Millionfold improvement in multivibration-feedback optomechanical refrigeration via auxiliary mechanical coupling

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The simultaneous ground-state refrigeration of multiple vibrational modes is a prerequisite for observing significant quantum effects of multiple-vibration systems. Here we propose how to realize a large amplification in the net-refrigeration rates based on cavity optomechanics and to largely improve the cooling performance of multivibration modes beyond the resolved-sideband regime. By employing an auxiliary mechanical coupling (AMC) between two mechanical vibrations, the dark mode, which is induced by the coupling of these vibrational modes to a common optical mode and cuts off cooling channels, can be fully removed. We use fully analytical treatments for the effective mechanical susceptibilities and net-cooling rates and find that when the AMC is turned on, the amplification of the net-refrigeration rates by more than six orders of magnitude can be observed. In particular, we reveal that the simultaneous ground-state cooling beyond the resolved-sideband regime arises from the introduced AMC, without which it vanishes. Our work paves the way for quantum control of multiple vibrational modes in the bad-cavity regime.

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I. INTRODUCTION

Cavity optomechanics [1–4] provides a promising platform to explore mechanical properties of quantum systems via optical means and to manipulate cavity-field statistics by mechanically changing the boundary of a cavity [5–29]. As a prominent application closely related to this optomechanical platform, optomechanical refrigeration has become a hot research topic [30–35]. This is due to the fact that to observe significantly quantum effects of systems, a prerequisite is to cool these systems to their quantum ground states by effectively suppressing their thermal noise. Until now, cooling a single mechanical mode to its quantum ground state by the coupling of multiple vibrational modes to a common optical mode and cuts off cooling channels, can be fully removed. We use fully analytical treatments for the effective mechanical susceptibilities and net-cooling rates and find that when the AMC is turned on, the amplification of the net-refrigeration rates by more than six orders of magnitude can be observed. In particular, we reveal that the simultaneous ground-state cooling beyond the resolved-sideband regime arises from the introduced AMC, without which it vanishes. Our work paves the way for quantum control of multiple vibrational modes in the bad-cavity regime.

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In this paper, based on the feedback-cooling mechanism, we propose a dark-mode-removing method to achieve the simultaneous ground-state cooling of the two mechanical modes in the unresolved-sideband regime. This is realized by employing auxiliary mechanical coupling (AMC) to break the symmetry of the system, and then, both dark and bright modes can be effectively controlled. By obtaining the exact analytical results of the net-cooling rates, effective mechanical susceptibilities, and steady-state mean phonon numbers, we find that when the AMC is turned on in the system, a millionfold amplification in the net-refrigeration rates can be observed. Specifically, these net-refrigeration rates can be amplified to more than six orders of magnitude by properly tuning the AMC strength.

In particular, we show that the tremendously amplified net-refrigeration rates can result in a giant improvement for the refrigeration performance of the mechanical modes.

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Without AMC, the two vibrational modes cannot be efficiently cooled due to an inefficient net-cooling rate. However, when the AMC is turned on, the simultaneous ground-state cooling of these vibrations is achieved beyond the resolved-sideband regime, owing to the millionfold amplification in the net-cooling rates. Remarkably, we reveal that the larger the feedback-loop strength of the resonator is, the better the cooling efficiency of this resonator is. Physically, the introduced AMC offers an effective strategy to remove the dark mode and, in turn, to rebuild cooling channels for extracting thermal phonons stored in the dark mode. This study could pave the way for studying quantum control and observing quantum-mechanical coherence effects involving multiple vibrational modes.

II. MODEL AND HAMILTONIAN

As shown in Fig. 1(a), we consider a three-mode optomechanical system in which two vibrational modes are optomechanically coupled to a common optical mode. An AMC between the two vibrational modes is introduced to improve the net-cooling rates and the cooling performance of the system. To control the system, an external control laser with amplitude $\Omega$ and frequency $\omega_L$ is applied to the optical cavity. The Hamiltonian of the system reads ($\hbar = 1$)

$$H_0 = \omega_c c^+ c + \sum_{j=1}^{2} \left( \frac{1}{2m_j} \dot{p}_{j}^2 + \frac{1}{2} m_j \dot{x}_j^2 \right) - \lambda_1 c^+ c x_1 - \lambda_2 c^+ c x_2 + \mu (x_1 - x_2)^2 + \Omega (c^+ e^{-i\omega_L t} + c e^{i\omega_L t}),$$

where $c$ ($c^+$) denotes the annihilation (creation) operator of the optical mode. The operators $p_{j}, x_{j}$ ($j = 1, 2$) and $x_j$ are, respectively, the momentum and position operators of the $j$th vibrational mode, with frequency $\omega_j$ and mass $m_j$. The $\lambda_j$ terms describe the optomechanical interactions between the optical mode and the $j$th vibrational mode, and the $\mu$ term denotes the AMC between the two vibrations. Note that the FFL denotes the strength of the first feedback loop, and the SFL describes the strength of the second feedback loop.

III. LANGEVIN EQUATIONS AND STEADY-STATE MEAN PHONON NUMBERS

In this section, we obtain the Langevin equations of the system, analyze a cold-damping feedback-cooling scheme, and derive the steady-state average phonon numbers.

