n-photon blockade with an n-photon parametric drive

Yan Hui Zhou,[1] Fabrizio Minganti,[2,3] Wei Qin,[2] Qi-Cheng Wu,[1] Jun-Long Zhao,[1] Yu-Liang Fang,[1]
Franco Nori,[2,4,5,*] and Chui-Ping Yang[1,6,†]
[1]Quantum Information Research Center, Shangrao Normal University, Shangrao 334001, China
[2]Theoretical Quantum Physics Laboratory, RIKEN Cluster for Pioneering Research, Wako-shi, Saitama 351-0198, Japan
[3]Institute of Physics, Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland
[4]RIKEN Center for Quantum Computing, Wako-shi, Saitama 351-0198, Japan
[5]Physics Department, The University of Michigan, Ann Arbor, Michigan 48109, USA
[6]Department of Physics, Hangzhou Normal University, Hangzhou 311121, China

(Received 16 December 2020; revised 26 September 2021; accepted 8 November 2021; published 22 November 2021)

We propose a mechanism to engineer an n-photon blockade in a nonlinear cavity with an *n-photon parametric drive* $\lambda(\hat{a}^{\dagger n} + \hat{a}^n)$. When an n-photon-excitation resonance condition is satisfied, the presence of n photons in the cavity suppresses the absorption of the subsequent photons. To confirm the validity of this proposal, we study the n-photon blockade in an atom-cavity system, a Kerr-nonlinear resonator, and two-coupled Kerr-nonlinear resonators. Our results demonstrate that n-photon bunching and $(n+1)$-photon antibunching can be simultaneously obtained in these systems. This effect is due both to the anharmonic energy ladder and to the nature of the n-photon drive. To show the importance of the drive, we compare the results of the n-photon drive with a coherent (one-photon) drive, proving the enhancement of antibunching in the parametric-drive case. This proposal is general and can be applied to realize the n-photon blockade in other nonlinear systems.

DOI: 10.1103/PhysRevA.104.053718

I. INTRODUCTION

In a nonlinear photonic cavity, the energy ladder of the harmonic oscillator is modified by the presence of photon-photon interactions. Even if the driving field can be tuned to be resonant with the cavity, a strong nonlinearity can significantly change the photon-number probability distribution, allowing sizable deviations from the Poissonian statistics. In the *conventional* photon blockade, a large nonlinearity changes the energy-level structure of the system, suppressing the simultaneous presence of photons in the resonator [1]. In the limit of large (infinite) interactions, the presence of a single photon (1P) in the cavity blocks the creation of a second photon [2,3]—an effect known as a conventional single-photon blockade.

Due to its potential applications in information and communication technology, the 1P blockade has been extensively studied in the past years [4–16]. For example, the 1P blockade has been predicted in cavity quantum electrodynamics [17–19], quantum optomechanical systems [20–23], and second-order nonlinear systems [24–26]. The conventional 1P blockade effect was first observed in an optical cavity coupled to a single trapped atom [27]. Since then, many experimental groups have observed this strong antibunching behavior in different systems, including a photonic crystal [28] and a superconducting circuit [29].

The photon blockade can also be enabled by quantum interference [30,31], a phenomenon called the *unconventional* photon blockade [32–38]. Indeed, two optical paths in, e.g., a dimer, can create interference, preventing the simultaneous excitation of two photons in a cavity [39–42]. This effect has been recently observed in quantum-dot cavities and superconducting architectures [43,44]. Although conceptually different, conventional and unconventional photon blockades are connected mechanisms [45,46], and they can even arise simultaneously [47,48]. In this paper, we will consider the conventional photon blockade, and for the sake of brevity we will denote the conventional photon blockade as a photon blockade.

In analogy to the 1P blockade, the n-photon blockade (nP blockade, $n \geqslant 2$) occurs when n photons in a nonlinear cavity suppress the creation of subsequent photons by the drive. The two-photon (2P) blockade (nP blockade with $n = 2$) was studied in several platforms, including a Kerr-type system driven by a laser [49], in a strong-coupling qubit-cavity system [50,51], and in a cascaded cavity QED system [52]. The 2P blockade can also be generated by squeezing [53].

Experimentally, the 2P blockade was realized in an optical cavity strongly coupled to a single atom [54], where driving the atom provides a larger optical nonlinearity than driving the cavity. The nP blockade with $n > 2$ has been studied in a cavity strongly coupled to two atoms [55], in a cavity with two cascade three-level atoms [56], and in a Kerr-type system driven by a laser [57,58]. Meanwhile, in analogy to the photon blockade, the phonon blockade has also been studied [59,60].

In this paper, we theoretically propose that the nP *blockade can be triggered in a nonlinear cavity with an n-photon parametric drive* (denoted for the sake of brevity as the nP drive). While the 2P drive is the parametric down-conversion (PDC) which characterizes $\chi^{(2)}$-type nonlinearities, the higher

*fnori@riken.jp
†yangcp@hznu.edu.cn

nP drive implies the simultaneous creation of n excitations in the system.

Similarly to PDC, the nP drive has been realized experimentally for $n = 3$ by using a superconducting parametric cavity [61]. Such a procedure relies on the fourth-order expansion of the nonlinearity of Josephson junctions, meaning that a high-frequency photon produced by a strong resonant field is transformed into three lower-frequency ones. In the same way in which PDC relies on $\chi^{(2)}$ nonlinearities, this effect relies on $\chi^{(3)}$. Thus, $n > 3$ parametric drives can be, in principle, realized exploiting higher-order nonlinearities. Other possible implementations of the nP drive rely on the multiphoton Jaynes-Cummings model [62] and the quantum Rabi model [63] by driving the atom with the method in Ref. [64], or on the use of generalized Rabi models in the ultrastrong-coupling regime, where a cavity photon can simultaneously excite n atoms, inducing an effective n-boson drive in the collective atomic degree of freedom [65–69].

In this paper, we first give a brief introduction of this proposal and then confirm its validity by considering three examples, i.e., an atom-cavity system, a Kerr-nonlinear resonator, and two-coupled Kerr-nonlinear resonators. This proposal is quite general and can be extended to other nonlinear systems that can display the nP blockade.

The study of the nP blockade in recent decades has mainly focused on a coherent (i.e., single-photon) drive. Compared with a proposal using a coherent driving, the use of an nP drive has the following advantages.

