Multimode optomechanical cooling via general dark-mode control

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The dark-mode effect is a stubborn obstacle for ground-state cooling of multiple degenerate mechanical modes optomechanically coupled to a common cavity-field mode. Here we propose an auxiliary-cavity-mode method for simultaneous ground-state cooling of two degenerate or near-degenerate mechanical modes by breaking the dark mode. We find that the introduction of the auxiliary cavity mode not only breaks the dark-mode effect, but also provides a new cooling channel to extract the thermal excitations stored in the dark mode. Moreover, we study the general physical-coupling configurations for breaking the dark mode in a generalized network-coupled four-mode optomechanical system consisting of two cavity modes and two mechanical modes. We find the analytical dark-mode-breaking condition in this system. This method is general and it can be generalized to break the dark-mode effect and to realize the simultaneous ground-state cooling in a multiple-mechanical-mode optomechanical system. We also demonstrate the physical mechanism behind the dark-mode breaking by studying the breaking of dark-state effect in the $N$-type four-level atomic system. Our results not only provide a general method to control various dark-mode and dark-state effects in physics, but also present an opportunity to the study of macroscopic quantum phenomena and applications in multiple-mechanical-resonator systems.

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I. INTRODUCTION

Considerable recent interest in cavity optomechanics [1–3] has been paid to multimode optomechanical systems involving two [4–14] or multiple [15–27] mechanical resonators; in particular, the two-mechanical-mode optomechanical systems have been realized in several experimental platforms [7–14]. The study of multiple-mechanical-mode optomechanical systems has significance in both fundamental quantum physics [28] and modern quantum technologies [3]. For example, generation of macroscopic mechanical entanglement in multimode optomechanical systems has been experimentally demonstrated [10,11,13,14]. Multimode optomechanical systems have also been considered to study quantum many-body effects [24–26], high performance sensors [29,30], precise measurement [31], and nonreciprocal phonon or photon transport [32–39].

The simultaneous ground-state cooling of multiple mechanical modes has become a desired task because it is a prerequisite for the manipulation of macroscopic mechanical coherence [28]. In particular, owing to the inherent structural features in multiple mechanical-mode optomechanical systems, people prefer to implement the simultaneous cooling of multiple mechanical resonators rather than to cool these resonators one by one using the single-resonator cooling techniques. Though great success has been made in cooling a single mechanical resonator in optomechanical systems [40–51], it remains a great challenge to perform ground-state cooling of multiple degenerate mechanical resonators coupled to a common cavity field, due to the mechanical dark-mode effect [7,52–55]. For the two-mechanical-mode case, the dark-mode effect has been theoretically found [52] and experimentally demonstrated [55]. Meanwhile, the dark mode formed in optomechanical systems involving two cavity modes and one mechanical mode has also been found [56–59]. So far, theoretical proposals for optomechanical cooling of multiple mechanical resonators coupled in series have been proposed [60,61] and cooling of multimodes in a resonator have been analyzed with the cold-damping feedback method [62,63]. In addition, ground-state cooling of multiple mechanical resonators has been proposed based on synthetic magnetism [64] and reservoir engineering [65]. In particular we mention that considerable attention [66–69] has been paid to the suppression of optomechanical backaction and to the improvement of restrictions for effective cooling in multimode optomechanical systems.

In this paper we propose an auxiliary-cavity-mode method for breaking the dark mode and further realizing ground-state cooling of two degenerate mechanical modes. We also explore the general coupling configurations for breaking the dark mode in a generalized four-mode optomechanical system, in which all the two-node couplings exist among two cavity modes and two mechanical modes. We find the
analytical general conditions for forming and breaking the dark mode. Moreover, we extend this method to break the dark modes and realize the ground-state cooling of the multiple mechanical resonators. Correspondingly, our scheme can also be used to cool multiple degenerate or near-degenerate vibrational modes in a resonator. We also describe the physical mechanism for breaking the dark-state effect in the N-type four-level atomic system. The physical mechanism of our scheme is general and it can be generalized to control other dark-mode and dark-state effects in various branches of physics. In this sense, our results not only open up a different route to the realization of simultaneous ground-state cooling of two degenerate or near-degenerate mechanical resonators and enrich the technique of few-body resonator cooling, but also initiate advances in dark-state engineering [70].

The rest of this paper is organized as follows. In Sec. II we introduce the physical model and present the quantum Langevin equations. In Sec. III we study ground-state cooling of the two mechanical modes by calculating the final mean phonon numbers. In Sec. IV we obtain the universal dark-mode-breaking conditions in a four-mode optomechanical system consisting of two cavity modes and two mechanical modes. We also analyze the quantum interference effect in the energy-level transitions of the system. In Sec. V we study the simultaneous ground-state cooling of multiple mechanical modes and the dark-mode breaking in a multiple-mechanical-mode optomechanical system. In Sec. VI we describe the physical mechanism for breaking the dark-state effect in the N-type four-level atomic system. We present a discussion of the experimental implementation of our scheme in Sec. VII and summarize this work in Sec. VIII.

II. PHYSICAL MODEL AND EQUATIONS OF MOTION

We consider an N-type four-mode optomechanical system consisting of two cavity modes (an intermediate cavity mode \(a\) and an auxiliary cavity mode \(a_s\)) and two mechanical modes \((b_1\) and \(b_2\)), as shown in Fig. 1(a). Here the intermediate cavity mode is coupled to the two mechanical modes via radiation-pressure interactions. When the frequencies of the two mechanical modes are degenerate, a dark mode is formed in this linearized optomechanical system [42]. This dark mode decouples from the intermediate cavity mode and hence the ground-state cooling of the two mechanical modes is largely suppressed. To break the dark-mode effect, we introduce the auxiliary cavity mode \(a_s\), which is optomechanically coupled to the mechanical mode \(b_1\). Moreover, two driving fields are applied to the cavity modes to control the optical and mechanical degrees of freedom. In a rotating frame defined by the operator \(\exp[-i(\omega_c a^\dagger a + \omega_{d1} a_s^\dagger a_s)\tau]\), the system Hamiltonian is given by (\(\hbar = 1\))

\[
H_T = \Delta_c a^\dagger a + \Delta_{a_s} a_s^\dagger a_s + \sum_{l=1,2} [\omega_l b_l^\dagger b_l + g_l a^\dagger a (b_l^\dagger + b_l)] + g_{s1} a_s^\dagger a_s (b_1^\dagger + b_1) + (\Omega a^\dagger + \Omega a_s^\dagger + H.c.),
\]

where \(\Delta_c = \omega_c - \omega_{d1}\) (\(\Delta_{a_s} = \omega_{a_s} - \omega_{d2}\)) is the driving detuning of the cavity frequency \(\omega_c\) (\(\omega_{a_s}\)) with respect to its driving frequency \(\omega_{d1}\) (\(\omega_{d2}\)) and \(a\) (\(a^\dagger\)), \(a_s\) (\(a_s^\dagger\)), and \(b_{l=1,2}(b_l^\dagger)\) are the annihilation (creation) operators of the intermediate cavity mode, the auxiliary cavity mode, and the \(l\)th mechanical mode, with the corresponding resonance frequencies \(\omega_c\), \(\omega_{a_s}\), and \(\omega_l\) respectively. The \(g_{l=1,2}\) term describes the optomechanical coupling between the mechanical mode \(b_{l=1,2}\) (\(b_l\)) and the cavity mode \(a\) \((a_s)\). The parameters \(\omega_{d1}\) \((\omega_{d2}\)) and \(\Omega\) \((\Omega_{a_s}\)) are the driving frequency and amplitude related to the driving field of the cavity mode \(a\) \((a_s)\), respectively.

To include the dissipations in this system, we assume that the two cavity fields are coupled to individual vacuum baths and that the two mechanical modes are coupled to individual heat baths. Then the evolution of this system is governed by the quantum Langevin equations

\[
\dot{a} = -(i\Delta_c + \kappa) a - i\Omega - \sum_{l=1,2} g_l a (b_l^\dagger + b_l) + \sqrt{2\kappa} a_{in},
\]

\[
\dot{a}_s = -(i\Delta_{a_s} + \kappa_s) a_s - i[\Omega + g_{s1} a_s (b_1^\dagger + b_1)] + \sqrt{2\kappa_s} a_{s,in},
\]

\[
\dot{b}_1 = -(\gamma_1 + i\omega_{d1}) b_1 - i(g_{a_s} a + g_{a_s} a_s^\dagger) + \sqrt{2\gamma_1} b_{1,in},
\]

\[
\dot{b}_2 = -(\gamma_2 + i\omega_{d2}) b_2 - i g_{2} a^\dagger a + \sqrt{2\gamma_2} b_{2,in},
\]
where $\kappa$, $\kappa_r$, and $\gamma_{\pm}=1,2$ are the decay rates of the intermediate cavity mode $a$, the auxiliary cavity mode $a_r$, and the $l$th mechanical mode, respectively. The operators $a_{in}$ ($a_{in}^\dagger$), $a_{r,in}$ ($a_{r,in}^\dagger$), and $b_{l,in}$ ($b_{l,in}^\dagger$) are the noise operators associated with the intermediate cavity mode, the auxiliary cavity mode, and the $l$th mechanical mode, respectively. These noise operators have zero mean values and obey the correlation functions \[ \langle a_{in}(t)a_{in}^\dagger(t') \rangle = \delta(t-t'), \] \[ \langle a_{r,in}^\dagger(t)a_{r,in}(t') \rangle = 0, \] \[ \langle a_{in}^\dagger(t)a_{r,in}^\dagger(t') \rangle = \delta(t-t'), \] \[ \langle a_{r,in}^\dagger(t)a_{r,in}(t') \rangle = 0, \] \[ \langle b_{l,in}^\dagger(t)b_{l,in}(t') \rangle = (\bar{n}_l + 1)\delta(t-t'), \] \[ \langle b_{l,in}(t)b_{l,in}^\dagger(t') \rangle = \bar{n}_l\delta(t-t') \] for $l = 1, 2$, where $\bar{n}_{l=1,2}$ is the average thermal-phonon occupation number associated with the $l$th mechanical mode.