A. Langevin equations

We consider the case where the two vibrational modes are subjected to quantum Brownian forces and the optical mode interacts with their vacuum baths. Then, the quantum Langevin equations can be used to describe the evolution of
the system:
\[
\dot{c} = [-\kappa + i(\Delta_c - \tilde{\lambda}_1 q_1 - \tilde{\lambda}_2 q_2)]c - i\Omega + \sqrt{2\kappa}c_m, \\
\dot{q}_j = \omega_j p_j, \quad j = 1, 2, \\
\dot{p}_1 = -\omega_1 q_1 + 2\mu q_2 + \tilde{\lambda}_1 c \cdot c - \gamma_1 p_1 + \xi_1, \\
\dot{p}_2 = -\omega_2 q_2 + 2\mu q_1 + \tilde{\lambda}_2 c \cdot c - \gamma_2 p_2 + \xi_2,
\]
where the operators $c_m$ and $\xi_j$ are, respectively, the input noise operator of the cavity-field mode and the Brownian noise operator resulting from the coupling of the corresponding vibrational modes to the thermal baths. These noise operators satisfy zero mean values and the following correlation functions:
\[
\langle c_m(t) c_m(t') \rangle = \delta(t - t'), \\
\langle c_m(t) c_m(t') \rangle = \langle c_m(t') c_m(t) \rangle = 0, \\
\langle \xi_j(t) \xi_j(t') \rangle = \frac{1}{2\pi\omega_j} \int_0^\infty \omega e^{-i\omega(t-t')} \left[ 1 + \coth \frac{\hbar \omega}{2k_B T_j} \right] d\omega,
\]
where $k_B$ is the Boltzmann constant and $T_j$ is the thermal bath temperature associated with the $j$th vibrational mode.

We assume that the cavity is strongly driven, and this allows us to linearize the dynamics of the system by writing each operator as the sum of their averages and fluctuations, i.e., $A = \langle A \rangle + \delta A$ for $A \in \{c, c', q_j, p_j\}$. By neglecting higher-order fluctuation terms, the linearized quantum Langevin equations, which are described by the quadrature fluctuations $\delta X = (\delta c^+ + \delta c)/\sqrt{2}$ and $\delta Y = i(\delta c^+ - \delta c)/\sqrt{2}$, are given by
\[
\delta \dot{X} = -\kappa \delta X + \Delta \delta Y + \sqrt{2\kappa}X_m, \\
\delta \dot{Y} = -\kappa \delta Y - \Delta \delta X + G_1 \delta q_1 + G_2 \delta q_2 + \sqrt{2\kappa}Y_m, \\
\delta \dot{q}_j = \omega_j \delta p_j, \quad j = 1, 2, \\
\delta \dot{p}_1 = -\omega_1 \delta q_1 + 2\mu \delta q_2 + G_1 \delta X - \gamma_1 \delta p_1 + \xi_1, \\
\delta \dot{p}_2 = -\omega_2 \delta q_2 + 2\mu \delta q_1 + G_2 \delta X - \gamma_2 \delta p_2 + \xi_2,
\]
where $X_m = (\delta c^+ + \delta c_m)/\sqrt{2}$ and $Y_m = i(\delta c^+ - \delta c_m)/\sqrt{2}$ are the corresponding Hermitian input noise quadratures and the parameter $\Delta = \Delta_c - \tilde{\lambda}(q_1)_{ss} - \tilde{\lambda}(q_2)_{ss}$ is the normalized effective driving detuning. Moreover, $G_1 = \sqrt{2\tilde{\lambda}_1(c)_{ss}}$ and $G_2 = \sqrt{2\tilde{\lambda}_2(c)_{ss}}$ are the effective optomechanical coupling strengths, with $(c)_{ss} = -i\Omega/(\kappa + i\Delta)$. Note that the phase reference of the cavity field $(c)_{ss}$ is assumed to be real and positive.

**B. Cold-damping feedback**

To realize the cold-damping feedback refrigeration, the case of $\Delta = 0$ is considered, so that the highest sensitivity for the position measurements of the vibrational modes can be achieved [36–45]. This feedback-refrigeration mechanism is essentially different from the sideband-cooling mechanism requiring the red-sideband resonance, i.e., $\Delta = \omega_j$ [30–35]. By using a negative-derivative feedback, the effective decay rate of the mechanical mode can be largely developed by the cold-damping feedback technique.