(i) There are systems where the nP blockade exists with an nP drive, while a coherent driving to the cavity will not induce the nP blockade in these systems (e.g., in an atom-cavity system as discussed in Sec. III and Ref. [54]). In this regard, the nP drive is a "more general" approach for the realization of the nP blockade.

(ii) For the same set of parameters characterizing a nonlinear system, we find that the nP drive approach exhibits a larger $(n+1)$-photon antibunching with respect to the coherent driving approach. Moreover, the nP drive mechanism leads to a larger photon number in the cavity.

The remainder of this paper is organized as follows. In Sec. II, we give a brief introduction of this proposal. In Sec. III, we analytically illustrate the nP blockade in an atom-cavity system, showing the differences between the nP drive, leading to the nP blockade, and the coherent drive which does not show any nP blockade. Similarly, in Sec. IV we show the nP blockade in a Kerr-nonlinear resonator, and we discuss the different features which occur with the nP drive and the coherent drive, respectively. In Sec. V, we focus on the 2P blockade in two-coupled Kerr-nonlinear resonators, showing that our analysis remains valid also for extended systems. Conclusions are drawn in Sec. VI.

II. PROPOSAL FOR AN *n*-PHOTON BLOCKADE WITH AN *n*-PHOTON PARAMETRIC DRIVE

The 2P drive has many applications [70–74], such as quantum computing [70], quantum metrology [71], cooling of micromechanical mirrors [72], and generation of long-lived cat states [73]. While the 2P drive is described by a Hamiltonian

$$\hat{H}_d = \lambda(\hat{a}^{\dagger 2}e^{-i\omega_p t} + \hat{a}^2 e^{i\omega_p t}), \tag{1}$$

the nP drive involved in our proposal is described by

$$\hat{H}_d = \lambda(\hat{a}^{\dagger n}e^{-i\omega_p t} + \hat{a}^n e^{i\omega_p t}), \tag{2}$$

where $\hat{a}$ is the cavity annihilation operator, λ is the parametric driving amplitude, and ω_p is the driving frequency.

Our idea is *to use the* nP *drive to induce the* nP *blockade in a nonlinear cavity*. Apart from the bosonic field of the cavity which is subject to an nP drive, an auxiliary nonlinear element (e.g., an atom, a Kerr-nonlinearity medium, or an auxiliary cavity) is required to realize the nP blockade.

The Hamiltonian of the auxiliary nonlinear system and of the undriven cavity is denoted by $\hat{H}_0$. The form of $\hat{H}_0$ is not unique, and it depends on the type of the nonlinear system. Generally speaking for U(1) symmetric (i.e., for particle-number conserving) Hamiltonians, $\hat{H}_0$ can be diagonalized and expressed as

$$\hat{H}_0 = \sum_{j=1}^{k_1}\omega_1^j|\psi_1^j\rangle\langle\psi_1^j| + \sum_{j=1}^{k_2}\omega_2^j|\psi_2^j\rangle\langle\psi_2^j| + \cdots + \sum_{j=1}^{k_n}\omega_n^j|\psi_n^j\rangle\langle\psi_n^j| + \cdots, \tag{3}$$

where ω_n^j is the jth eigenfrequency of $\hat{H}_0$ for the photon excitation number n, and we have assumed that the ground-state energy is zero. The corresponding eigenstate $|\psi_n^j\rangle$ is constructed by the k_n basis vectors for an n-photon excitation manifold. This basis forms a closed space under the action of $\hat{H}_0$ due to the $U(1)$ model symmetry. The set of eigenfrequencies $\{\omega_1^j\}, \{\omega_2^j\}\cdots, \{\omega_n^j\}, \cdots$ is anharmonic due to the nonlinear interaction. Among these eigenfrequencies, $\{\omega_n^j\}$ (where j is from 1 to k_n) is crucial to the nP blockade because the corresponding eigenstate $\{|\psi\rangle_n^j\}$ includes an n-photon state. When the parametric drive frequency ω_p is tuned to the $\{\omega_n^j\}$, the parametric drive resonantly excites n photons in the cavity. As a result, the system occupies the state $\{|\psi\rangle_n^j\}$. This gives rise to a sizable nP blockade. We deduce the following conditions for the nP blockade:

$$\omega_p = \omega_n^j, \tag{4}$$

where j ranges from 1 to k_n. The n-photon resonance excitation by the nP drive ensures that the nP blockade is triggered in the nonlinear cavity due to the system nonlinearity, strongly suppressing higher-order processes which excite more photons.

So far, we have considered the eigenvalues of the Hamiltonian $\hat{H}_0$ to provide the conditions for the blockade. Obviously, to excite an n-photon state, the drive and the dissipation should be correctly taken into account, and in the following numerical simulations we will explicitly compute their effects. To do that, we assume that the systems we consider are described by the Lindblad master equation with the form [75–77]

$$\frac{\partial\hat{\rho}}{\partial t} = -i[\hat{H}, \hat{\rho}] + \sum_j \kappa_j \ell(\hat{o}_j)\hat{\rho}, \tag{5}$$

where κ_j denotes the decay rates and the Lindblad superoperators $\ell(\hat{o}_j)$ act as

$$\ell(\hat{o}_j)\hat{\rho} = \hat{o}_j\hat{\rho}\hat{o}_j^\dagger - \tfrac{1}{2}\hat{o}_j^\dagger\hat{o}_j\hat{\rho} - \tfrac{1}{2}\hat{\rho}\hat{o}_j^\dagger\hat{o}_j. \quad (6)$$

We are interested in characterizing the steady state $\hat{\rho}_{SS}$ under the Lindblad master equation, which is reached by the system once it has evolved for a sufficiently long time. $\hat{\rho}_{SS}$ can be obtained by

$$\frac{\partial \hat{\rho}_{SS}}{\partial t} = 0. \quad (7)$$

In the examples below, if not specified otherwise, any plot in the following refers to quantities computed for the steady state.

The simple Hamiltonian picture in Eq. (4) allows us to capture the main idea behind the mechanism in the regime of the weak drive. Even if the drive and the dissipation modify the system properties, in the limit which we consider the drive can be seen as a perturbation of $\hat{H}_0$, and therefore the resonance condition is well captured by $\hat{H}_0$. In this regard, increasing the drive strength can reduce the accuracy of our prediction. Furthermore, the blockade effect can become weaker by increasing n. For example, we verify that, to have a non-negligible photon number, a four-photon (4P) blockade needs a stronger drive than a three-photon (3P) blockade.