To cool the mechanical modes, we assume that the two cavity modes are driven strongly and then the mean photon numbers in the two cavities are large enough and this four-mode optomechanical system can be processed by the linearization procedure. In this way, the operators $\hat{o} \in \{a, a_r, a_r^\dagger, b_l, b_l^\dagger\}$ can be expressed as a summation of steady-state average values and quantum fluctuation operators, i.e., $\hat{o} = \langle \hat{o} \rangle_{ss} + \delta \hat{o}$. By separating the steady-state average values and the quantum fluctuation operators, we can obtain the linearized Langevin equations for the quantum fluctuation operators

\[ \delta \dot{a} = -(\kappa + i\Delta_a^\prime)\delta a - i \sum_{l=1,2} [G_l \langle \delta b_l + \delta b_l^\dagger \rangle] + \sqrt{2\kappa} a_{in}, \] \[ \delta \dot{a}_r = -(\kappa_r + i\Delta_a^\prime)\delta a_r - iG_{11} \langle \delta b_1 + \delta b_1^\dagger \rangle + \sqrt{2\kappa_r} a_{r,in}, \] \[ \delta \dot{b}_1 = -(i\omega_1 + \gamma_1)\delta b_1 - iG_{11}^\prime \delta a - iG_{12} \delta a_r - iG_{11}^\dagger \delta a_r - iG_{11}^\prime \delta a_r, \] \[ \delta \dot{b}_2 = -(i\omega_2 + \gamma_2)\delta b_2 - iG_{22}^\prime \delta a - iG_{22}^\dagger \delta a_r + \sqrt{2\gamma_2} b_{2,in}, \]

where the parameters $\Delta_a^\prime = \Delta_a + 2g_1 \text{Re}(\beta_1) + 2g_2 \text{Re}(\beta_2)$ and $\Delta_a^\prime = \Delta_a + 2g_1 \text{Re}(\beta_1)$ are the effective driving detunings of the cavity mode $a$ and the auxiliary cavity mode $a_r$, respectively, with $\text{Re}(\beta_{l=1,2})$ taking the real part of $\beta_l$. The parameter $G_1 = g_1 a$ ($G_{11} \equiv g_{11} a_r$) is the linearized optomechanical-coupling strength between the cavity mode $a$ ($a_r$) and the $l$th mechanical mode (the mechanical mode $b_l$). In the steady-state case, the average values of the system operators can be obtained as

\[ \alpha \equiv \langle a \rangle_{ss} = -i\Omega \frac{\kappa}{\kappa + i\Delta_a^\prime}, \] \[ \alpha_r \equiv \langle a_r \rangle_{ss} = -i\Omega_r \frac{\kappa_r}{\kappa_r + i\Delta_a^\prime}, \]

with the intermediate cavity mode, the auxiliary cavity mode, and the $l$th mechanical mode, respectively. These noise operators have zero mean values and obey the correlation functions \[ \langle a_{in}(t)a_{in}^\dagger(t') \rangle = \delta(t-t'), \] \[ \langle a_{r,in}^\dagger(t)a_{r,in}(t') \rangle = 0, \] \[ \langle a_{in}^\dagger(t)a_{r,in}^\dagger(t') \rangle = \delta(t-t'), \] \[ \langle a_{r,in}^\dagger(t)a_{r,in}(t') \rangle = 0, \] \[ \langle b_{l,in}^\dagger(t)b_{l,in}(t') \rangle = (\bar{n}_l + 1)\delta(t-t'), \] \[ \langle b_{l,in}(t)b_{l,in}^\dagger(t') \rangle = \bar{n}_l\delta(t-t') \] for $l = 1, 2$, where $\bar{n}_{l=1,2}$ is the average thermal-phonon occupation number associated with the $l$th mechanical mode.

To cool the mechanical modes, we assume that the two cavity modes are driven strongly and then the mean photon numbers in the two cavities are large enough and this four-mode optomechanical system can be processed by the linearization procedure. In this way, the operators $\hat{o} \in \{a, a_r, a_r^\dagger, b_l, b_l^\dagger\}$ can be expressed as a summation of steady-state average values and quantum fluctuation operators, i.e., $\hat{o} = \langle \hat{o} \rangle_{ss} + \delta \hat{o}$. By separating the steady-state average values and the quantum fluctuation operators, we can obtain the linearized Langevin equations for the quantum fluctuation operators

\[ \delta \dot{a} = -(\kappa + i\Delta_a^\prime)\delta a - i \sum_{l=1,2} [G_l \langle \delta b_l + \delta b_l^\dagger \rangle] + \sqrt{2\kappa} a_{in}, \] \[ \delta \dot{a}_r = -(\kappa_r + i\Delta_a^\prime)\delta a_r - iG_{11} \langle \delta b_1 + \delta b_1^\dagger \rangle + \sqrt{2\kappa_r} a_{r,in}, \] \[ \delta \dot{b}_1 = -(i\omega_1 + \gamma_1)\delta b_1 - iG_{11}^\prime \delta a - iG_{12} \delta a_r - iG_{11}^\dagger \delta a_r - iG_{11}^\prime \delta a_r, \] \[ \delta \dot{b}_2 = -(i\omega_2 + \gamma_2)\delta b_2 - iG_{22}^\prime \delta a - iG_{22}^\dagger \delta a_r + \sqrt{2\gamma_2} b_{2,in}, \]

where the parameters $\Delta_a^\prime = \Delta_a + 2g_1 \text{Re}(\beta_1) + 2g_2 \text{Re}(\beta_2)$ and $\Delta_a^\prime = \Delta_a + 2g_1 \text{Re}(\beta_1)$ are the effective driving detunings of the cavity mode $a$ and the auxiliary cavity mode $a_r$, respectively, with $\text{Re}(\beta_{l=1,2})$ taking the real part of $\beta_l$. The parameter $G_1 = g_1 a$ ($G_{11} \equiv g_{11} a_r$) is the linearized optomechanical-coupling strength between the cavity mode $a$ ($a_r$) and the $l$th mechanical mode (the mechanical mode $b_l$). In the steady-state case, the average values of the system operators can be obtained as

\[ \beta_1 \equiv \langle b_1 \rangle_{ss} = \frac{-ig_1 |\alpha|^2 - ig_{11} |\alpha_r|^2}{\gamma_1 + i\omega_1}, \] \[ \beta_2 \equiv \langle b_2 \rangle_{ss} = \frac{-i\gamma_2 |\alpha_r|^2}{\gamma_2 + i\omega_2}. \]
From Eqs. (10) we can see that the two hybrid modes $B_-$ and $B_+$ are decoupled from each other ($\zeta = 0$) when $\omega_1 = \omega_2$. Moreover, the hybrid mode $B_-$ also decouples from the cavity mode $a$, which means that the hybrid mode $B_-$ becomes a dark mode. At this time, the ground-state cooling of the two mechanical modes is largely suppressed.

To break the dark-mode effect in this optomechanical system, we introduce an auxiliary cavity mode, which is coupled to the mechanical mode $b_1$ via the radiation-pressure interaction. By substituting operators $B_+ (B_+^\dagger)$ and $B_- (B_-^\dagger)$ into Eq. (6), the Hamiltonian $H_{\text{RWA}}$ becomes

$$H_{\text{RWA}} = \Delta_1 ^\zeta \delta a_0 \delta a_0^\dagger + \omega_1 B_0^\dagger B_0 + \omega_2 B_-^\dagger B_- + \zeta (B_0^\dagger B_- + B_-^\dagger B_0) + G_+ (\delta a B_0^\dagger + B_0^\dagger \delta a^\dagger) + G_- (\delta a B_-^\dagger + B_-^\dagger \delta a^\dagger).$$

(11)

where we introduce two new coupling strengths $G_+$ and $G_-$. The coupling configuration associated with the approximate Hamiltonian (11) is described by Fig. 1(b). Here the couplings are expressed in the representation of the mechanical hybrid modes. We can see from Eqs. (12) that $G_\pm > 0$, i.e., the hybrid modes $B_\pm$ are always coupled with the auxiliary cavity mode $a_0$. Therefore, even if the hybrid mode $B_-$ is decoupled from both the intermediate cavity mode $a$ and the hybrid mode $B_+$, the ground-state cooling of the two mechanical modes can also be achieved via the cooling channel associated with the auxiliary cavity mode $a_0$.

For studying quantum cooling of the mechanical modes, we are interested in the steady-state properties of the system. Therefore, we should analyze the stability condition of this linearized system. To this end, we rewrite the linearized Langevin equations (4) as the contact form

$$\dot{u}(t) = Au(t) + N(t),$$

(13)

where

$$\begin{align*}
N(t) & = \sqrt{2} [\sqrt{\kappa} b_0, \sqrt{\kappa} a_0, \sqrt{\gamma_1} b_1, \sqrt{\gamma_2} b_2, \sqrt{\kappa} a_0, \sqrt{\gamma_2} b_2] \end{align*} $$

and

$$E = \begin{pmatrix}
\kappa & i \Delta_1 & 0 & i G_1 & i G_2 \\
0 & \kappa_+ & i \Delta_1 & i G_1 & 0 \\
i G_1^\dagger & i G_2^\dagger & \gamma_1 + i \omega_1 & 0 & 0 \\
i G_2^\dagger & 0 & 0 & \gamma_2 + i \omega_2 & 0 \\
0 & 0 & 0 & 0 & 0 
\end{pmatrix}$$

(14)

and $F$ is defined by the nonzero elements $F_{13} = -i G_1$, $F_{14} = -i G_2$, $F_{23} = -i G_1$, $F_{11} = -i G_1$, $F_{22} = -i G_1$, and $F_{11} = -i G_2$. The eigensystem of the coefficient matrix $A$ determines the stability of the system. By using the Routh-Hurwitz criterion [73], we can find the stability condition. In the following derivation, all the parameters used satisfy the stability conditions.

### III. GROUND-STATE COOLING OF THE TWO MECHANICAL MODES

In this section we study the cooling performance of the two mechanical modes by calculating the final mean phonon numbers. The formal solution of the linearized Langevin equations (13) can be obtained as

$$u(t) = M(t)u(0) + \int_{0}^{t} M(t-s)N(s)ds,$$

(15)

where we introduce the matrix $M(t) = \exp(At)$. The final mean phonon numbers of the two mechanical modes can be calculated by solving the steady state of the system.