Physically, the position of the two mechanical modes is measured through the phase-sensitive detection of the cavity output field, and then, the readout of the cavity output field is fed back into the two vibrational modes by applying feedback forces. The intensity of these feedback forces is proportional to the time derivative of the output signal and, therefore, to the velocity of the mechanical modes. Then, the linearized quantum Langevin equations in Eq. (5) become
\[
\delta \dot{X} = -\kappa \delta X + \sqrt{2\kappa}X_m, \\
\delta \dot{Y} = -\kappa \delta Y - \Delta \delta X + G_1 \delta q_1 + G_2 \delta q_2 + \sqrt{2\kappa}Y_m, \\
\delta \dot{q}_j = \omega_j \delta p_j, \\
\delta \dot{p}_1 = -\omega_1 \delta q_1 + 2\mu \delta q_2 + G_1 \delta X - \gamma_1 \delta p_1 + \xi_1, \\
\delta \dot{p}_2 = -\omega_2 \delta q_2 + 2\mu \delta q_1 + G_2 \delta X - \gamma_2 \delta p_2 + \xi_2,
\]
where the convolution term $\int_{-\infty}^{t} g_j(t - s) \delta Y_{\text{out}}(s) ds$ $(j = 1, 2)$ denotes the feedback force acting on the $j$th vibrational mode. These feedback forces depend on the past dynamics of the detected quadrature $\delta Y_j$, which is driven by a weighted sum of the fluctuations of the vibrational modes. Here the causal kernel is defined by
\[
g_j(t) = g_{cd,j} \frac{d}{dt} \theta(t)\omega_{th}e^{-i\omega_{\text{out}}t},
\]
where $g_{cd,j}$ and $\omega_{th}$ are the dimensionless feedback gain and feedback bandwidth associated the $j$th vibrational mode, respectively. In Eq. (7), we have assumed that the electronic loop can provide an instantaneous feedback in the system, and this assumption is included in the argument of the Heaviside function $\theta(t)$ [32,43,45]. This assumes fast electronics that can respond much quicker than the oscillation time of the system [32,43,45]. The estimated intracavity phase quadrature $\delta Y_{\text{out}}$ results from the homodyne measurement of the output quadrature $\delta Y_{\text{out}}(t)$. Here we generalize the usual input-output relation,
\[
\delta Y_{\text{out}}(t) = \sqrt{2\kappa}\delta Y(t) - Y_{\text{in}}(t),
\]
which is the case of a nonunit detection efficiency by modeling a detector with quantum efficiency $\vartheta$ with an ideal detector preceded by a beam splitter (with transmissivity $\sqrt{\vartheta}$), which mixes the incident field with the uncorrelated vacuum field $Y_{\text{v}}(t)$. Then, we obtain the generalized input-output relation,
\[
Y_{\text{out}}(t) = \sqrt{\vartheta}[\sqrt{2\kappa}\delta Y(t) - Y_{\text{in}}(t)] - \sqrt{1 - \vartheta}Y_{\text{v}}(t).
\]
The estimated phase quadratures $\delta Y_{\text{est}}$ are obtained as
\[
\delta Y_{\text{est}}(t) = \frac{Y_{\text{out}}(t)}{\sqrt{2\kappa} - \delta Y(t) - \sqrt{\vartheta - 1}Y_{\text{v}}(t)}.
\]
Below, we seek the steady-state solution of Eq. (6) by solving it in the frequency domain with the Fourier transform. $r(t) = (1/2\pi)^{1/2} \int_{-\infty}^{\infty} e^{-i\omega t} \tilde{r}(\omega) d\omega$ (for $r = \delta X, \delta Y, \delta q_j, \delta p_j, \xi_j, X_m, \text{and } Y_m$), and consequently, the quantum Langevin
equations in Eq. (6), with the cold-damping feedback, can be solved in the frequency domain. Based on the steady-state solution, we can calculate the spectra of the position and momentum operators for two mechanical modes, and then, the steady-state mean phonon numbers in these resonators can be obtained by integrating the corresponding fluctuation spectra.

C. Final average phonon numbers

The final steady-state average phonon numbers in the $j$th vibrational mode can be obtained with

$$
n_j = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{q_j}(\omega) d\omega,
$$

(11)

where $\langle \delta q_j^2 \rangle$ and $\langle \delta p_j^2 \rangle$ are the variances of the position and momentum operators, respectively. We solve Eq. (6) in the frequency domain and integrate the corresponding fluctuation spectra, and then, the corresponding variances can be obtained as

$$
\langle \delta q_j^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{q_j}(\omega) d\omega,
$$

$$
\langle \delta p_j^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 S_{q_j}(\omega) d\omega.
$$

(12)

The fluctuation spectra of the position and momentum operators are defined by

$$
S_A(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \langle \delta A(t + \tau) \delta A(t) \rangle_s d\tau \quad (A = q_j, p_j),
$$

(13)

where $\langle \cdot \rangle_s$ denotes the steady-state average of the system. The fluctuation spectra in the frequency domain are expressed as

$$
\langle \delta \tilde{A}(\omega) \delta \tilde{A}(\omega') \rangle_s = S_A(\omega) \delta(\omega - \omega').
$$

(14)

Thus, in the frequency domain, we can solve this system and obtain the analytical results of the steady-state average phonon numbers.

D. Dark-mode effect and its removal

We next show the dark-mode effect and its removal from the two-vibrational-mode optomechanical system. For convenience, we introduce the annihilation (creation) operators of the two vibrational modes: $b_j = (q_j + ip_j)/\sqrt{2}$, $\tilde{b}_j = (q_j - ip_j)/\sqrt{2}$. In the process of optomechanical cooling, the beam-splitting-type interactions (corresponding to the rotating-wave interaction term) between these bosonic modes dominate the linearized couplings in this system. By considering a red-detuned driving of the cavity and performing the rotating-wave approximation (RWA), the Hamiltonian of the system can be simplified as (discarding the noise terms)

$$
H_{RWA} = \Delta \delta c \delta c + \sum_{j=1}^{2} \left[ \omega_j \delta b_j^\dagger \delta b_j + G_j (\delta b_j^\dagger \delta c^\dagger + \delta b_j \delta c^\dagger) \right] + \eta (\delta b_1^\dagger \delta b_2 + \delta b_2^\dagger \delta b_1).
$$

(15)

To show the dark-mode effect and its removal from the system, we discuss in detail the physical system when the AMC is absent ($\eta = 0$) and present ($\eta \neq 0$).