To demonstrate the validity of the above proposal, we study three examples of the nP blockade arising in three different systems: an atom-cavity system, a Kerr resonator, and two-coupled Kerr cavities. Deriving an analytical condition for the nP blockade via Eq. (4), we refine our analysis including the drive and the dissipation, confirming that the nP blockade can be triggered in a nonlinear cavity with the nP drive.

By solving the master equation and obtaining the steady state, we numerically compute the nth-order equal-time correlation function and demonstrate the presence of the nP blockade. Here the nth-order equal-time correlation function is defined as $g^{(n)}(0) = \langle \hat{a}^{\dagger n}\hat{a}^n\rangle/\langle \hat{a}^\dagger\hat{a}\rangle^n$, and $\hat{a}$ is a bosonic annihilation operator. The nth-order correlation function $g^{(n)}(0)$ can be reexpressed as

$$g^{(n)}(0) \simeq \frac{n!P_n}{\left(\sum_n nP_n\right)^n}, \quad (8)$$

where P_n is the n-photon occupation probability of a bosonic mode [78]. Thus, $g^{(n)}(0) \geqslant 1$ implies a large n-photon occupation probability, which indicates n-photon bunching. In contrast, $g^{(n+1)}(0) < 1$ implies a small $(n+1)$-photon occupation probability, which indicates $(n+1)$-photon antibunching. That is, the conditions $g^{(n)}(0) \geqslant 1$ and $g^{(n+1)}(0) < 1$ simultaneously prove the presence of the nP blockade [54]. The nP blockade can be interpreted as the fact that n photons in a nonlinear cavity suppress the presence of the $(n+1)$th photon, and thus a large $(n+1)$-photon antibunching is a requirement for a good n-photon blockade.

Before proceeding further, let us also remark that a strong n-photon drive can trigger instabilities and leads to nonphysical results if the appropriate renormalization terms are not taken into account. Therefore, one has to verify that the occupation of the system is not divergent, which ultimately leads to a boundary on the intensity of the driving. In our case, we are interested in the weak drive and the few-photon regime, where these parametric instabilities never take place.

III. ATOM-CAVITY SYSTEM

We begin our investigation by considering an atom-cavity system described by the Jaynes-Cummings Hamiltonian, where the cavity is driven by an nP drive. In a frame rotating at the parametric drive frequency ω_p/n, the Hamiltonian is (assuming $\hbar = 1$ hereafter)

$$\hat{H} = \hat{H}_0 + \hat{H}_d = \hat{H}_0 + \lambda(\hat{a}^{\dagger n} + \hat{a}^n),$$
$$\hat{H}_0 = \Delta_a\hat{a}^\dagger\hat{a} + \Delta_e\hat{\sigma}_+\hat{\sigma}_- + g(\hat{a}^\dagger\hat{\sigma}_- + \hat{\sigma}_+\hat{a}) \quad (9)$$

where $\hat{a}$ is the cavity annihilation operator, $\hat{\sigma}_\pm$ are the atom raising and lowering operators, g is the coupling strength of the atom and the cavity mode, λ is the amplitude of the nP drive, and Δ_a (Δ_e) is the detuning between the cavity frequency ω_a (the atom frequency ω_e) and the rescaled driving frequency ω_p/n, such that

$$\Delta_a = \omega_a - \omega_p/n, \quad \Delta_e = \omega_e - \omega_p/n. \quad (10)$$

If $\omega_a \gg \omega_e$ or $\omega_a \ll \omega_e$, and g is sufficiently small, the qubit degree of freedom can be traced out, inducing some (possibly nonlinear) energy shifts. Thus, we focus on the resonant case $\omega_a = \omega_e$, resulting in $\Delta_a = \Delta_e$. The Hamiltonian (9) with $n = 2$ can be used to exponentially enhance the light-matter coupling in a generic cavity QED [79–81].

In the absence of the nP drive, the atom-cavity Hamiltonian $\hat{H}_0$ in Eq. (9) can be analytically diagonalized and brought in the form of Eq. (3) with $k_n = 2$ for all n. The eigenstates of $\hat{H}_0$ are

$$|\psi_n^{1,2}\rangle = \frac{1}{\sqrt{2}}(|n-1, e\rangle \mp |n, g\rangle) \quad (11)$$

and the corresponding energies are

$$\omega_n^{1,2} = n\omega_a \mp \sqrt{n}g, \quad (12)$$

where $|g\rangle$ ($|e\rangle$) is the ground (excited) state of the atom, and n denotes the photon excitation number. The energy-level diagram of the system is shown in Fig. 1(a).

Using Eq. (4), *the optimal conditions for the* nP *blockade* are

$$g = \pm\sqrt{n}\Delta, \quad (13)$$

where $\Delta = \Delta_a = \Delta_e$. When one of the above conditions is met, the atom-cavity system will occupy the state $|\psi_n^1\rangle$ or $|\psi_n^2\rangle$ due to resonance excitation. There is one path for the system to reach the state $|\psi_n^{1,2}\rangle$: the system first arrives at an n-photon state by the nP drive, then goes to the state $|\psi_n^{1,2}\rangle$ via the coupling g, i.e., $|0g\rangle \xrightarrow{\lambda} |ng\rangle \xrightarrow{g} |\psi_n^{1,2}\rangle$. The system has difficulty reaching the other manifolds due to the large energy-level splitting once it occupies the states $|\psi_n^{1,2}\rangle$, and the blockade occurs.