Based on Eq. (15), we can obtain the final mean phonon numbers by solving the Lyapunov equation [74]. To this end, we introduce the covariance matrix $V$ of the system by defining the matrix elements as

$$V_{ij} = \frac{1}{2} [\langle u_i(\infty) u_j(\infty) \rangle + \langle u_j(\infty) u_i(\infty) \rangle], \quad i, j = 1, \ldots, 8.$$  

(16)

In the linearized optomechanical system, the covariance matrix $V$ satisfies the Lyapunov equation

$$AV + VA^T = -Q.$$  

(17)

Here the matrix $Q$ is defined by

$$Q = \frac{1}{2}(C + C^T),$$  

(18)

where $C$ is the correlation matrix related to the noise operators. The matrix elements of $C$ are defined by

$$\langle N_k(s) N_l(s') \rangle = C_{kl} \delta(s - s').$$  

(19)

In this work we consider the Markovian dissipation case. Then the constant matrix $C$ can be obtained with the nonzero elements $C_{15} = 2\kappa$, $C_{26} = 2\kappa_+$, $C_{37} = 2\gamma_1 (\bar{\omega}_1 + 1)$, $C_{48} = 2\gamma_2 (\bar{\omega}_2 + 1)$, $C_{73} = 2\gamma_1 \bar{\omega}_1$, and $C_{84} = 2\gamma_2 \bar{\omega}_2$. By solving the Lyapunov equation, we obtain the covariance matrix $V$ defined in Eq. (16). Then the final mean phonon numbers of the two mechanical modes can be obtained as

$$n_i^f = \langle \delta b_i^\dagger \delta b_i \rangle = V_{33} - \frac{1}{2},$$  

(20a)

$$n_i^f = \langle \delta b_i^\dagger \delta b_i \rangle = V_{44} - \frac{1}{2},$$  

(20b)

where $V_{33}$ and $V_{44}$ are the matrix elements of the covariance matrix $V$.

Here we first consider the cooling performance in the two-degenerate-resonator case ($\omega_1 = \omega_2$). In Fig. 2 we plot the final mean phonon numbers $n_1^f$ and $n_2^f$ as functions of the scaled driving detuning $\Delta_c/\omega_1$ ($\Delta_c/\omega_1$) and the scaled cavity-field decay rate $\kappa/\omega_1$ ($\kappa/\omega_1$). To better analyze the influence of the driving detuning and the sideband-resolution condition on the cooling performance, here we choose the mechanical frequency $\omega_1$ as the scale unit. In Figs. 2(a) and 2(b) we can see that ground-state cooling of the two mechanical modes can be realized in the resolved-sideband limit ($\kappa/\omega_1 \ll 1$) and around $\Delta_c/\omega_1 \sim 1$. Moreover, the cooling efficiency of the first mechanical mode is much better than that of the second
one even in unresolved-sideband limit ($\kappa/\omega_1 \geq 1$). This is because the first mechanical mode is simultaneously connected to two cooling channels and the coupling strength between the auxiliary cavity mode and the first mechanical mode is large enough ($G_{31}/\omega_1 = 0.08$). For a given decay rate $\kappa/\omega_1$, the optimal driving detuning is about $\Delta/\omega_1 = 1$, which corresponds to the red-sideband resonance. In Figs. 2(c) and 2(d) we can see that the ground-state cooling of the two mechanical modes can be realized in the resolved-sideband limit ($\kappa_1/\omega_1 \ll 1$) and the cooling performance is the best at the optimal driving detuning $\Delta/\omega_1 \approx 1$. These results are consistent with the sideband cooling results in typical optomechanical systems [40,41,52]. In addition, we perform numerical calculations with several sets of parameters when $\Delta/\omega_1 \gg 1$ or $\kappa_1/\omega_1 \gg 1$. We find that though these mechanical modes can be cooled significantly, but they cannot be cooled to the ground state in the $N$-type optomechanical system.

Next we analyze the influence of the frequency mismatch between the two mechanical modes on the cooling efficiency. In Figs. 3(a) and 3(b) we plot the phonon numbers $n_1^f$ and $n_2^f$ versus the frequency ratio $\omega_2/\omega_1$ and the scaled cavity-field decay rate $\kappa/\omega_1$ in the absence of the auxiliary cavity mode ($G_{31}/\omega_1 = 0$). Here we see that the final mean phonon numbers in the two mechanical modes cannot be efficiently decreased around $\omega_2 = \omega_1$, which means that the two mechanical modes cannot be cooled to their ground states when their frequencies are degenerate or nearly degenerate (in a finite-detuning window). This phenomenon can be explained according to the dark-mode effect. In the degenerate-resonator case ($\omega_1 = \omega_2$), a bright mode and a dark mode are formed in this optomechanical system. Physically, the two mechanical modes have an obvious spectral overlap and become effectively degenerate in the presence of dissipation; thus the dark-mode effect works in the near-degenerate case. The dark mode decouples from both the cavity mode and the bright mode, so the phonon excitations stored in the dark mode cannot be extracted through the optomechanical-cooling channel [64]. However, the dark-mode effect disappears when the two mechanical modes are far-off-resonance, thus achieving ground-state cooling. In Figs. 3(c) and 3(d) we plot the phonon numbers $n_1^f$ and $n_2^f$ versus the frequency ratio $\omega_2/\omega_1$ and the scaled cavity-field decay rate $\kappa/\omega_1$ when the auxiliary cavity field is present ($G_{31}/\omega_1 = 0.08$). Here the ground-state cooling of the two mechanical modes can be achieved ($n_1^f \ll 1$) when the system works in the resolved-sideband regime ($\kappa \ll \omega_1$). Here the first mechanical resonator has a better cooling efficiency ($n_1^f < n_2^f$) because it is connected to two cooling channels at the same time.

Since the auxiliary cavity mode not only provides the direct channel to extract the thermal excitations from the mechanical mode $b_1$, but also provides a cooling channel to extract the thermal excitations from the mechanical mode $b_2$, the coupling strength between the auxiliary cavity mode $a_1$ and the mechanical mode $b_1$ is an important factor to the cooling efficiency. To see this effect more clearly, in Figs. 4(a) and 4(b) we plot the phonon numbers $n_1^f$ and $n_2^f$ as functions of the linearized optomechanical-coupling strength $G_{31}$ between the auxiliary cavity mode $a_1$ and the mechanical mode $b_1$ when the scaled cavity-field decay rate $\kappa_1/\omega_1$ takes various values. Here we see that the phonon numbers decrease with the increase of the coupling strength $G_{31}$, which means that the increase of the coupling strength $G_{31}$ is beneficial to the ground-state cooling of the two mechanical resonators. Moreover, the final mean phonon numbers are smaller for smaller values of the decay rate $\kappa_1/\omega_1$ for a certain parameter.
range, which is consistent with the analyses of the cooling efficiency on the sideband-resolution condition [40,41,52]. In addition, we can see from Fig. 4 that the ground-state cooling can be realized in the N-type optomechanical system and the cooling limit cannot be broken compared with the single-resonator optomechanical system under the same parameters.

IV. GROUND-STATE COOLING AND UNIVERSAL DARK-MODE-BREAKING CONDITIONS IN THE FOUR-MODE OPTOMECHANICAL SYSTEM

In previous sections we have shown that, by introducing an optomechanical coupling between the auxiliary cavity mode $a_s$ and the mechanical mode $b_1$, the dark mode in this system can be broken and then ground-state cooling of the two mechanical modes can be realized. However, in practice, the diverse interactions among these degrees of freedom in this system are more complicated [25], so it is an interesting topic to study the universal condition for breaking the dark-mode effect in a more general four-mode optomechanical system. In this section we analyze the parameter conditions under which the dark-mode effect works and study how to break the dark mode by controlling the couplings in the four-mode optomechanical system. We also analyze the interference effect by studying the energy-level transition of this four-mode optomechanical system.

A. Ground-state cooling in the general four-mode optomechanical system

We now consider a network-coupled four-mode optomechanical system consisting of an intermediate-coupling cavity mode and an auxiliary cavity mode, which are both coupled to two mechanical modes via the radiation-pressure interaction. Here the two cavity (mechanical) modes are coupled to each other via a photon-hopping (phonon-hopping) interaction, as shown in Fig. 5(a). In a rotating frame defined by the operator

$$\exp[-i(\omega_L a^\dagger a + \omega_d a^\dagger a_s)\tau]$$

under $\omega_L = \omega_d$, the transformed Hamiltonian becomes

$$H_I = \Delta_c a^\dagger a + \Delta_s a^\dagger a_s + J(a^\dagger a_s + a^\dagger a) + \eta(b_1^\dagger b_2 + b_2^\dagger b_1) + \sum_{l=1,2} [\omega_l b_1^\dagger b_1 + g_{l1} a^\dagger a (b_1^\dagger + b_1) + g_{l2} a^\dagger a_s (b_2^\dagger + b_2)] + (\Omega a^\dagger + \Omega_s a^\dagger_s + H.c.),$$

where some operators and variables have been defined in Eq. (1). We also introduce the $g_{l2}$ coupling term, the $J$ coupling term, and the $\eta$ coupling term, which correspond to the optomechanical coupling between modes $a_s$ and $b_2$, the photon-hopping coupling, and the phonon-hopping coupling, respectively.