(i) To study the dark-mode effect, we first assume that the AMC is turned off (i.e., $\eta = 0$). In this case, the system can induce a bright mode and a dark mode:

$$
B_+ = (G_1 \delta b_1 + G_2 \delta b_2)/G_0, \quad \text{and} \quad B_- = (G_2 \delta b_1 - G_1 \delta b_2)/G_0,
$$

(16a)

(16b)

respectively, where $G_0 = \sqrt{G_1^2 + G_2^2}$. Then, the Hamiltonian in Eq. (15) can be rewritten with the bright and dark modes as

$$
H_{yb} = \Delta \delta c \delta c + \sum_{j=\pm} \omega_j B_j^\dagger B_j + G_+ (\delta c B_+^\dagger + B_+ \delta c^\dagger) + G_- (B_-^\dagger - B_-^\dagger B_+).
$$

(17)

We see from Eq. (18) that when $\omega_1 = \omega_2$, the mode $B_-$ is decoupled from the system and it becomes a dark mode.

(ii) We then turn on the AMC (i.e., $\eta \neq 0$), so that the dark-mode effect can be efficiently removed. We introduce two new bosonic modes $\tilde{B}_\pm$ associated with the AMC, defined by

$$
\delta b_1 = f B_+ + h \tilde{B}_+, \quad \delta b_2 = -h B_+ + f \tilde{B}_-,
$$

(19)

and then, the Hamiltonian in Eq. (15) becomes

$$
H_{RWA} = \Delta \delta c \delta c + \sum_{j=\pm} \left[ \tilde{\omega}_j \delta \tilde{B}_j^\dagger \delta \tilde{B}_j + (\tilde{G}_j \delta \tilde{B}_j^\dagger + \tilde{G}_j \delta \tilde{B}_j \delta c^\dagger) \right],
$$

(20)

where the resonance frequencies $\tilde{\omega}_\pm = (\omega_1 + \omega_2 \pm \sqrt{(\omega_1 - \omega_2)^2 + 4\eta^2})/2$ and the redefined-coupling strengths $\tilde{G}_\pm$ are

$$
\tilde{G}_+ = f \omega_1 - h \omega_2, \quad \tilde{G}_- = h \omega_1 + f \omega_2,
$$

(21)

with $f = \frac{|\omega_1 - \omega_2|}{\sqrt{(\omega_1 - \omega_2)^2 + 4\eta^2}}$, $h = \frac{\eta f}{\omega_1 - \omega_2}$. When $\omega_1 = \omega_2 = \omega_m$, the coupling strengths in Eq. (21) can be simplified as

$$
\tilde{G}_\pm = (G_2 \pm G_1)/\sqrt{2}.
$$

(22)

Equations (20) and (22) show that the dark mode $\tilde{B}_-$. can be fully removed when the strengths of the two optomechanical couplings are different (i.e., $G_1 \neq G_2$). The underlying physical mechanism behind our proposed method can be explained as follows: By tuning the coupling strength between the optical mode and each mechanical mode, the symmetry of the system is broken, and both bright and dark mechanical modes can be effectively manipulated.

IV. COOLING OF TWO MECHANICAL MODES

In this section, we derive the analytical expressions of the effective mechanical susceptibilities and net-refrigeration rates and study the cooling performance of the two vibrational modes.
A. Analytical results for effective susceptibilities, cooling rates, and noise spectra

We obtain the position fluctuation spectra of the two vibrational modes as

\[
S_{j_{\text{th}}}(\omega) = |\chi_{j_{\text{th}}, j}(\omega)|^2 \left[ S_{\text{ph}, j}(\omega) + S_{\text{me}, j}(\omega) + S_{\text{th}, j}(\omega) \right].
\]

In the coordinate fluctuation spectra, we introduce the effective susceptibility of the \( j \)th vibrational mode as

\[
\chi_{j, \text{eff}}(\omega) = \omega_1 \left[ \Omega_{j, \text{eff}}^2(\omega) - \omega^2 - i\omega \Gamma_{j, \text{eff}}(\omega) \right]^{-1},
\]

where \( \Omega_{j, \text{eff}}(\omega) \) and \( \Gamma_{j, \text{eff}}(\omega) \) are, respectively, the effective mechanical resonance frequency and the effective mechanical decay rate of the \( j \)th vibrational mode, defined as

\[
\Gamma_{j, \text{eff}}(\omega) = \gamma_j + \gamma_j(\omega),
\]

\[
\Omega_{j, \text{eff}}(\omega) = \omega_j + \omega_j(\omega).
\]

The net refrigeration rates \( \gamma_{j, \text{C}} \) of the \( j \)th vibrational modes are

\[
\gamma_{j, \text{C}} = \frac{-[G_1 \gamma(\omega, 2 \tilde{\mu}_j F_j)]}{A_1(\omega) + A_2(\omega)} \frac{[C_1(\omega) + C_2(\omega)]}{C_1(\omega) + C_2(\omega)},
\]

\[
\gamma_{j, \text{C}} = \frac{-[G_2 \gamma(\omega, 2 \tilde{\mu}_j F_j)]}{A_1(\omega) + A_2(\omega)} \frac{[C_1(\omega) + C_2(\omega)]}{C_1(\omega) + C_2(\omega)}.
\]

and the frequency shifts \( \delta \omega_j(\omega) \) of the \( j \)th vibrational mode are caused by the optical spring effect, given by

\[
\delta \omega_1 = \sqrt{\omega_1^2 + \frac{E_1(\omega)}{A_1(\omega) + A_2(\omega)}} - \omega_1,
\]

\[
\delta \omega_2 = \sqrt{\omega_2^2 + \frac{T_2(\omega)}{A_1(\omega) + A_2(\omega)}} - \omega_2.
\]