Next, we numerically study the nP blockade effect in the presence of the drive and the dissipation. We obtain the steady state of the Lindblad master equation in Eq. (5), which for our system reads

$$\frac{\partial\hat{\rho}}{\partial t} = -i[\hat{H}, \hat{\rho}] + \kappa\ell(\hat{a})\rho + \gamma\ell(\hat{\sigma_-})\rho, \quad (14)$$

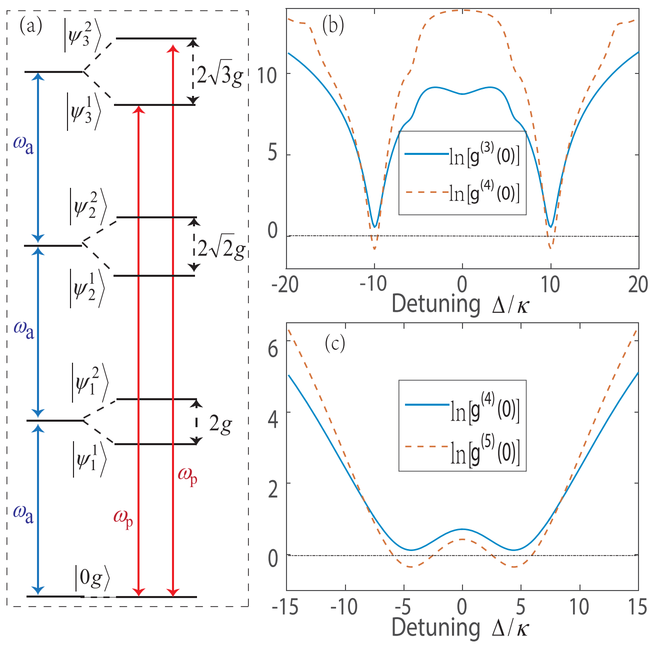

FIG. 1. (a) Schematic energy-level diagram explaining the occurrence of a three-photon blockade. (b) Logarithmic plot of the third-order correlation function $g^{(3)}(0)$ and the fourth-order correlation function $g^{(4)}(0)$ as a function of the detuning Δ/κ, for $g/\kappa = 10\sqrt{3}$, $\gamma/\kappa = 0.1$, and $\lambda/\kappa = 0.3$. (c) $g^{(4)}(0)$ and $g^{(5)}(0)$ as a function of Δ/κ, for $g/\kappa = 10$, $\gamma/\kappa = 0.1$, and $\lambda/\kappa = 1.5$.

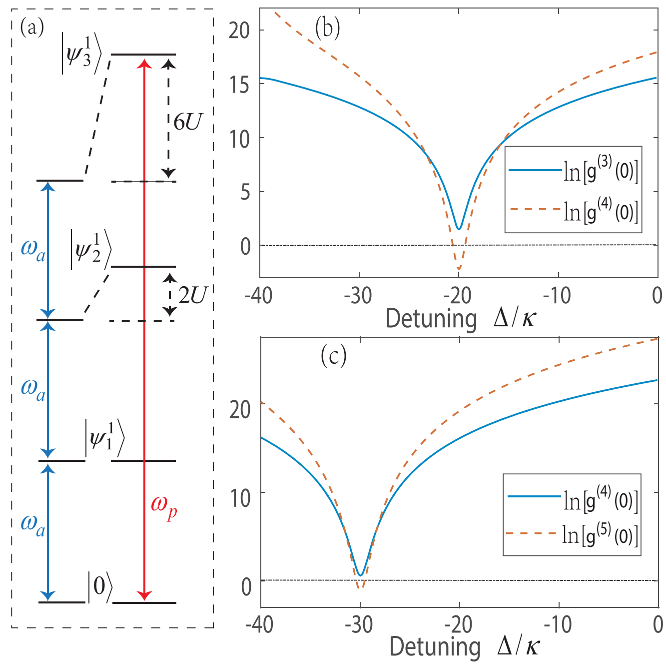

FIG. 2. (a) Energy spectrum of the single mode Kerr-nonlinearity system leading to a three-photon blockade via a three-photon parametric drive. (b) Logarithmic plot of $g^{(3)}(0)$ and $g^{(4)}(0)$ as a function of Δ/κ. (c) Logarithmic plot of $g^{(4)}(0)$ and $g^{(5)}(0)$ as a function of Δ/κ. In (b, c), the parameters are $U/\kappa = 10$ and $\lambda/\kappa = 0.1$.

where κ denotes the decay rate of the cavity and γ is the atomic spontaneous emission rate.

In Fig. 1(b), we show the emergence of the 3P blockade by plotting $g^{(3)}(0)$ and $g^{(4)}(0)$ versus Δ/κ with $g/\kappa = 10\sqrt{3}$. Clearly, the 3P blockade appears for $\Delta/\kappa = \pm 10$, which agrees well with the conditions in Eq. (13). The case of the 4P blockade is studied in Fig. 1(c). We set $g/\kappa = 10$, and the 4P blockade appears when $\Delta/\kappa = \pm 5$, which also agrees with the prediction of Eq. (13) with $n = 4$. The numerical results confirm the analytic conditions and the corresponding analysis.

We here remark that for the Jaynes-Cummings Hamiltonian in Eq. (9) it has been proved that the nP blockade cannot emerge with a coherent drive (i.e., a drive with $n = 1$ in $\hat{H}_d$) [54], but the nP blockade can exist for this system with an nP drive. This prediction shows the nontrivial effect of the nP drive.

IV. KERR-NONLINEAR RESONATOR

The Kerr-nonlinear resonator with a 2P drive has been extensively studied due to its rich physics [82–87]. Here, we investigate the nP blockade with the nP drive. In the frame rotating at the pump frequency, the Hamiltonian of this model reads [84]

$$\begin{aligned} \hat{H} &= \hat{H}_0 + \hat{H}_d = \hat{H}_0 + \lambda(\hat{a}^{\dagger n} + \hat{a}^n), \\ \hat{H}_0 &= \Delta \hat{a}^\dagger \hat{a} + U \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}, \end{aligned} \tag{15}$$

where

$$\Delta = \omega_a - \omega_p/n \tag{16}$$

is the pump-to-cavity detuning, U is the Kerr nonlinear strength, and λ is the amplitude of the nP drive.

The Hamiltonian $\hat{H}_0$ in Eq. (15) is diagonal in the Fock basis, and the eigenstates and eigenfrequencies are $|\psi_n^1\rangle = |n\rangle$ and

$$\omega_n^1 = \omega_a n + U(n^2 - n), \tag{17}$$

respectively. Hence, Eq. (3) has $k_n = 1$, for all n. The nP blockade can be triggered by the n-photon-excitation resonance, where the $|0\rangle \rightarrow |n\rangle$ transition is enhanced. According to Eq. (4), the condition for the nP blockade is

$$U = -\frac{\Delta}{n-1}. \tag{18}$$

To demonstrate the presence of an nP blockade, we consider the Lindblad master Eq. (5) with dissipation $\kappa\ell(\hat{a})\rho$. The energy-level diagram for the 3P blockade is shown in Fig. 2(a), and the corresponding numerical simulation is shown in Fig. 2(b), where we plot $g^{(3)}(0)$ and $g^{(4)}(0)$ as a function of Δ/κ with $U/\kappa = 10$. These results show that the 3P blockade can be obtained at $\Delta/\kappa = -20$, as predicted in Eq. (18) for $n = 3$. Similarly, the 4P blockade depicted in Fig. 2(c) appears at $\Delta/\kappa = -30$, which also agrees with Eq. (18) with $n = 4$.