Based on the Hamiltonian (21), we can obtain the Langevin equations by adding the decay and noise terms to the Heisenberg equations. Following a linearization procedure similar to the one performed in Sec. II, we obtain the linearized Langevin equations

$$\delta \dot{a} = -(\kappa + i\Delta_c')\delta a - iJ\delta a_s - i\sum_{l=1,2} [G_l (\delta b_1 + \delta b_1^\dagger)] + \sqrt{2\kappa a_{in}},$$

$$\delta \dot{a}_s = -(\kappa_s + i\Delta_s')\delta a_s - iJ\delta a - i\sum_{l=1,2} [G_{sl} (\delta b_1 + \delta b_1^\dagger)] + \sqrt{2\kappa_s a_{in}},$$

$$\delta \dot{b}_1 = -(i\omega_l + \gamma_1)\delta b_1 - iG_{l1}^* \delta a - iG_{l1} \delta a_s - iG_{l1} a^\dagger_s - iG_{l1}^* a_s - iG_{l2} a^\dagger_s - iG_{l2} a_s,$$

$$\delta \dot{b}_2 = -(i\omega_2 + \gamma_2)\delta b_2 - iG_{s1}^* \delta a - iG_{s1} \delta a_s - iG_{s1} a^\dagger_s - iG_{s1}^* a_s - iG_{s2} a^\dagger_s - iG_{s2} a_s.$$
where $\Delta'_2 = \Delta_2 + 2g_{s2}Re(\beta_1) + 2g_{s2}Re(\beta_2)$ is the renormalized driving detuning of the auxiliary cavity mode $\alpha$. It should be pointed out that the parameter $\Delta'_2$ and the linearized optomechanical-coupling strengths $G_l$ and $G_{l1}$ for $l = 1, 2$ have the same definition as those defined in Sec. II. However, the coherent displacements of the steady state $\alpha, \alpha_1, \beta_1,$ and $\beta_2$ in the network-coupled four-mode optomechanical system should be replaced by the relations

$$\alpha \equiv \langle a \rangle_{ss} = \frac{-i(Ja + \Omega_0)}{\kappa + i\Delta'_2},$$

$$\alpha_1 \equiv \langle a_1 \rangle_{ss} = \frac{-i(Ja + \Omega_0)}{\kappa + i\Delta'_2},$$

and $\mathbf{F}'$ is defined by the nonzero elements $F_{13} = -iG_{s1}, F_{14} = -iG_{s2}, F_{23} = -iG_{s1}, F_{24} = -iG_{s2}, F_{31} = -iG_{s1}, F_{32} = -iG_{s2}, F_{41} = -iG_{s2},$ and $F_{42} = -iG_{s2}$.

Following the same procedure as that performed in Sec. III, we can also obtain the steady-state expression of the covariance matrix $\mathbf{V}'$, which is defined by $\mathbf{A}' \mathbf{V}' + \mathbf{V}' \mathbf{A}' = -\mathbf{Q}$, where $\mathbf{Q}$ is given by Eq. (18). Then the mean phonon numbers in the two mechanical resonators can be obtained.

To clearly analyze the influence of these couplings in the four-mode optomechanical system on the ground-state cooling, below we consider various cases of different coupling configurations, as shown in Fig. 6. To analyze the dark-mode effect in this system when the frequencies of the two mechanical modes are degenerate, we consider the case where the two coupling channels $g_1$ and $g_2$ always exist. Then we study various cases of coupling configurations by controlling the four coupling channels $J, g_{s1}, g_{s2},$ and $\eta$. In Figs. 6(a)–6(d) and Figs. 6(e)–6(j) we show that one or two coupling channels of $J, g_{s1}, g_{s2},$ and $\eta$ are closed, respectively. In Figs. 6(k)–6(n), three of the four coupling channels $J, g_{s1}, g_{s2},$ and $\eta$ are closed. Therefore, when the coupling channels $g_1$ and $g_2$ still exist, there are 14 cases of coupling configurations, as shown in Fig. 6.

Corresponding to the 14 cases depicted in Fig. 6, we plot the final mean phonon numbers $n'_1$ and $n'_2$ as functions of the scaled decay rate $\kappa/\omega_1$ in Fig. 7. We can see from Figs. 7(a) and 7(b) that ground-state cooling of the two mechanical modes can (cannot) be realized in the cases of $G_{s1} = 0$ or $G_{s2} = 0$ ($J = 0$ or $\eta = 0$), which implies that the dark mode can (cannot) be broken. In Figs. 7(c) and 7(d) we plot the phonon numbers $n'_1$ and $n'_2$ as functions of $\kappa/\omega_1$ when two of the four coupling channels ($J, g_{s1}, g_{s2},$ and $\eta$) are closed. Based on the cooling performance, we know that the dark mode cannot be broken when the coupling channels $J = \eta = 0$ or $G_{s1} = G_{s2} = 0$. In the four cases $J = G_{s1} = 0, J = G_{s2} = 0, \eta = G_{s1} = 0,$ and $\eta = G_{s2} = 0$, the dark mode can be broken. In Figs. 7(e) and 7(f) we also plot the phonon numbers $n'_1$ and $n'_2$ as functions of $\kappa/\omega_1$ when three of the four coupling channels are closed, corresponding to the cases shown in Figs. 6(k)–6(n). Here we can see that the dark mode cannot be broken in the cases of $J = G_{s1} = G_{s2} = 0$ or $\eta = G_{s1} = G_{s2} = 0$. However, in the two cases of $J = \eta = G_{s1} = 0$ or $J = \eta = G_{s2} = 0$, the dark mode can be broken.

In the study of quantum cooling of the mechanical modes, the linearized Langevin equations (22) can be written as a compact form $\mathbf{u}(t) = \mathbf{A}' \mathbf{u}(t) + \mathbf{N}(t)$, where the form of the fluctuation operator vector $\mathbf{u}(t)$ and the noise operator vector $\mathbf{N}(t)$ is the same as those defined in Sec. III, while the coefficient matrix is given by $\mathbf{A}' = \begin{pmatrix} \mathbf{F}' & \mathbf{F} \\ \mathbf{F}' & \mathbf{F} \end{pmatrix}$, where

\begin{equation}
\beta_1 \equiv \langle b_1 \rangle_{ss} = \frac{-i(\eta \beta_2 + g_1|\alpha|^2 + g_{s1}|\alpha|^2)}{\gamma_1 + i\omega_1},
\end{equation}

\begin{equation}
\beta_2 \equiv \langle b_2 \rangle_{ss} = \frac{-i(\eta \beta_1 + g_2|\alpha|^2 + g_{s2}|\alpha|^2)}{\gamma_2 + i\omega_2}.
\end{equation}

In this case, the ground-state cooling can be realized by choosing $G_{s2}/G_{s1} < 0.4$.
the four-mode optomechanical system. Concretely, the one-coupling-closed cases include the following: (a) The coupling channel $g_{s1} \eta$ is closed ($g_{s1} = 0$), (c) the coupling channel $g_{s2}$ is closed ($g_{s2} = 0$), and (d) the coupling channel $\eta$ is closed ($\eta = 0$). The two-coupling-closed cases include the following: (e) The coupling channels $J$ and $g_{s1}$ are closed ($J = g_{s1} = 0$), (f) the coupling channels $J$ and $g_{s2}$ are closed ($J = g_{s2} = 0$), (g) the coupling channels $J$ and $\eta$ are closed ($J = \eta = 0$), (h) the coupling channels $g_{s1}$ and $g_{s2}$ are closed ($g_{s1} = g_{s2} = 0$), (i) the coupling channels $g_{s1}$ and $\eta$ are closed ($g_{s1} = \eta = 0$), and (j) the coupling channels $g_{s2}$ and $\eta$ are closed ($g_{s2} = \eta = 0$). In addition, the cases in which three couplings are closed include the following: (k) The coupling channels $J$, $g_{s1}$, and $g_{s2}$ are closed ($J = g_{s1} = g_{s2} = 0$), (l) the coupling channels $J$, $g_{s1}$, and $\eta$ are closed ($J = g_{s1} = \eta = 0$), (m) the coupling channels $J$, $g_{s2}$, and $\eta$ are closed ($J = g_{s2} = \eta = 0$), and (n) the coupling channels $g_{s1}$, $g_{s2}$, and $\eta$ are closed ($g_{s1} = g_{s2} = \eta = 0$). We point out that the single-photon optomechanical-coupling strengths $g_{1}$, $g_{2}$, $g_{s1}$, and $g_{s2}$ are related to the linearized optomechanical-coupling strengths $G_{1}$, $G_{2}$, $G_{s1}$, and $G_{s2}$ in the four-mode optomechanical system.

B. Universal conditions for breaking the dark mode

In this section we analyze the parameter conditions under which the dark mode is formed in the network-coupled optomechanical system. We also study the method for breaking the dark-mode effect. Based on Eqs. (22), we can derive the approximate linearized Hamiltonian. By discarding the noise terms, the linearized optomechanical Hamiltonian under the rotating-wave approximation can be written as

$$
\tilde{H}_{\text{RWA}} = \Delta' \delta a'_1 \delta a + \Delta' \delta a'_2 \delta a + J (\delta a'_1 \delta a + \delta a'_2 \delta a) + \sum_{l=1,2} [\omega_l \delta b'_l \delta b_l + G_l (\delta a \delta b'_l + \delta a'_l \delta b_l)] + \eta (\delta b'_1 \delta b_2 + \delta b'_2 \delta b_1).
$$

(25)
where one coupling is closed, i.e., \( J = 0 \) (green dashed curves), and \( J = \eta = 0 \) (red circles), \( G_{a1} = 0 \) (cyan dash-dotted curves), and \( G_{a2} = 0 \) (blue solid curves), \( G_{a1} = G_{a2} = 0 \) (red circles), \( \eta = G_{a1} = 0 \) (green dashed curves), \( J = G_{a1} = 0 \) (cyan dash-dotted curves), \( J = G_{a2} = 0 \) (brown squares), and \( \eta = G_{a2} = 0 \) (purple triangles); and (e) and (f) three-coupling-closed cases, i.e., \( J = G_{a1} = G_{a2} = 0 \) (red solid curves), \( \eta = G_{a1} = G_{a2} = 0 \) (red dash-dot curves), and \( J = \eta = G_{a1} = G_{a2} = 0 \) (purple dashed curves). The other parameters are \( \omega_{s}/\omega_{1} = 1 \), \( \gamma_{1}/\omega_{1} = \gamma_{2}/\omega_{1} = 10^{-5} \), \( \kappa/\omega_{1} = \kappa_{s}/\omega_{1} = 0.1 \), \( J/\omega_{1} = \eta/\omega_{1} = 0.03 \), \( \Delta J/\omega_{1} = \Delta J_{1}/\omega_{1} = 1 \), \( G_{1}/\omega_{1} = G_{2}/\omega_{1} = 0.05 \), \( G_{1}/\omega_{1} = G_{2}/\omega_{1} = 0.08 \), and \( \bar{n}_{1} = \bar{n}_{2} = 1000 \). Note that the values of \( J, \eta, G_{a1}, \) and \( G_{a2} \) presented here work when the corresponding coupling channels are open.