In Eq. (23), we introduced the feedback-induced noise spectrum \( S_{\text{ph}, j}(\omega) \), the radiation-pressure noise spectrum \( S_{\text{me}, j}(\omega) \), the mechanical-coupling-induced noise spectrum \( S_{\text{me}, j}(\omega) \), and the thermal noise spectrum \( S_{\text{th}, j}(\omega) \) of the \( j \)th vibrational mode, which are given by

\[
S_{\text{ph}, j}(\omega) = \frac{\gamma_{j, \text{C}}}{\omega_j} \coth \beta_j,
\]

\[
S_{\text{me}, j}(\omega) = \frac{N_j(\omega)}{\omega_j} \coth \beta_j,
\]

\[
S_{\text{me}, j}(\omega) = \frac{M_j(\omega)}{\omega_j} \coth \beta_j,
\]

\[
S_{\text{th}, j}(\omega) = \frac{\omega_j^3 (\kappa^2 + \omega_j^2 \omega_0^2)}{4 \kappa \theta} N_j(\omega),
\]

\[
S_{\text{th}, j}(\omega) = \frac{\omega_j^3 (\kappa^2 + \omega_j^2 \omega_0^2)}{4 \kappa \theta} M_j(\omega),
\]

where \( \beta_j = \hbar \omega_j/(2k_B T_j) \) and the other parameters are given in the Appendix A.

B. Giant amplification of both mechanical decay rates and net-cooling rates via the AMC

Here we study how to achieve a giant enhancement of both effective decay rates \( \Gamma_{j, \text{eff}} \) and net-cooling rates \( \gamma_{j, \text{C}} \) of the \( j \)th vibrational mode by introducing the AMC. In Fig. 1(b), we plot the effective mechanical decay rates \( \Gamma_{j, \text{eff}} \) as a function of the frequency \( \omega \) when system operates without (i.e., \( \tilde{\mu} = 0 \); see the solid curves) and with (i.e., \( \tilde{\mu}/\omega_m = 0.02 \); see the dashed curves) the AMC. We find that by introducing the AMC, the effective mechanical decay rates \( \Gamma_{j, \text{eff}} \) are largely enhanced at resonance \( \omega = \pm \omega_m \). Specifically, the effective mechanical decay rates \( \Gamma_{j, \text{eff}} \) without the AMC (i.e., \( \tilde{\mu} = 0 \)) are approximately equal to \( 2 \gamma_j \) at \( \omega = \pm \omega_m \), and this means the cooling of these vibrational modes is inefficient (see the solid curves). However, when the AMC is switched on (i.e., \( \tilde{\mu} \neq 0 \)), the effective mechanical decay rates \( \Gamma_{j, \text{eff}} \) at \( \omega = \pm \omega_m \) can be amplified from \( \approx 2 \gamma_j \) to \( \gg 10^4 \gamma_j \) [see the dashed curves in Fig. 1(b)]. This indicates that, by employing the AMC, a giant amplification of the effective mechanical decay rates can be realized, which makes the simultaneous refrigeration of the two vibrational modes feasible.

To further illustrate the underlying physics of the multimode refrigeration under the AMC mechanism, we plot the net-refrigeration rate \( \gamma_{j, \text{C}} \) of the \( j \)th vibrational mode versus the AMC strength \( \tilde{\mu} \) at the resonance \( \omega = \omega_m \); as shown in Fig. 1(c). We find that when turning off the AMC (i.e., \( \tilde{\mu} = 0 \)), the net-refrigeration rates are extremely small (i.e., \( \gamma_{j, \text{C}} \approx \gamma_j \)). These results indicate that all the vibrational modes cannot be cooled when the AMC is absent, i.e., \( \tilde{\mu} = 0 \). However, when the AMC is turned on (i.e., \( \tilde{\mu} 
eq 0 \)), the net-refrigeration rates \( \gamma_{j, \text{C}} \) are significantly enhanced with the increase of the AMC strength \( \tilde{\mu} \). For example, the net-refrigeration rates \( \gamma_{j, \text{C}} \) can be increased from \( \gamma_{j, \text{C}}/\gamma_j \approx 1 \) to more than \( 10^6 \) in our simulations.

C. Dependence of the multimode optomechanical cooling on the system parameters

In cavity optomechanics, the cold-damping feedback refrigeration of a single vibrational mode can be achieved using the cold-damping effect, which employs a designed feedback force applied to this vibrational mode, and this leads to the freezing of their thermal fluctuations [32,36-45]. Correspondingly, in principle, the feedback refrigeration of multiple vibrational modes can also be realized based on this feedback-cooling mechanism.