To prove the efficiency of the nP drive mechanism for the realization of the nP blockade, we compare it with the case of a coherent drive (one-photon drive) $F(\hat{a}^\dagger + \hat{a})$, where F is the coherent driving strength. As an example, we compare the 3P blockade based on the 3P drive with that based on the coherent drive in Fig. 3. To this end, we plot $g^{(3)}(0)$ and $g^{(4)}(0)$ versus the 3P drive strength λ and coherent drive strength F under

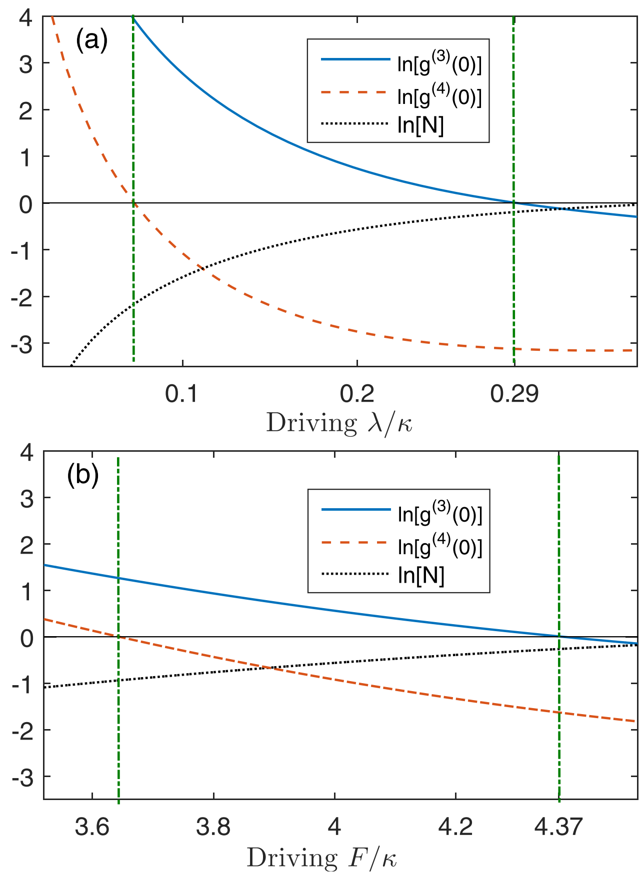

FIG. 3. Logarithmic plot of the correlation functions $g^{(3)}(0)$, $g^{(4)}(0)$, and the occupation N as a function of (a) 3P driving strength λ/κ, and (b) coherent driving strength F/κ. In (a, b), the parameters are $U/\kappa = 10$ and $\Delta/\kappa = -20$.

the blockade condition of Eq. (18) ($U/\kappa = 10$, $\Delta/\kappa = -20$). Notice that the blockade condition obtained in Eq. (18) is independent of the form of the drive, and therefore we expect that this set of parameters allows us to observe the maximal 3P blockade possibility.

The 3P blockade due to the 3P drive is obtained in a region of small λ [the region between the vertical dash-dotted lines in Fig. 3(a)], while the implementation of the 3P blockade with the coherent driving needs a larger F [the region between the vertical dash-dotted lines in Fig. 3(b)]. Comparing with the coherent driving approach, we see that the nP drive approach leads to a larger suppression of $g^{(n+1)}(0)$, which corresponds to a stronger $(n+1)$-photon antibunching.

Furthermore, the occupation $N = \langle\hat{a}^\dagger\hat{a}\rangle$ is an important indicator to evaluate the proposed protocol for the nP blockade, and a larger N indicates a brighter emission. In general, from the interpretation of the nP blockade that n photons in a nonlinear cavity suppress the presence of the $(n+1)$th photon, one could define an optimal blockade point as (i) $g^{(n)}(0) \geqslant 1$ and (ii) $g^{(n+1)}(0)$ is minimal. However, this criterion does not take into account the occupation N. As such, a "better" photon blockade point would require that (iii) N is maximal. Thus the perfect blockade point needs to satisfy (i), (ii), and (iii). Sometimes, however, these three conditions cannot be met simultaneously. If such a perfect blockade point cannot be found out, we can relax the condition (iii) in order to look for an optimal blockade point only satisfying the conditions (i) and (ii). In Fig. 3(a), we show that $g^{(4)}(0) \simeq 0.044$, $g^{(3)}(0) = 1$, i.e., $\ln[g^{(3)}(0)] \simeq 0$, and $N \simeq 0.83$ for $\lambda/\kappa = 0.29$ (the right vertical line), which is a perfect 3P blockade point. The perfect 3P blockade point for the coherent driving appears at $F/\kappa = 4.37$; instead, $g^{(4)}(0) \simeq 0.197$ and $N \simeq 0.766$. The nP drive approach has a larger $g^{(4)}(0)$ and also a larger N than the coherent driving approach for the perfect blockade point.

We conclude that the nP drive triggers the nP blockade better than the coherent driving. The difference between the two cases (nP drive versus coherent driving) lies in the efficiency for exciting n photons; this can be understood using a perturbation theory approach. The nP drive resonantly drives the nth-excitation manifold: it requires just one action of the drive to pass from the vacuum to a state with n photons. As such, even a relatively weak drive can accomplish this task. In contrast, for the coherent driving, the drive has to act n times to bring the photon from the vacuum to the state with n photons. This is a process of order n passing through many nonresonant Hamiltonian states. As such, (i) it requires a much stronger drive to produce a comparable number of excitations and (ii) it causes a higher probability of exciting other undesired states other than $|n\rangle$.