By substituting the operators \( B_{+} \) (\( B_{+}^{\dagger} \)) and \( B_{-} \) (\( B_{-}^{\dagger} \)) into Eq. (25), the Hamiltonian \( \hat{H}_{RWA} \) becomes [see Fig. 5(b)]

\[
\hat{H}_{RWA} = \Delta J \delta a^{\dagger} \delta a + \Delta J_{1} \delta a_{s}^{\dagger} \delta a_{s} + J(\delta a^{\dagger} \delta a_{s} + \delta a_{s}^{\dagger} \delta a) + \tilde{\omega}_{s} B_{+}^{\dagger} B_{+} + \tilde{\omega}_{B} B_{-}^{\dagger} B_{-} + \tilde{\zeta}(B_{1}^{\dagger} B_{-} + B_{+}^{\dagger} B_{+}) + G_{s}(\delta a B_{+}^{\dagger} + B_{+}^{\dagger} \delta a_{s}) + \tilde{G}_{s+(}\delta a B_{+}^{\dagger} + \delta a_{s}^{\dagger} B_{+}) + \tilde{G}_{s-}(\delta a B_{-}^{\dagger} + \delta a_{s}^{\dagger} B_{-}),
\]

where the resonance frequencies of the two hybrid modes should be replaced by

\[
\tilde{\omega}_{+} = \frac{\omega_{1} G_{1}^{2} + \omega_{2} G_{2}^{2} + 2\eta G_{a1} G_{a2}}{G_{+}^{2}},
\]

\[
\tilde{\omega}_{-} = \frac{\omega_{1} G_{1}^{2} + \omega_{2} G_{2}^{2} - 2\eta G_{a1} G_{a2}}{G_{-}^{2}},
\]

and the three new coupling strengths \( \tilde{\zeta} \), \( \tilde{G}_{s+} \), and \( \tilde{G}_{s-} \) are defined by

\[
\tilde{\zeta} = \frac{(\omega_{1} - \omega_{2})G_{1}G_{2} + \eta(G_{2}^{2} - G_{1}^{2})}{G_{1}^{2} + G_{2}^{2}},
\]

\[
\tilde{G}_{s+} = \frac{G_{a1} G_{1} + G_{2} G_{a2}}{\sqrt{G_{1}^{2} + G_{2}^{2}}},
\]

\[
\tilde{G}_{s-} = \frac{G_{a2} G_{2} - G_{a1} G_{1}}{\sqrt{G_{1}^{2} + G_{2}^{2}}}.
\]

From Eqs. (26) and (28) we can see that the two hybrid modes \( B_{+} \) and \( B_{-} \) are decoupled from each other when \( \omega_{1} = \omega_{2} \) and \( \eta = 0 \) (\( G_{1} = G_{2} \)). Moreover, the hybrid mode \( B_{+} \) also decouples from the auxiliary cavity mode \( a_{s} \) when \( G_{s-} = 0 \). In this case, the hybrid mode \( B_{-} \) becomes a dark mode and the ground-state cooling of the two mechanical modes cannot be realized. All in all, the parameter conditions for the appearance of the dark mode are \( \tilde{\zeta} = 0 \) and \( \tilde{G}_{s-} = 0 \), which lead to the conditions

\[
(\omega_{1} - \omega_{2})G_{1}G_{2} + \eta(G_{2}^{2} - G_{1}^{2}) = 0,
\]

\[
G_{a1} G_{2} - G_{a2} G_{1} = 0.
\]
Therefore, the universal dark-mode-breaking condition is that both $\tilde{\zeta}$ and $G_{a,\omega}$ cannot be zero at the same time [see Fig. 5(b)]. In the following we analyze various cases in which the dark mode appears or disappears in the degenerate-resonator case ($\omega_1 = \omega_2$).

(i) In the case of $\eta = 0$, i.e., $\tilde{\zeta} = 0$, it can be seen from Eq. (28) that when $G_{a1}(G_2 - G_{a2}G_1 = 0 (G_1/G_2)G_1$), the hybrid mechanical mode $B_-$ (the dark mode) decouples from both the cavity mode $a$ and the auxiliary cavity mode $a_s$. In this situation, the excitation energy is stored in the dark mode and cannot be extracted through the cooling channel. In this case, the parameter $J$ has no effect on the breaking of the dark-mode effect. This analysis is consistent with the results obtained by numerical calculations in Sec. IV A.

(ii) In the case of $\eta \neq 0$ and $G_1 \neq G_2$, i.e., $\tilde{\zeta} \neq 0$, we can find that the dark mode can be broken regardless of whether the auxiliary cavity mode appears or not (namely, there is no dark mode). (iii) In the case of $\eta \neq 0$ and $G_1 = G_2$, i.e., $\tilde{\zeta} = 0$, there are two different situations. First, in the case of symmetric coupling ($G_{a1} = G_{a2}$), we can see from Eq. (28) that $\tilde{G}_{a,\omega} = 0$. At this time, the hybrid mechanical mode $B_-$ decouples from both the cavity mode $a$ and the auxiliary cavity mode $a_s$, i.e., $B_-$ becomes a dark mode. However, in the case of asymmetric coupling ($G_{a1} \neq G_{a2}$), i.e., $\tilde{G}_{a,\omega} \neq 0$, we can see that the two hybrid mechanical modes $B_-$ and $B_s$ are coupled with the auxiliary cavity mode $a_s$. Even if the hybrid mode $B_-$ is decoupled from both the cavity mode $a$ and the hybrid mode $B_s$ at the same time, the ground-state cooling of the two mechanical resonators becomes accessible through the cooling channel associated with auxiliary cavity mode $a_s$. Obviously, when one of the two coupling strengths $G_{a1}$ and $G_{a2}$ is 0, the dark-mode effect can naturally be broken. This analysis is consistent with the results we obtained in Sec. IV A. Generally speaking, to break the dark mode formed in a three-mode optomechanical system consisting of a cavity mode and two mechanical modes, the easiest way is to introduce an auxiliary cavity mode to couple with one of the two mechanical modes.

C. Analyzing the quantum interference effect in the energy-level transitions of the optomechanical system

To clarify the physical mechanism behind the dark-mode breaking, in this section we analyze the quantum interference effect in the energy-level transitions of the system. For the optomechanical system, the cavity modes provide the cooling channels to extract the thermal excitations from the mechanical resonators. However, when a single cavity mode is used to cool multiple degenerate mechanical resonators, the phonon modes decaying through the same cooling channel will interfere with each other, similar to the quantum interference effect in electromagnetic induced transparency [75,76]. In this case, some of the mechanical normal modes are decoupled from the cavity mode and then the cooling channel of these decoupled modes (dark modes) is closed and the excitations stored in these mechanical dark modes cannot be extracted. As a result, these dark modes cannot be cooled to their ground states. When multiple cooling channels exist, the phonon dissipation prohibited by one cooling channel can take place via another cooling channel; then the dark-mode effect is broken and the ground-state cooling of multiple mechanical resonators can be realized.

To further understand the quantum interference effect in the energy-level transitions of the optomechanical system, in Figs. 9(a) and 9(b) we plot the energy levels and transitions of the system including two mechanical modes in either the absence or presence of the auxiliary cavity mode $a_s$. Here the states $|m, j, k\rangle$ and $|m, n, j, k\rangle$ correspond to the states $|m, j, k, a, b_1, b_2\rangle$ and $|m, n, j, k, a, b_1, b_2\rangle$, respectively. The states $|m, n, 0, j, k\rangle$, $|n, 0, j, k\rangle$, and $|k, 0\rangle$ denote the number states of the cavity mode $a$, auxiliary cavity mode $a_s$, mechanical mode $b_1$, and mechanical mode $b_2$, respectively.

FIG. 9. Part of the energy levels and transitions of the system in either the (a) absence or (b) presence of the auxiliary cavity mode $a_s$. Here the states $|m, j, k\rangle$ and $|m, n, j, k\rangle$ correspond to the states $|m, j, k, a, b_1, b_2\rangle$ and $|m, n, j, k, a, b_1, b_2\rangle$, respectively. The states $|m, n, 0, j, k\rangle$, $|n, 0, j, k\rangle$, and $|k, 0\rangle$ denote the number states of the cavity mode $a$, auxiliary cavity mode $a_s$, mechanical mode $b_1$, and mechanical mode $b_2$, respectively.

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between two neighboring mechanical modes and the optomechanical dark modes. Introduce the phonon-hopping coupling between all the neighboring mechanical modes and the optomechanical interaction between the auxiliary cavity mode and the mechanical mode \( b_i \) are introduced into the \( N \)-mechanical-mode optomechanical system.

Driving is needed. In addition, though the dark mode appears theoretically in the degenerate mechanical-resonator case, our scheme works well even for near-degenerate resonators within a detuning window with the width of the cavity-field decay rate. Physically, the resonators with these detuned frequencies cannot be distinguished via the cavity emission spectrum.

V. GROUND-STATE COOLING AND DARK-MODE BREAKING IN A MULTIPLE-MECHANICAL-MODE OPTOMECHANICAL SYSTEM

In this section we generalize the auxiliary-cavity-mode method to realize the simultaneous ground-state cooling of \( N \) mechanical modes in a multiple-mechanical-mode optomechanical system, in which an intermediate cavity mode is coupled to \( N \) (\( N \geq 3 \)) mechanical modes [see Fig. 10(a)]. Concretely, we introduce an auxiliary cavity mode optomechanically coupled to the first mechanical mode. We also introduce the phonon-hopping coupling between all the neighboring two mechanical modes [see Fig. 10(b)]. Moreover, we analyze the parameter conditions for forming and breaking the dark modes.