However, in contrast to this anticipation, we find that by using this feedback refrigeration mechanism, a counterintuitive cooling phenomenon emerges; that is, the multiple vibrational modes cannot be cooled. Physically, the dark mode, which is induced by the coupling of the multiple vibrational modes to a common optical mode, cuts off the thermal-phonon extraction channels. Since the dark mode leads to this counterintuitive uncooling phenomenon, it is natural to ask whether one can remove this dark mode to further cool these vibrational modes to their quantum ground states. To this end, the AMC is introduced to our system to remove the dark mode and control the refrigeration performance of these vibrational modes.
Specifically, when the AMC is on, we plot the steady-state mean phonon numbers $n_1$ and $n_2$ versus the optomechanical-coupling-strength ratio $G_2/G_1$ and the feedback-gain ratio $g_{cd2}/g_{cd1}$ of the two vibrational modes in Figs. 2(a) and 2(b). It is seen that the two vibrational modes can be efficiently cooled to their quantum ground states ($n_1^g, n_2^g < 1$) when the system operates in the regimes for SFL < FFL (i.e., $g_{cd2} > g_{cd1}$ and $G_2 < G_1$) or SFL > FFL (i.e., $g_{cd2} < g_{cd1}$ and $G_2 > G_1$). In contrast to the above ground-state refrigeration results, we find that when SFL = FFL (i.e., $g_{cd2} = g_{cd1}$ and $G_2 = G_1$; see the red disks), the two vibrational modes are not cooled. The physical origin behind this no-cooling phenomenon is due to the dark mode, which decouples from the system and prevents the extraction of the phonons. These results mean that the simultaneous ground-state refrigeration of these vibrational modes is achievable owing to the breaking of the system symmetry by introducing the AMC. Breaking the system symmetry leads to the removal of the dark mode.

To further elucidate how the refrigeration performance of the two vibrational modes depends on the parameters of the SFL and FFL, we plot $n_1$ and $n_2$ versus $G_2/G_1$ [see Figs. 2(c) and 2(d)] and $g_{cd2}/g_{cd1}$ [see Figs. 2(e) and 2(f)]. We can see from Figs. 2(c) and 2(d) that when SFL < FFL (i.e., $g_{cd2} < g_{cd1}$ and $G_2 < G_1$), these mechanical modes can be cooled effectively and that the refrigeration performance of the first vibrational mode is better than that of the second one. Correspondingly, when SFL > FFL (i.e., $g_{cd2} > g_{cd1}$ and $G_2 > G_1$), the simultaneous ground-state refrigeration of these vibrational modes can also be realized, and the cooling performance of the second vibrational mode is better than that of the first one. Physically, the strength of the feedback loop directly governs the feedback cooling performance, and this means that the larger the feedback-loop strength of the resonator is, the better the cooling efficiency of this resonator is.

Since the AMC plays a key role in removing the dark mode and achieving simultaneous refrigeration of the two vibrational modes, the effect of the AMC on the refrigeration performance should be studied in detail. To this end, we plot the steady-state mean phonon numbers $n_1$ and $n_2$ as functions of the AMC strength $\tilde{\mu}$ when SFL > FFL and SFL < FFL in Figs. 3(a) and 3(b). We find that in the absence of the AMC (i.e., $\tilde{\mu} = 0$), neither of the two vibrational modes can be cooled. In contrast, the simultaneous ground-state refrigeration of these vibrational modes is achieved (i.e., $(n_1^g, n_2^g < 1)$) by introducing the AMC. This is because by employing the AMC, the dark mode can be completely removed and the refrigeration channels of these vibrational modes can be opened.

In addition, in Figs. 3(c) and 3(d), we plot $n_1$ and $n_2$ versus the feedback bandwidth $\omega_{fb}$ when the AMC is on. We find that the simultaneous ground-state refrigeration of the two vibrational modes is realized (i.e., $(n_1^g, n_2^g < 1)$) under the proper parameter conditions and that the optimal refrigeration of these vibrational modes can be observed for the parameter $\omega_{fb,j}/\omega_m > 2$. In particular, we demonstrate that, with decreasing the feedback bandwidth, i.e., $\omega_{fb,j} \rightarrow 0$, the refrigeration of the two vibrations becomes inefficient. The physical origin behind this is that a smaller feedback bandwidth corresponds to a longer time delay of the feedback loop, and it leads to a lower cooling efficiency for the vibrational modes.

In particular, we find from Figs. 3(c) and 3(d) that, when SFL < FFL (SFL > FFL), the refrigeration performance of the first (second) vibrational mode is better than that of the second (first) one with increasing feedback bandwidth $\omega_{fb,j}$. This asymmetrical cooling is directly induced by the asymmetrical feedback-loop strength, which indicates that the cooling performance is better for a stronger feedback loop.

Furthermore, in Figs. 3(e) and 3(f), the final average phonon numbers $n_1$ and $n_2$ are plotted as a function of

![Figure 2](image_url)
FIG. 3. Steady-state mean thermal occupations $n_f^1$ (blue solid curves) and $n_f^2$ (red dashed curves) versus (a) and (b) the AMC strength $\tilde{\mu}$, (c) and (d) the feedback bandwidth $\omega_{fb}$, and (e) and (f) the cavity-field decay rate $\kappa$ for the cases when SFL $<\text{FFL}$ and SFL $>\text{FFL}$. Other parameters are the same as those in Fig. 2.

We find that, surprisingly, the dark-mode effect can also cause a cooling suppression for the near-degenerate-vibration case. To see the width of the frequency-detuning window associated with this dark-mode effect, in Fig. 4 we plot $n_f^j$ and $n_s^j$ versus the ratio $\omega_2/\omega_1$ without (i.e., $\tilde{\mu} = 0$) and with (i.e., $\tilde{\mu}/\omega_1 = 0.02$) the AMC. We find that without the AMC, the simultaneous ground-state refrigeration of the two vibrational modes is impossible in the frequency-detuning range $0.98 < \omega_2/\omega_1 < 1.02$. However, when the AMC is turned on, the simultaneous ground-state cooling can be realized in the corresponding region. Our findings mean that the AMC mechanism can lead to the simultaneous ground-state refrigeration of both near-degenerate and degenerate vibrational modes.