V. TWO-COUPLED KERR-NONLINEAR RESONATORS

Two-coupled cavities with Kerr nonlinearity have been considered to study the 1P blockade [31]. Here, we label the two cavities as a and b. The Hamiltonian in the frame rotating at the drive frequency is

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{H}_d = \hat{H}_0 + \lambda(\hat{a}^{\dagger n} + \hat{a}^n),\\ \hat{H}_0 &= \Delta(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}) + J(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})\\ &\quad + U(\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}),\end{aligned} \tag{19}$$

where $\hat{a}$ ($\hat{b}$) is the photon annihilation operator for the cavity a (b) with frequency ω_a (ω_b),

$$\Delta = \omega_a - \omega_p/n = \omega_b - \omega_p/n, \tag{20}$$

J is the coupling strength of the two cavities, U is the Kerr nonlinear strength, and λ is the nP drive strength.

Since the total number of photons is conserved by $\hat{H}_0$ in Eq. (19), we can deduce that there are $k_n = n + 1$ states in Eq. (3). In this case, we cannot analytically determine all the eigenfrequencies $\{\omega_n^j\}$ for all n, but the case of $n = 2$ can still be analytically solved. We now focus on the $n = 2$ subspace, and to diagonalize it we project the Hamiltonian onto the two-photon states $|20\rangle$, $|11\rangle$, and $|02\rangle$, where $|\alpha\beta\rangle \equiv |\alpha\rangle \otimes |\beta\rangle$ is a Fock state with α (β) photons in the cavity a (b). In the two-photon subspace, $\hat{H}_0$ is expressed as

$$\hat{H}_2 = \begin{bmatrix} 2\omega_a + 2U & \sqrt{2}J & 0 \\ \sqrt{2}J & 2\omega_a & \sqrt{2}J \\ 0 & \sqrt{2}J & 2\omega_a + 2U \end{bmatrix}. \tag{21}$$

The three eigenfrequencies are

$$\omega_2^{1,3} = 2\omega_a + U \mp \sqrt{4J^2 + U^2}, \quad \omega_2^2 = 2(U + \omega_a). \tag{22}$$

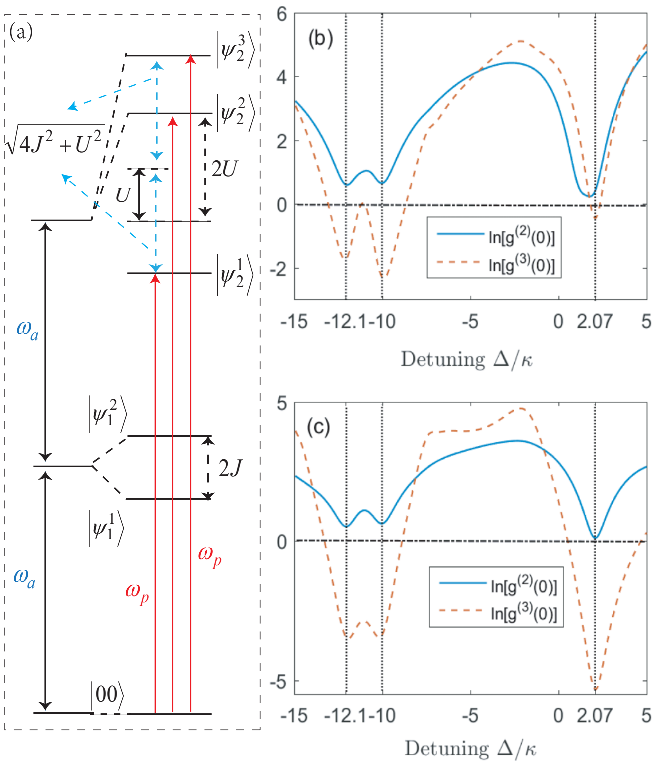

FIG. 4. (a) Energy spectrum for two-coupled cavities with Kerr nonlinearity. The blue arrows are half the distance of ω_2^1 and ω_2^3, which are calculated by ω_2^1 and ω_2^3. (b, c) Logarithmic plot of $g^{(2)}(0)$ and $g^{(3)}(0)$ as a function of Δ/κ for (b) the cavity a and (c) the cavity b, respectively. In (b, c), the parameters are $U/\kappa = 10$, $J/\kappa = 5$, and $\lambda/\kappa = 0.5$.

The corresponding un-normalized eigenstates are

$$\left|\psi_2^{1,3}\right\rangle = |20\rangle - [\sqrt{2}U \mp \sqrt{2(4J^2+U^2)}]/(2J)|11\rangle + |02\rangle,$$
$$\left|\psi_2^2\right\rangle = -|20\rangle + |02\rangle. \tag{23}$$

The energy-level diagram is shown in Fig. 4(a).

The conditions for the 2P blockade, obtained from Eq. (4), are satisfied if

$$\Delta = -U, \quad \Delta = \frac{-U \pm \sqrt{4J^2+U^2}}{2}. \tag{24}$$

Under these resonance conditions, the 2P blockade can be triggered via the resonant transitions $|00\rangle \rightarrow \{|\psi_2^2\rangle, |\psi_2^{1,3}\rangle\}$. The two cavities occupy the two-photon states $|20\rangle$ and $|02\rangle$, which ensures that the 2P blockade is simultaneously realized in the two cavities when the conditions in Eq. (24) are satisfied.

The numerical study of the 2P blockade is the same as before. In Figs. 4(b) and 4(c), we plot $g^{(2)}(0)$ and $g^{(3)}(0)$ as a function of Δ/κ, for the cavity a and the cavity b, respectively. The results indicate that the 2P blockade occurs for $\Delta/\kappa = -12.7$, -10, and 2.07, which are predicted by the three 2P blockade conditions given in Eq. (24). Thus, it is seen that the 2P blockade is simultaneously realized in both cavities due to the feature of the system and the nP drive.

In Fig. 5, we compare the 2P drive case with the coherent drive case by showing $g^{(2)}(0)$ and $g^{(3)}(0)$ as a function of λ/κ (F/κ) for the three blockade points shown in Figs. 4(b) and 4(c). We focus on the undriven cavity b, where $g^{(3)}(0)$ is smaller than that of the cavity a. The blockade regions are shown between the green vertical lines. Similarly to the single Kerr-nonlinear resonator case, the effect of the drive leads to two different blockade regions. Again, the 2P drive approach shows a better three-photon antibunching property than the coherent drive approach.