A. Simultaneous ground-state cooling of \( N \) mechanical modes

We consider a multiple-mechanical-mode optomechanical system, which is composed of an intermediate cavity mode, an auxiliary cavity mode, and \( N \) (\( N \geq 3 \)) mechanical modes [see Fig. 10(b)]. In a rotating frame defined by the transformation operator \( \exp[-i(\omega_0a^\dagger a + \omega_0a^\dagger a^\dagger)] \), the Hamiltonian of this system is written as

\[
H_i = \Delta_i a^\dagger a + \sum_{l=1}^{N} \{\omega_i b^\dagger_l b_l + g_i a^\dagger a b^\dagger_l + b_l + b^\dagger_l\} \\
+ \sum_{l=1}^{N-1} \eta_i (b^\dagger_l b_{l+1} + b^\dagger_{l+1} b_l) + (\Omega a^\dagger + \text{H.c.}) \\
+ \Delta a^\dagger a + g a^\dagger a (b^\dagger_1 + b_1) + (\Omega a^\dagger + \text{H.c.}), \tag{30}
\]

where the operators and variables for the cavity modes have been defined before and \( b_i \) (\( b^\dagger_i \)) are the annihilation (creation) operators of the \( l \)-th mechanical mode. The parameter \( g_i \) (\( g_i \)) denotes the single-photon optomechanical-coupling strength between the intermediate (auxiliary) cavity mode and the \( l \)-th mechanical mode (mechanical mode \( b_i \)). The cavity mode \( a \) (\( a^\dagger \)) is strongly driven by the laser field with the driving frequency \( \omega_l \) and \( \omega_d \) and the driving amplitude \( \Omega \) (\( \Omega^2 \)).

Similar to the two-mechanical-mode case, the two cavity fields are strongly driven by two laser fields; then the dynamics of this system can be treated by the linearization method. Based on the Hamiltonian (30), the linearized Langevin equations for the quantum fluctuations are given by

\[
\delta \dot{a} = -i\delta A \delta a - i\{g_a^\dagger (b_1 + b^\dagger_1) \}
+ g_a (b_2 + b^\dagger_2) + \cdots + ig_N (b_N + b^\dagger_N)\]
+ \( \sqrt{2\kappa a_{\text{in}}^\dagger} \),

\[
\delta \dot{a}_s = (-\delta \kappa \omega) \delta a_s - i\{g_a^\dagger (b_1 + b^\dagger_1) + \sqrt{2\kappa a_{\text{in}}^\dagger} \},
\delta \dot{b}_1 = (\gamma_1 + \omega_0) b_1 - ig_a^\dagger \delta a - ig_a \delta a^\dagger - \sqrt{2\gamma_1 b_{1,\text{in}}^\dagger} \delta a,
- ig_a \delta a^\dagger - \eta_1 b_2 + \sqrt{2\gamma_1 b_{2,\text{in}}^\dagger} \delta a,
\delta \dot{b}_2 = (\gamma_2 + \omega_0) b_2 - ig_a^\dagger \delta a + ig_a \delta a^\dagger - \eta_1 b_1 - \eta_2 b_3 + \sqrt{2\gamma_2 b_{3,\text{in}}^\dagger} \delta a,
\vdots
\delta \dot{b}_{N-1} = (\gamma_{N-1} + \omega_0) b_{N-1} - ig_{N-1}^\dagger \delta a - ig_{N-1} \delta a^\dagger - \eta_{N-2} b_{N-2} + \sqrt{2\gamma_{N-1} b_{N-2,\text{in}}^\dagger} \delta a,
\delta \dot{b}_N = (\gamma_N + \omega_0) b_N - ig_{N}^\dagger \delta a + ig_{N} \delta a^\dagger - \eta_{N-1} b_{N-1} + \sqrt{2\gamma_{N} b_{N-1,\text{in}}^\dagger} \delta a, \tag{31}
\]

where \( \Delta_i = \Delta + \sum_{l=1}^{N} g_l (\beta_l + \beta^\dagger_l) \) [\( \Delta'_i = \Delta_s + g_i (\beta_l + \beta^\dagger_l) \)] is the effective driving detuning of the cavity mode \( a \) (\( a^\dagger \)) and \( G_l = g_l |\alpha| \) (\( G_l = g_l |\alpha| \)) is the linearized optomechanical-coupling strength between the cavity mode \( a \) (\( a^\dagger \)) and the \( l \)-th mechanical mode (mechanical mode \( b_l \)).

Below we study the ground-state cooling of \( N \) mechanical modes. The cooling performance of mechanical modes can be verified by calculating the final mean phonon numbers. Therefore, we rewrite the linearized Langevin equations (31) as

\[
\hat{\mathbf{u}}(t) = \hat{A} \hat{\mathbf{u}}(t) + \hat{\mathbf{N}}(t), \tag{32}
\]

where we introduce the vector of the quantum fluctuations \( \hat{\mathbf{u}}(t) = [\delta a, \delta a_s, \delta b_1, \ldots, \delta b_{N-1}, \delta b_N, \delta a_1, \delta a_2, \ldots, \delta b_{N-1}, \delta b_N] \) and the vector of the quantum noise \( \hat{\mathbf{N}}(t) = \sqrt{2/\kappa a_{\text{in}}^\dagger} \sqrt{\kappa a_{\text{in}}^\dagger}, \sqrt{\gamma_1 b_{1,\text{in}}^\dagger} \cdots, \sqrt{2\gamma_{N-1} b_{N-1,\text{in}}^\dagger} \), \( \sqrt{\gamma_{N} b_{N,\text{in}}^\dagger} \). The corresponding coefficient matrix \( \hat{A} = \left( \begin{array}{c|c} \hat{F} & \hat{E}^\dagger \\ \hline \hat{E} & -\hat{F} \end{array} \right) \).
where

\[
\mathbf{E} = \begin{pmatrix}
\kappa + i\Delta_r & 0 & iG_1 & \cdots & iG_{N-1} & iG_N \\
0 & \kappa_s + i\Delta_s' & 0 & \cdots & 0 & 0 \\
iG_1^* & iG_s^* & \eta_1 + i\omega_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
iG_{N-1}^* & 0 & 0 & \cdots & \gamma_{N-1} + i\omega_{N-1} & i\eta_{N-1} \\
iG_N^* & 0 & 0 & \cdots & i\eta_{N-1} & \gamma_N + i\omega_N
\end{pmatrix}
\]  

(33)

and

\[
\mathbf{F} = \begin{pmatrix}
0 & 0 & -iG_1 & \cdots & -iG_{N-1} & -iG_N \\
0 & 0 & -iG_s & 0 & \cdots & 0 \\
-iG_1 & -iG_s & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-iG_{N-1} & 0 & 0 & \cdots & 0 & 0 \\
-iG_N & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]  

(34)

The formal solution of Eq. (32) is given by

\[
\mathbf{u}(t) = \mathbf{M}(t)\mathbf{u}(0) + \int_0^t \mathbf{M}(t-s)\mathbf{N}(s)ds,
\]

where the matrix \(\mathbf{M}(t) = \exp(\mathbf{A}t)\). Note that our simulations should satisfy the stability conditions, which can be obtained by analyzing the Routh-Hurwitz criterion \[73\].

Based on Eq. (35), we can obtain the final mean phonon numbers by solving the Lyapunov equation. For this reason, we introduce the covariance matrix \(\mathbf{V}\) of the system by defining the matrix elements as

\[
\mathbf{V}_{ij} = \frac{1}{2}[\mathbf{u}_i(\infty)\mathbf{u}_j(\infty)] + \langle \mathbf{u}_i(\infty)\mathbf{u}_j(\infty)\rangle.
\]

(36)

Under the stability conditions, the covariance matrix \(\mathbf{V}\) is determined by the Lyapunov equation

\[
\mathbf{AV} + \mathbf{VA}^T = -\mathbf{Q},
\]

where

\[
\mathbf{Q} = \frac{1}{2}(\mathbf{C} + \mathbf{C}^T),
\]

(38)

with the correlation matrix \(\mathbf{C}\) related to the noise operators. In the Markovian-dissipation case, the correlation matrix \(\mathbf{C}\) can be obtained as \(\mathbf{C} = 2\mathbf{R} \mathbf{P} \mathbf{R}^T\), where

\[
\mathbf{P} = \begin{pmatrix}
\kappa & 0 & 0 & \cdots & 0 & 0 \\
0 & \kappa_s & 0 & \cdots & 0 & 0 \\
0 & 0 & \gamma_1(\bar{n}_1 + 1) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \gamma_{N-1}(\bar{n}_{N-1} + 1) & 0 \\
0 & 0 & 0 & \cdots & \gamma_N(\bar{n}_N + 1)
\end{pmatrix}
\]  

(39)

and

\[
\mathbf{R} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \gamma_1(\bar{n}_1) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \gamma_{N-1}(\bar{n}_{N-1}) & 0 \\
0 & 0 & 0 & \cdots & \gamma_N(\bar{n}_N)
\end{pmatrix}
\]  

(40)

According to the covariance matrix \(\mathbf{V}\) defined in Eq. (36), we can derive the final mean phonon numbers of the \(l\)th mechanical mode as

\[
\langle \delta b_l^\dagger \delta b_l \rangle = \mathbf{V}_{N+l,4,4} + \frac{1}{2},
\]

(41)
and (c) and (d) of the scaled driving detuning $\tilde{\Delta}_c/\omega_1$ in the dark-mode-unbreaking ($G_s/\omega_1 = 0$ and $\eta_l/\omega_1 = 0.06$) cases for (a) $N = 3$ and (b) $N = 4$. (c) and (d) Final mean phonon numbers $N_l^f$ as functions of the scaled cavity-field decay rate $\kappa/\omega_1$ for (c) $N = 3$ and (d) $N = 4$. The other parameters are $\omega_0/\omega_1 = 1$, $\Delta_i^c/\omega_1 = 1$, $\kappa_l/\omega_1 = 0.1$, $\eta_l/\omega_1 = 0.06$, $\gamma/\omega_1 = 10^{-5}$, $G_s/\omega_1 = 0.05$, $G_i/\omega_1 = 0.1$, $h_i = 1000$, and (a) and (b) $\kappa/\omega_1 = 0.1$ and (c) and (d) $\Delta_c/\omega_1 = 1$.

$(G_s/\omega_1 = 0.1$ and $\eta_l/\omega_1 = 0.06$) and the dark-mode-unbreaking case $(G_s = \eta_l = 0)$. We can see that the ground-state cooling cannot be realized for these mechanical modes when the auxiliary cavity mode and phonon-hopping interactions are absent $(G_s = \eta_l = 0)$ [see the upper curves in Figs. 11(a) and 11(b)]. This is because the thermal excitations stored in the dark modes cannot be extracted through the cooling channel related to cavity mode $a$.

