^R_{m,\alpha_2}(j_2)[\varphi^L_{m,\alpha_2}(j_2)]^*/\bar{\mathcal{N}}_m}{z-\varepsilon_m+i\kappa/2}\left|e_2\right\rangle\left\langle e_2\right|. \tag{S104}$$

We now proceed to calculate the non-unitary real-time dynamics governed by $\left|\psi_t\right\rangle = e^{-i\hat{\mathcal{H}}_{\text{eff}}t}\left|\psi_0\right\rangle$ for two emitters (labeled as 1 and 2) coupled to sites j_{1,α_1} and j_{2,α_2} ($\alpha_1,\alpha_2 = a$ or b) of the bath with $j_{2,\alpha_2} > j_{1,\alpha_1}$, respectively. The initial state is chosen as one excited emitter $\left|e_1\right\rangle$ or $\left|e_2\right\rangle$ with $\left|\psi_0\right\rangle = \left|e_n\right\rangle\left|\text{vac}\right\rangle$ ($n = 1$ or 2), and the time-evolved state can be expanded as

$$\left|\psi_t\right\rangle = \left(\sum_{m=1}^{2N}c_m(t)\left|\varphi^R_m\right\rangle\left\langle\text{vac}\right| + \sum_{n=1}^{2}c_{e_n}(t)\left|e_n\right\rangle\left\langle g\right|\right)\left|gg\right\rangle\otimes\left|\text{vac}\right\rangle. \tag{S105}$$

Then, the component $\mathcal{P}\left|\psi_t\right\rangle$ can be evaluated by the resolvent method[S17] as

$$\mathcal{P}\left|\psi_t\right\rangle = \frac{i}{2\pi}\int_{-\infty}^{+\infty}dE\mathcal{G}_p\left(E+i0^+\right)e^{-iEt}\left|\psi_0\right\rangle. \tag{S106}$$

Using Eq. (S106), we can express $\boldsymbol{c}_e(t) = [c_{e_1}(t),\ c_{e_2}(t)]^T$ as

$$\boldsymbol{c}_e(t) = \frac{i}{2\pi}\int_{-\infty}^{+\infty}dE\mathcal{G}_p\left(E+i0^+\right)e^{-iEt}\boldsymbol{c}_e(0), \tag{S107}$$

where, according to Eqs. (S102)-(S104), we explicitly write $\mathcal{G}_p(z)$ as

$$\mathcal{G}_p(E) = \begin{pmatrix} \frac{1}{E-\Delta-\mathcal{T}(\alpha_1,\alpha_1)} & \frac{1}{E-\mathcal{F}(\alpha_1,\alpha_2)\mathcal{T}(\alpha_1,\alpha_2)} \\ \frac{1}{E-\mathcal{F}(\alpha_2,\alpha_1)\mathcal{T}(\alpha_1,\alpha_2)} & \frac{1}{E-\Delta-\mathcal{T}(\alpha_2,\alpha_2)} \end{pmatrix}, \tag{S108}$$

where

$$\mathcal{T}(\alpha_1,\alpha_2) = g^2\sum_{m=1}^{2L}\frac{\varphi_{m,\alpha_1}(j_{1,\alpha_1})\varphi_{m,\alpha_2}(j_{2,\alpha_2})}{(E-\varepsilon_m+i\kappa/2)\mathcal{N}_m}, \tag{S109}$$

and

$$\mathcal{F}(\alpha_1,\alpha_2) = \left(\frac{J_1+\kappa/2}{J_1-\kappa/2}\right)^{\frac{\delta_{\alpha_1,b}}{2}}\left(\frac{J_1-\kappa/2}{J_1+\kappa/2}\right)^{\frac{\delta_{\alpha_2,b}}{2}}\left(\frac{J_1+\kappa/2}{J_1-\kappa/2}\right)^{\frac{j_{1,\alpha_1}-j_{2,\alpha_2}}{2}}. \tag{S110}$$

We assume a small g, a large band gap of the topological bath under OBCs and $\Delta = -i\kappa/2$. According to Eqs. (S107)-(S110) and Eq. (S81), the main contribution from the diagonal elements of the Green function $\mathcal{G}_p(z)$ to the time evolution is the dressed state for small g and $\Delta = -i\kappa/2$. The off-diagonal elements contribute to the state exchanges between two emitters. Remarkably, such state exchange is asymmetric [see Eq. (S110)]. To be specific, when the emitter at the site j_{2,α_2} is initially excited, there is no excitation transferred to the emitter at site j_{1,α_1} for the large distance $|j_{1,\alpha_1} - j_{2,\alpha_2}|$ between them, due to the power-law decay of $\mathcal{F}(\alpha_1,\alpha_2)$. In principle, according to Eq. (S110), as the J_1 approaches $\kappa/2$ (while ensuring $J_1 \neq \kappa/2$), the directional long-range emitter-emitter interaction is enhanced. However, as J_1 gets closer to $\kappa/2$, the band gap of the open-boundary condition (OBC) spectrum of the SSH bath diminishes. A smaller band gap reduces the robustness of the system against disorder and increases the likelihood of coupling between the emitters and the bulk modes, which can undermine the desired directional transport properties.

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