In particular, when SFL $< \text{FFL}$ (SFL $> \text{FFL}$), the cooling efficiency of the first (second) vibrational mode is better than that of the second (first) one. This is because the cooling is governed by the feedback loop, and the cooling performance is better for a larger feedback loop.

V. DISCUSSION AND CONCLUSION

Here we present a discussion to compare the cooling performance based on both bare and squeezed quadratures. It is obvious that the quadratures $q_j$ and $p_j$ in Eq. (2) are squeezed with respect to the bare quadratures and that our present cooling pertains to the squeezed quadratures. Due to the fact that the antisqueezing may have an effect on phonon number, it is worth discussing further the cooling results in the bare-quadrature-based case. To this end, we plot the final mean phonon numbers $n_f^j$ as a function of the AMC strength $\tilde{\mu}$ in both bare- and squeezed-quadrature-based cases, as shown in Figs. 5(a) and 5(b). The blue solid curves show the cooling results in the squeezed-quadrature-based case, while the red dashed curves correspond to those in the bare-quadrature-based case. We find that the two mechanical resonators can be simultaneously cooled to their quantum ground states by introducing the AMC. Physically, the introduced AMC offers an effective strategy to remove the dark mode and, in turn,
to rebuild cooling channels for extracting thermal phonons stored in the dark mode. This implies that no actual cooling is observed for the two mechanical resonators when $\mu = 0$.

Moreover, we find that when $\bar{\mu}/\omega_m \leq 0.025$, excellent agreement is observed between the bare-quadrature-based (red dashed curves) and squeezed-quadrature-based (blue solid curves) cooling results; that is, the cooling performances in both bare- and squeezed-quadrature cases are approximately the same in the region $\bar{\mu}/\omega_m \leq 0.025$. This can be explained by the fact that a weaker AMC strength leads to a smaller squeezing effect with respect to the bare quadratures. In the region $\bar{\mu}/\omega_m > 0.025$, the cooling performance of the bare-quadrature-based case becomes worse, while that of the squeezed-quadrature-based case becomes better with increasing $\bar{\mu}$. Physically, by increasing the AMC strength, the quadratures $q_j$ and $p_j$ in Eq. (2) are significantly squeezed with respect to the bare quadratures, and thus, the cooling performance in the squeezed-quadrature-based case can be improved. These results mean that, when the AMC strength $\bar{\mu}$ is properly chosen (i.e., $\bar{\mu}/\omega_m \leq 0.025$), the squeezing effect caused by the AMC has little effect on cooling performance. However, when $\bar{\mu}/\omega_m > 0.025$, the difference in the cooling results in the bare- and squeezed-quadrature-based cases cannot be neglected owing to a significant squeezing effect.

Namely, the squeezed-quadrature-based cooling is more efficient than that with the bare quadratures for larger values of the AMC strength, but the differences between these two cases are negligible if the strength is $\bar{\mu}/\omega_m \leq 0.025$. Note that in our other simulations, we set $\bar{\mu}/\omega_m = 0.02 < 0.025$, and this indicates that squeezed-quadrature-based cooling can be used to implement bare-quadrature-based cooling when $\bar{\mu}/\omega_m \leq 0.025$.

In summary, we proposed a method to achieve the simultaneous ground-state refrigeration of multiple vibrational modes beyond the resolved-sideband regime and to realize a millionfold amplification in the net-refrigeration rates. This is realized by introducing an AMC to break the symmetry of the system, which then leads to removing the dark-mode effect. Using fully analytical treatments, we showed that when the AMC is switched on, the amplification of the net-refrigeration rates can be observed for more than six orders of magnitude. Remarkably, we revealed that without the AMC, the simultaneous ground-state refrigeration of the two vibrational modes is unfeasible; however, with the AMC, these vibrational modes can be efficiently cooled to their quantum ground states. Our work could potentially be used for observing

FIG. 4. Steady-state mean phonon numbers $n_i^f$ (blue curves) and $n_i^s$ (red curves) as a function of the frequency ratio $\omega_2/\omega_1$ of the two vibrational modes without ($\bar{\mu} = 0$, solid curves) and with ($\bar{\mu} = 0.02\omega_1$, dashed curves) the AMC. The parameters used for (a) are $g_{\text{out}} = 0.5$ and $G_2 = 0.5G_1$, and those for (b) are $g_{\text{out}} = 1.9$ and $G_2 = 1.9G_1$. Other parameters are set as in Fig. 1.

FIG. 5. Steady-state average phonon numbers (a) $n_1^f$ and (b) $n_2^f$ as a function of the AMC $\bar{\mu}$ for the cooling cases of the squeezed (blue solid curves) and bare (red dashed curves) quadratures under SFL $< FFL$. Here the bare- and squeezed-quadrature-based cases are, respectively, based on the quadratures in Eqs. (1) and (2).
quantum-mechanical effects and controlling macroscopic mechanical coherence in the unresolved-sideband regime.