When the auxiliary cavity mode and the phonon-hopping interactions are introduced, a new cooling channel is formed and the dark modes are broken. Then the ground-state cooling of multiple mechanical modes can be realized, as shown in Figs. 11(a) and 11(b). Moreover, in Figs. 11(c) and 11(d) we plot the final mean phonon numbers $N_l^f$ as functions of the cavity-field decay rate $\kappa/\omega_1$. We find that the cooling performance of the first mechanical mode is the best, because it is directly connected to the auxiliary cavity mode. The cooling performance of other mechanical modes is almost the same, because all the other mechanical modes have similar coupling connections with the cooling baths.

### B. Parameter conditions for breaking dark modes

To study the parameter conditions for breaking the dark modes, we can derive the approximate linearized Hamiltonian, which governs the dynamics of the system. To implement the cooling scheme, the system should work in the red-sideband resonance regime, in which the rotating-wave approximation can be safely performed. By discarding the noise terms, the linearized optomechanical Hamiltonian is given by

$$H_l = \tilde{\Delta}_c \delta a^\dagger \delta a + \sum_{l=1}^N \omega_l \delta b_l^\dagger \delta b_l + \sum_{l=1}^N G_l (\delta a^\dagger \delta b_l + \delta b_l^\dagger \delta a)$$

$$+ \Delta_l^c \delta a^\dagger \delta a_s + \sum_{l=1}^{N-1} [\eta_l (\delta b_l^\dagger \delta b_{l+1}^\dagger + \delta b_{l+1} \delta b_l)]$$

$$+ G_s (\delta a^\dagger \delta b_1 + \delta b_1^\dagger \delta a_s).$$

To clearly see the dark-mode effect in the multiple-mechanical-mode optomechanical system, we first consider the situation where the auxiliary cavity and the phonon-hopping interactions between two neighboring mechanical modes are absent, i.e., $\Delta_l^c = 0$, $\eta_l = 0$, and $G_s = 0$ [see Fig. 10(a)]. Then the Hamiltonian (42) becomes

$$H_l' = \tilde{\Delta}_c \delta a^\dagger \delta a + \sum_{l=1}^N \omega_l \delta b_l^\dagger \delta b_l + \sum_{l=1}^N G_l (\delta a^\dagger \delta b_l + \delta b_l^\dagger \delta a).$$

For simplicity, we consider that all the mechanical modes have the same resonance frequencies $(\omega_0 = \omega_m)$, and the optomechanical-coupling strengths between the intermediate cavity mode and all mechanical modes are also the same $(G_l = G)$. In this case, there are a bright mode $B_+ = \sum_{l=1}^N \delta b_l / \sqrt{N}$ and $(N-1)$ dark modes decoupled from the intermediate cavity mode. Therefore, the thermal excitations stored in the dark modes cannot be extracted through the cooling channel of the cavity mode and then the ground-state cooling of these mechanical modes cannot be realized.

To break the dark mode and achieve the ground-state cooling of $N$ mechanical modes, we introduce an auxiliary cavity mode optomechanically coupled to the mechanical mode $b_1$ and phonon-hopping interactions between the neighboring mechanical modes, as shown in Fig. 10(b). For convenience, we consider the case where all the phonon-hopping coupling strengths are the same $(\eta_l = \eta)$. Thus, the Hamiltonian associated with these coupled mechanical modes can be diagonalized as

$$H_{\text{mph}} = \omega_m \sum_{l=1}^N \delta b_l^\dagger \delta b_l + \sum_{l=1}^{N-1} (\delta b_l^\dagger \delta b_{l+1}^\dagger + \delta b_{l+1} \delta b_l^\dagger)$$

$$+ \sum_{k=1}^N \Omega_k B_k^\dagger B_k,$$

where $\Omega_k$ is the resonance frequency of the $k$th hybrid mechanical mode $B_k$ and is defined by

$$\Omega_k = \omega_m + 2\eta \cos \left( \frac{k\pi}{N+1} \right), \quad k = 1, 2, 3, \ldots, N.$$  

Meanwhile, the relationship between the hybrid mode $B_k$ and the mechanical mode $\delta b_l$ can be expressed as

$$\delta b_l = \frac{1}{D} \sum_{k=1}^N \sin \left( \frac{l k \pi}{N+1} \right) B_k,$$

with $D = \sqrt{(N+1)/2}$. When the auxiliary cavity mode is absent, by substituting the hybrid mode $B_k$ into the Hamiltonian
\[
H_I = \tilde{\Delta}_e \delta a^\dagger \delta a + \sum_{k=1}^{N} \Omega_k B_k^\dagger B_k + H_{\text{OM}}.
\]
(47)

where the optomechanical-interaction Hamiltonian \( H_{\text{OM}} \) is given by
\[
H_{\text{OM}} = \sum_{k=1}^{N} \left[ \frac{G}{D} \sum_{l=1}^{N} \sin \left( \frac{lk\pi}{N+1} \right) \delta a B_k^\dagger + \text{H.c.} \right].
\]
(48)

From Eq. (48) we can see that the coupling strength between the cavity mode \( a \) and the hybrid mode \( B_k \) is determined by the coefficient \( \frac{G}{D} \sum_{l=1}^{N} \sin \left( \frac{lk\pi}{N+1} \right) \). Hence, we next analyze the dependence of this coefficient on the variables \( k \) and \( N \).

First, we consider the situation of \( N = 2 \). In this case, the system is reduced to the two-mechanical-mode optomechanical system, which has been analyzed in detail in previous sections. When \( N = 2 \), the optomechanical interaction becomes
\[
H_{\text{OM}} = \sqrt{2}G\delta a B_1^\dagger + \sqrt{2}G^* B_1 \delta a.
\]
(49)

It is obvious that the hybrid mechanical mode \( B_2 \) becomes a dark mode, which decouples from both the cavity mode \( a \) and the hybrid mechanical mode \( B_1 \), so the ground-state cooling of the two mechanical modes becomes inaccessible.

In the case of \( N \geq 3 \), the coupling coefficient between the cavity mode \( a \) and the \( k \)-th hybrid mechanical mode \( B_k \) defined in Eq. (48) is given by \( \frac{G}{D} \sum_{l=1}^{N} \sin \left( \frac{lk\pi}{N+1} \right) \). Since the forms of the coupling coefficients are different when \( N \) is either an odd number or an even number, below we will analyze two cases corresponding to odd and even \( N \), respectively. (i) When \( N \) is an odd number, the form of the coupling coefficient depends on \( k \). If \( k \) is also an odd number, we get \( \frac{G}{D} \sum_{l=1}^{N} \sin \left( \frac{l\pi}{2N+1} \right) \). If \( k \) is even, we obtain \( \frac{G}{D} \sum_{l=1}^{N} \sin \left( \frac{(k+1)\pi}{2N+1} \right) = 0 \). For an even \( N \), when \( k \) is an odd number, we obtain \( \frac{G}{D} \sum_{l=1}^{N} \sin \left( \frac{lk\pi}{2N+1} \right) \). When \( k \) is also an even number, we get \( \frac{G}{D} \sum_{l=1}^{N} \sin \left( \frac{lk\pi}{2N+1} \right) = 0 \).

Based on the above discussion, we can find that for an odd \( k \), the coupling strength between the intermediate cavity mode \( a \) and the \( k \)-th hybrid mechanical mode \( B_k \) is nonzero. However, when \( k \) is an even number, the intermediate cavity mode \( a \) is decoupled from the \( k \)-th hybrid mechanical mode \( B_k \). In this situation, all the even hybrid mechanical modes are decoupled from the intermediate cavity mode. Thus, the ground-state cooling of multiple mechanical modes cannot be realized under the influence of the dark-mode effect. Nevertheless, we can introduce an auxiliary cavity mode \( a \), to break the dark-mode effect, which is coupled to the mechanical mode \( b_1 \) via the radiation-pressure interaction. By substituting the hybrid mechanical modes \( B_k \) into the optomechanical Hamiltonian \( H_{\text{OM}} = G_{\text{s}}(\delta a^\dagger \delta b_1 + \delta b_1^\dagger \delta a) \), we can get
\[
H_{\text{OM}} = G_{\text{s}} \sum_{k=1}^{N} \left[ \sin \left( \frac{k\pi}{N+1} \right) \delta a^\dagger B_k + \text{H.c.} \right].
\]
(50)

It can be seen from Eq. (50) that all the hybrid mechanical modes \( B_k \) are coupled with the auxiliary cavity mode \( a \). Even when the even hybrid mechanical modes are decoupled from the intermediate cavity mode \( a \), the ground-state cooling of the \( N \) mechanical modes can still be achieved through the cooling channel associated with the auxiliary cavity mode \( a \).

**VI. PHYSICAL MECHANISM FOR BREAKING THE DARK-STATE EFFECT IN THE N-TYPE FOUR-LEVEL ATOMIC SYSTEM**

To further investigate the generality of the physical mechanism for breaking the dark-mode effect, in this section we consider the dark-state effect in an atomic-level system. Concretely, we demonstrate the dark-state effect in a \( \Lambda \)-type three-level system and show how to break this dark-state effect by introducing an auxiliary state coupled to one of the two lower states, namely, forming an \( N \)-type four-level system (Fig. 12). For the three-level system, the Hamiltonian reads
\[
H_{\text{TLS}}(t) = E_e|e\rangle\langle e| + E_f|f\rangle\langle f| + E_d|g\rangle\langle g| + (\Omega_1|e\rangle\langle g| e^{-i\Omega_1 t} + \Omega_2|e\rangle\langle f| f e^{-i\Omega_2 t} + \text{H.c.}),
\]
(51)

where \( E_e \), \( E_f \), and \( E_g \) denote the energies of energy levels \( |e\rangle \), \( |f\rangle \), and \( |g\rangle \), respectively. For convenience, hereafter we assume that the energy of the ground state is 0 \( (E_g = 0) \). The two atomic transitions \( |g\rangle \leftrightarrow |e\rangle \) and \( |f\rangle \leftrightarrow |e\rangle \) are coupled to two monochromatic fields with the frequencies \( \omega_1 \) and \( \omega_2 \) and the transition amplitudes \( \Omega_1 \) and \( \Omega_2 \), respectively. In a rotating frame with respect to \( \omega_1|e\rangle\langle e| + \omega_2|f\rangle\langle f| + E_g|g\rangle\langle g| \), the Hamiltonian in Eq. (51) becomes
\[
V_I = \Delta_1|e\rangle\langle e| + \Omega_1(|e\rangle\langle g| |g\rangle + |e\rangle) + \Omega_2(|e\rangle\langle f| |f\rangle + |e\rangle),
\]
(52)

where \( \Delta_1 = E_e - \omega_1 \) and \( \Delta_2 = E_e - E_f - \omega_2 \) are the transition detunings.