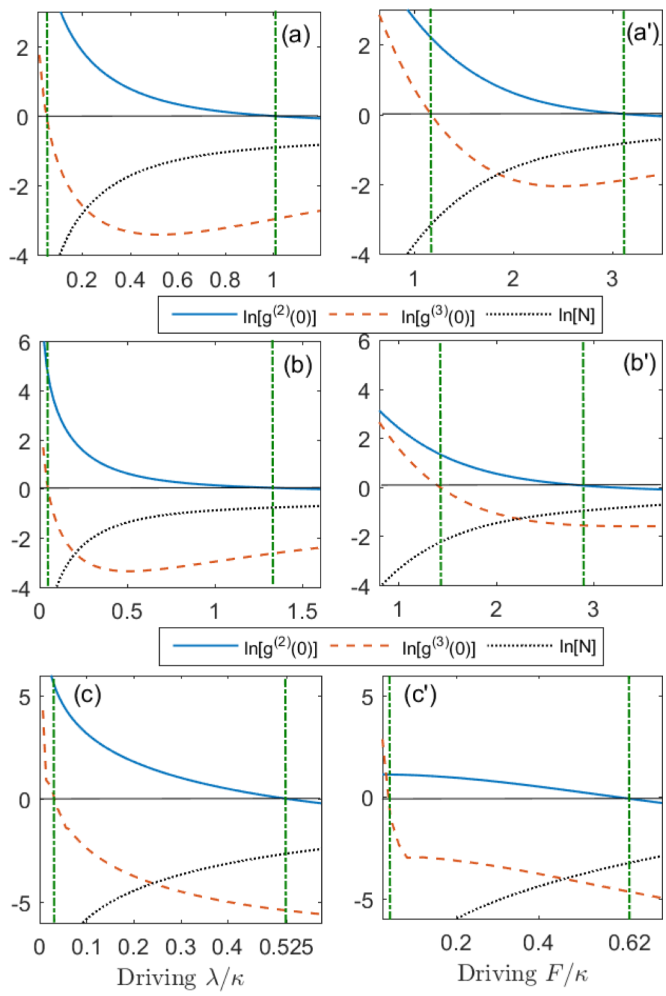

FIG. 5. Logarithmic plot of the correlation functions $g^{(2)}(0)$, $g^{(3)}(0)$, and the occupation N of the cavity b as a function of λ/κ (F/κ) for $U/\kappa = 10$ and $J/\kappa = 5$. The column on the left corresponds to the 3P driving strength λ/κ, and that on the right corresponds to the coherent driving strength F/κ. (a, a') $\Delta/\kappa = -12.1$. (b, b') $\Delta/\kappa = -10$. (c, c') $\Delta/\kappa = 2.07$.

Finally, we also look for the perfect blockade point defined in the previous section. As opposed to the scheme of the single Kerr-nonlinear resonator in Sec. IV, $g^3(0)$ is not a monotone function when $\Delta/\kappa = -12.7$ and -10 as shown in Figs. 5(a) and 5(b) for the 2P drive case. And we can only find the optimal blockade point, which corresponds to a minimal $g^3(0)$. For $\Delta/\kappa = 2.07$, instead, the correlation function $g^3(0)$ is monotone decreasing and the occupation N is monotone increasing, shown in Figs. 5(c) and 5(c'). A perfect 2P blockade point with the 2P drive appears at $\lambda/\kappa = 0.525$, where $g^{(3)}(0) \simeq 0.0045$ and $N = 0.0725$. For the coherent driving, we find $g^{(3)}(0) \simeq 0.0094$ and $N \simeq 0.0387$ for $F/\kappa = 0.62$. Thus, the 2P drive approach has a stronger three-photon antibunching and a larger occupation on this perfect blockade point.

VI. CONCLUSIONS

We have proposed that an *n*-photon blockade can be realized in a nonlinear cavity with an *n*-photon parametric drive. The validity of this proposal has been confirmed by three examples, i.e., the *n*-photon blockade in an atom-cavity system, in a single-mode Kerr nonlinear device, and in a two-coupled Kerr-nonlinear resonator. By solving the master equation in the steady-state limit and computing the correlation functions $g^{(n)}(0)$ and $g^{(n+1)}(0)$, we have shown that the *n*P blockade can be realized, and the optimal conditions for the *n*P blockade are in good agreement with the numerical simulations, thus supporting the validity of our proposal. Although we focused on cases where the Hamiltonian $\hat{H}_0$ can be diagonalized analytically (and therefore the resonance condition can be expressed as an algebraic equation of the system parameters), the proposed procedure to realize the *n*P blockade with the *n*P drive remains valid for more complex Hamiltonian systems.

ACKNOWLEDGMENTS

This work is supported by the Jiangxi Natural Science Foundation (Grant No. 20192ACBL20051), the National Natural Science Foundation of China (Grants No. 11965017, No. 11705025, No. 11804228, and No. 11774076), and the Jiangxi Education Department Fund (Grant No. GJJ180873). This work is also supported in part by Nippon Telegraph and Telephone Corporation Research, the Japan Science and Technology Agency (via the Quantum Leap Flagship Program, Moonshot R&D Grant No. JPMJMS2061, and Centers of Research Excellence in Science and Technology Grant No. JPMJCR1676), the Japan Society for the Promotion of Science (JSPS) (via a Grant-in-Aid for Scientific Research KAKENHI Grant No. JP20H00134 and JSPS-RFBR Grant No. JPJSBP120194828), the U.S. Army Research Office (Grant No. W911NF-18-1-0358), the Asian Office of Aerospace Research and Development (Grant No. FA2386-20-1-4069), and the Foundational Questions Institute Fund (Grant No. FQXi-IAF19-06).

[1] A. Imamoğlu, H. Schmidt, G. Woods, and M. Deutsch, Strongly Interacting Photons in a Nonlinear Cavity, Phys. Rev. Lett. **79**, 1467 (1997).

[2] C. Lang, D. Bozyigit, C. Eichler, L. Steffen, J. M. Fink, A. A. Abdumalikov, M. Baur, S. Filipp, M. P. da Silva, A. Blais, and A. Wallraff, Observation of Resonant Photon Blockade at Microwave Frequencies Using Correlation Function Measurements, Phys. Rev. Lett. **106**, 243601 (2011).

[3] A. Ridolfo, M. Leib, S. Savasta, and M. J. Hartmann, Photon Blockade in the Ultrastrong Coupling Regime, Phys. Rev. Lett. **109**, 193602 (2012).