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APPENDIX: CALCULATION OF THE STEADY-STATE MEAN PHONON NUMBERS

In this Appendix, we show the remaining expressions for the parameters used in the Eqs. (27)–(35). These expressions are defined as follows:

$$A_1 = \omega (\kappa + \omega_h), \quad A_2 = \kappa \omega_h - \omega^2, \quad A_3 = \omega (\gamma \omega^2 - \kappa \Delta_2), \quad A_4 = \omega^2 (\gamma \kappa + \Delta_2),$$
$$A_5 = \omega \omega_h (\gamma - \omega^2), \quad A_6 = \omega^2 \omega_h (\gamma + \kappa), \quad A_7 = G_2 g_{cd\omega \omega} \omega \omega_h, \quad A_8 = \omega \omega^2 \omega_h, \quad A_9 = \kappa \omega \omega_h,$$
$$W_1 = \omega (\gamma \omega^3 - \kappa \Delta_1), \quad W_2 = \omega^2 (\gamma \kappa + \Delta_1), \quad W_3 = \omega \omega_h (\gamma \kappa - \omega^2), \quad W_4 = \omega^2 \omega_h (\gamma + \kappa),$$
$$W_5 = G_1 g_{cd\omega \omega} \omega \omega_h, \quad W_6 = \omega \omega^2 \omega_h, \quad W_7 = \kappa \omega^2 \omega_h, \quad B_1 = A_1 A_3 - A_2 A_4,$$
$$B_2 = A_2 A_3 + A_1 A_4, \quad B_3 = A_1 A_3 + A_2 A_4 + A_1 A_2 - A_1 A_4 - A_2 A_3,$$
$$B_4 = A_2 A_5 - A_1 A_6 + A_2 A_7 + A_2 A_8 + A_1 A_9, \quad C_1 = B_1 - B_3,$$
$$C_2 = B_2 - B_4, \quad D_1 = 2 \mu A_1 - G_2 g_{cd\omega \omega} \omega \omega_h, \quad D_2 = 2 \mu \omega (A_1 - G_1 g_{cd\omega \omega} \omega \omega_h), \quad D_3 = 2 \mu A_2,$$
$$E_j = C_i D_j D_{j+1} + D_j D_{j+1} C_{j+1} - (1)^{j} (D_j C_j - C_j D_{j+1}) - (1)^{j} (D_j Y_j - Y_j D_{j+1}),$$
$$T_j = D_j D_{j+1} Y_j - (1)^{j} (D_j Y_j - Y_j D_{j+1}),$$
$$M_j = 2 \gamma D_j Y_j + (1)^{j} D_j Y_j D_{j+1} + D_j D_{j+1} Y_{j+1},$$

$$E_{j+2} = \omega_1 \left[ E_{j+2} \left( A_{j+2}^2 + A_j^2 \right) + (1)^{j+1} A_j G_1 g_{cd\omega \omega} \omega \omega_h \left( C_{j+2}^2 + C_j^2 \right) \right], \quad L_{j+2} = A_j (W_5 + W_3 + W_6) + (1)^{j+1} A_{j-1} (W_4 - W_7),$$
$$F_{j+2} = \omega_1 \omega_2 \chi^2 \left[ \left( -G_2 g_{cd\omega \gamma} \omega \omega_h + 2 G_3 g_{cd\omega \omega \omega} \omega \omega_h \right) + \left( G_2 g_{cd\omega \omega} \omega \omega_h \right) \right],$$
$$N_2 = \omega_3 \chi^2 \left[ \left( -G_2 g_{cd\omega \omega} \omega \omega_h \right) + \left( G_2 g_{cd\omega \omega} \omega \omega_h \right) \right],$$
$$N_3 = \omega_4 \chi^2 \left[ \left( -G_2 g_{cd\omega \omega} \omega \omega_h \right) + \left( G_2 g_{cd\omega \omega} \omega \omega_h \right) \right],$$
$$N_4 = \left( G_2 g_{cd\omega \omega} \omega \omega_h \right),$$
$$M_2 = \omega_5 \chi^2 \left[ \left( -G_2 g_{cd\omega \omega} \omega \omega_h \right) + \left( G_2 g_{cd\omega \omega} \omega \omega_h \right) \right],$$

$$\omega = \omega_1 \omega_2 \chi^2 \left[ \left( -G_2 g_{cd\omega \omega} \omega \omega_h \right) + \left( G_2 g_{cd\omega \omega} \omega \omega_h \right) \right], \quad \omega_2 = \omega_1 \omega_2 \chi^2 \left[ \left( -G_2 g_{cd\omega \omega} \omega \omega_h \right) + \left( G_2 g_{cd\omega \omega} \omega \omega_h \right) \right],$$
$$\omega_3 = \omega_1 \omega_2 \chi^2 \left[ \left( -G_2 g_{cd\omega \omega} \omega \omega_h \right) + \left( G_2 g_{cd\omega \omega} \omega \omega_h \right) \right],$$
$$\omega_4 = \omega_1 \omega_2 \chi^2 \left[ \left( -G_2 g_{cd\omega \omega} \omega \omega_h \right) + \left( G_2 g_{cd\omega \omega} \omega \omega_h \right) \right],$$

where $j = 1, 2, \Delta_j = \omega^2 - \omega^2, \Delta_{3-j} = \omega^2 - \omega^2, x = \kappa + \omega^2$, and $y = \omega^2 + \omega^2$.