To better study the eigenvalues and eigenstates of the Hamiltonian (52), we introduce three basis states defined by
the vectors
\[ |e\rangle = (1 \ 0 \ 0)^T, \quad |f\rangle = (0 \ 1 \ 0)^T, \]
\[ |g\rangle = (0 \ 0 \ 1)^T. \]  
(53)
To demonstrate the dark-state effect in this three-level system, we consider the single- and two-photon resonance cases, i.e., \( \Delta_1 = \Delta_2 = 0 \). Then the Hamiltonian \( V_I \) can be written as
\[ V_I = \Omega_2 \begin{pmatrix} 0 & 1 & \xi \\ 1 & 0 & 0 \\ \xi & 0 & 0 \end{pmatrix}, \]  
(54)
where \( \xi = \Omega_1/\Omega_2 \) is the amplitude ratio.

The dark-state effect of this three-level system can be analyzed by calculating the eigensystem of the matrix \( V_I \) given in Eq. (54). The eigenequation of \( V_I \) reads
\[ V_I |\lambda_s\rangle = \lambda_s |\lambda_s\rangle, \quad s = 1, 2, 3. \]  
(55)
The eigenvalues are given by \( \lambda_1 = 0, \lambda_2 = -\Omega_2 \sqrt{1 + \xi^2} \), and \( \lambda_3 = \Omega_2 \sqrt{1 + \xi^2} \). The corresponding eigenstates can be obtained as
\[ |\lambda_1\rangle = \frac{1}{\sqrt{1 + \xi^2}} (|e\rangle - \xi |f\rangle + |g\rangle), \]
\[ |\lambda_2\rangle = \frac{1}{\sqrt{2(1 + \xi^2)}} (-|e\rangle - \sqrt{1 + \xi^2} |f\rangle + |g\rangle + \xi |g\rangle), \]
\[ |\lambda_3\rangle = \frac{1}{\sqrt{2(1 + \xi^2)}} (|e\rangle + \sqrt{1 + \xi^2} |f\rangle + |g\rangle + \xi |g\rangle). \]  
(56)
To study the dark-state effect in this system, we can calculate the probability of the excited state \( |e\rangle \) in these eigenstates,
\[ p^e_\varepsilon = |\langle e |\lambda_s \rangle|^2. \]  
(57)
By combining Eqs. (56) and (57), we find that the probability of the eigenstate \( |\lambda_1\rangle \) is always zero, which means that the eigenstate \( |\lambda_1\rangle \) is the dark state.

To break the dark-state effect in the \( \Lambda \)-type three-level system, we introduce an auxiliary state coupled to the lower states \( |f\rangle \), forming an \( N \)-type four-level system. The Hamiltonian of this system can be expressed as
\[ H_{\text{TLS}}(t) = H_{\text{TLS}}(t) + E_d |d\rangle \langle d| + \Omega_3 (|f\rangle \langle e| + \Omega_2 |e\rangle \langle f| + \Omega_2 |e\rangle \langle g| + |g\rangle \langle e|) |d\rangle \langle d| + H.c., \]  
(58)
where \( \omega_3 \) is the frequency of the monochromatic field and \( |d\rangle \) is the auxiliary state coupled to the lower state \( |f\rangle \) with the coupling strength \( \Omega_3 \). In a rotating frame with respect to \( \omega_3 \), the Hamiltonian \( H_{\text{TLS}} \) becomes
\[ V_I^\prime = \Delta_1 |e\rangle \langle e| + \Delta_3 |d\rangle \langle d| + \Omega_3 (|e\rangle \langle g| + |g\rangle \langle e|) + \Omega_2 (|f\rangle \langle f| + |f\rangle \langle e| + \Omega_3 (|d\rangle \langle f| + |f\rangle \langle d|)), \]  
(59)
where \( \Delta_3 = E_d - E_f - \omega_3 \) is the transition detuning.

By defining the four basis states with the vectors
\[ |e\rangle = (1 \ 0 \ 0 \ 0)^T, \quad |f\rangle = (0 \ 1 \ 0 \ 0)^T, \]
\[ |g\rangle = (0 \ 0 \ 1 \ 0)^T, \quad |d\rangle = (0 \ 0 \ 0 \ 1)^T, \]  
(60)
the Hamiltonian \( V_I^\prime \) can be expressed as
\[ V_I^\prime = \Omega_2 \begin{pmatrix} 0 & 1 & \xi & 0 \\ 1 & 0 & 0 & \xi \\ \xi & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 \end{pmatrix}. \]  
(61)
Here we introduce the parameter \( \Omega_1 = \Omega_2 = \Omega' \) and \( \Omega_3/\Omega = \xi^2 \) and consider the case \( \Delta_1 = \Delta_3 = 0 \). Following the standard procedure as performed in the three-level system, we can obtain the eigenvalues of the matrix \( V_I^\prime \) as
\[ \lambda'_{1,2,3,4} = \pm \Omega' \sqrt{2 + \xi^2 \pm \sqrt{(2 + \xi^2)^2 - 4 \xi^2}}. \]  
(62)
The corresponding eigenstates are given by
\[ |\lambda'_s\rangle = |\delta [\lambda'_s (\lambda'_s^2 - \xi^2)|e\rangle + \lambda'_s (\lambda'_s^2 - \xi^2)|g\rangle + \lambda'_s \xi |d\rangle |, \]  
(63)
where \( |\delta |^2 = 1/(\lambda'_s^2 + 1) \lambda'_s^2 - \xi^2)^2 + \lambda'_s^2 (\lambda'_s^2 - \xi^2) \) is the normalization constant. To check the dark-state effect in this system, we also calculate the probability of these eigenstates,
\[ p^e_\varepsilon = |\langle e |\lambda'_s \rangle|^2, \]  
(64)
where \( p_e \) is the probability of the excited state \( |e\rangle \) in the \( N \)-type four-level system.

In Fig. 13(a) we plot the probability \( p^e_\varepsilon \) of these three eigenstates \( |\lambda_s \rangle \) \((s = 1, 2, 3, \) and \( 4)\) as a function of the amplitude ratio \( \Omega_1/\Omega_2 \) in the three-level system. We can find that the probability \( p^e_1 \) is always zero no matter how the ratio changes, which means that the eigenstate \( |\lambda_1\rangle \) is the dark state. In Fig. 13(b) we plot the probability \( p^e_\varepsilon \) of these four eigenstates \( |\lambda'_s \rangle \) as a function of the amplitude ratio \( \Omega_3/\Omega_2 \) in the \( N \)-type four-level system. Here we can see that the probabilities \( p^e_1 \) and \( p^e_3 \) are zero when the amplitude ratio \( \Omega_3/\Omega_2 = 0 \), which means that the two eigenstates \( |\lambda'_1 \rangle \) and \( |\lambda'_3 \rangle \) are dark states when the transition channel between the auxiliary state \( |d\rangle \) and the lower state \( |f\rangle \) is closed. However, with the increase of the ratio \( \Omega_3/\Omega_2 \), the excited-state probability of the four eigenstates is nonzero, which means that the dark-state effect is broken when the auxiliary state is introduced to the system.
VII. DISCUSSION OF EXPERIMENT IMPLEMENTATION

In this section we present a discussion of the experimental implementation of this scheme. This system only involves the linearized optomechanical couplings and photon-phonon-hopping coupling, which are experimentally accessible in current optomechanical systems [2]. In the simulations, we consider the model in the resolved-sideband regime \( \omega_{l} = \omega_{c} \approx 2 \pi \times 4.2 \text{ GHz} \). The two mechanical modes are two drum resonators that function as compliant modes coupled to two mechanical modes, while the other microwave cavity mode is only coupled to one of the two mechanical modes. Here the mechanical modes of two drums have the resonance frequency \( \omega_{l} = \omega_{d} \approx 2 \pi \times 10 \times 10 \text{ MHz} \). The intrinsic energy decay rates of two microwave cavities and two drum resonators are \( \kappa = \kappa_{l} \approx 2 \pi \times 1 \text{ MHz} \) and \( \gamma_{l} = \gamma_{d} \approx 2 \pi \times 100 \text{ Hz} \), respectively. The effective coupling rates between the intermediate-coupling microwave cavity and the two drum resonators are \( G_{1l} = G_{2l} \approx 2 \pi \times 0.5 \text{ MHz} \) and the effective coupling rates between the auxiliary microwave cavity and the two drum resonators are \( G_{1d} = G_{2d} \approx 2 \pi \times 0.8 \text{ MHz} \). In Table I we present the reported experimental parameters in the circuit electromechanical systems [11] and the scaled parameters used in our simulations. By comparing these scaled parameters and the experimental parameters, we find that the experimental implementation of this scheme should be within the reach of current or near-future experimental conditions.

VIII. CONCLUSION

We have proposed an auxiliary-cavity-mode method to realize simultaneous ground-state cooling of two degenerate or nearly degenerate mechanical modes. We have also studied the general physical coupling configuration for breaking the dark mode in the network-coupled four-mode optomechanical system. The analytical parameter conditions for breaking the dark-mode effect have been found. Moreover, we have generalized this method to realize ground-state cooling of multiple mechanical modes, which was achieved by introducing the auxiliary cavity mode and phonon-hopping couplings between nearest-neighbor mechanical modes. We also described the physical mechanism for breaking the dark-state effect in the \( N \)-type four-level atomic system. Our results pave a way toward the demonstration of macroscopic quantum coherence and quantum manipulation in multiple-mechanical-mode optomechanical systems.

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