[4] B. Dayan, A. S. Parkins, T. Aoki, E. P. Ostby, K. J. Vahala, and H. J. Kimble, A photon turnstile dynamically regulated by one atom, Science **319**, 1062 (2008).

[5] A. Reinhard, T. Volz, M. Winger, A. Badolato, K. J. Hennessy, E. L. Hu, and A. Imamoğlu, Strongly correlated photons on a chip, Nat. Photonics **6**, 93 (2012).

[6] S.-L. Su, Y. Tian, H. Z. Shen, H. Zang, E. Liang, and S. Zhang, Applications of the modified Rydberg antiblockade regime with simultaneous driving, Phys. Rev. A **96**, 042335 (2017).

[7] S.-L. Su, Y. Gao, E. Liang, and S. Zhang, Fast Rydberg antiblockade regime and its applications in quantum logic gates, Phys. Rev. A **95**, 022319 (2017).

[8] A. Miranowicz, J. c. v. Bajer, M. Paprzycka, Y.-x. Liu, A. M. Zagoskin, and F. Nori, State-dependent photon blockade via quantum-reservoir engineering, Phys. Rev. A **90**, 033831 (2014).

[9] Y.-x. Liu, X.-W. Xu, A. Miranowicz, and F. Nori, From blockade to transparency: Controllable photon transmission through a circuit-QED system, Phys. Rev. A **89**, 043818 (2014).

[10] D. G. Angelakis, M. F. Santos, and S. Bose, Photon-blockade-induced Mott transitions and XY spin models in coupled cavity arrays, Phys. Rev. A **76**, 031805 (2007).

[11] K. Müller, A. Rundquist, K. A. Fischer, T. Sarmiento, K. G. Lagoudakis, Y. A. Kelaita, C. Sánchez Muñoz, E. del Valle, F. P. Laussy, and J. Vučković, Coherent Generation of Nonclassical Light on Chip via Detuned Photon Blockade, Phys. Rev. Lett. **114**, 233601 (2015).

[12] S. Ghosh and T. C. H. Liew, Dynamical Blockade in a Single-Mode Bosonic System, Phys. Rev. Lett. **123**, 013602 (2019).

[13] S. Rosenblum, S. Parkins, and B. Dayan, Photon routing in cavity QED: Beyond the fundamental limit of photon blockade, Phys. Rev. A **84**, 033854 (2011).

[14] H. Zheng, D. J. Gauthier, and H. U. Baranger, Cavity-Free Photon Blockade Induced by Many-Body Bound States, Phys. Rev. Lett. **107**, 223601 (2011).

[15] H. Carmichael, Breakdown of Photon Blockade: A Dissipative Quantum Phase Transition in Zero Dimensions, Phys. Rev. X **5**, 031028 (2015).

[16] Y.-P. Gao, X.-F. Liu, T.-J. Wang, C. Cao, and C. Wang, Photon excitation and photon-blockade effects in optomagnonic microcavities, Phys. Rev. A **100**, 043831 (2019).

[17] Y.-x. Liu, A. Miranowicz, Y. B. Gao, J. c. v. Bajer, C. P. Sun, and F. Nori, Qubit-induced phonon blockade as a signature of quantum behavior in nanomechanical resonators, Phys. Rev. A **82**, 032101 (2010).

[18] Y. H. Zhou, H. Z. Shen, X. Y. Zhang, and X. X. Yi, Zero eigenvalues of a photon blockade induced by a non-Hermitian Hamiltonian with a gain cavity, Phys. Rev. A **97**, 043819 (2018).

[19] J. M. Fink, A. Dombi, A. Vukics, A. Wallraff, and P. Domokos, Observation of the Photon-Blockade Breakdown Phase Transition, Phys. Rev. X **7**, 011012 (2017).

[20] A. Nunnenkamp, K. Børkje, and S. M. Girvin, Single-Photon Optomechanics, Phys. Rev. Lett. **107**, 063602 (2011).

[21] P. Rabl, Photon Blockade Effect in Optomechanical Systems, Phys. Rev. Lett. **107**, 063601 (2011).

[22] H. Wang, X. Gu, Y.-x. Liu, A. Miranowicz, and F. Nori, Tunable photon blockade in a hybrid system consisting of an optomechanical device coupled to a two-level system, Phys. Rev. A **92**, 033806 (2015).

[23] J.-Q. Liao and F. Nori, Photon blockade in quadratically coupled optomechanical systems, Phys. Rev. A **88**, 023853 (2013).
[24] Y. H. Zhou, X. Y. Zhang, Q. C. Wu, B. L. Ye, Z.-Q. Zhang, D. D. Zou, H. Z. Shen, and C.-P. Yang, Conventional photon blockade with a three-wave mixing, Phys. Rev. A **102**, 033713 (2020).
[25] H. Z. Shen, Y. H. Zhou, and X. X. Yi, Quantum optical diode with semiconductor microcavities, Phys. Rev. A **90**, 023849 (2014).
[26] A. Majumdar and D. Gerace, Single-photon blockade in doubly resonant nanocavities with second-order nonlinearity, Phys. Rev. B **87**, 235319 (2013).
[27] K. M. Birnbaum, A. Boca, R. Miller, A. D. Boozer, T. E. Northup, and H. J. Kimble, Photon blockade in an optical cavity with one trapped atom, Nature (London) **436**, 87 (2005).
[28] A. Faraon, I. Fushman, D. Englund, N. Stoltz, P. Petroff, and J. Vučković, Coherent generation of non-classical light on a chip via photon-induced tunnelling and blockade, Nat. Phys. **4**, 859 (2008).
[29] A. J. Hoffman, S. J. Srinivasan, S. Schmidt, L. Spietz, J. Aumentado, H. E. Türeci, and A. A. Houck, Dispersive Photon Blockade in a Superconducting Circuit, Phys. Rev. Lett. **107**, 053602 (2011).
[30] T. C. H. Liew and V. Savona, Single Photons from Coupled Quantum Modes, Phys. Rev. Lett. **104**, 183601 (2010).
[31] M. Bamba, A. Imamoğlu, I. Carusotto, and C. Ciuti, Origin of strong photon antibunching in weakly nonlinear photonic molecules, Phys. Rev. A **83**, 021802 (2011).
[32] H. J. Carmichael, Photon Antibunching and Squeezing for a Single Atom in a Resonant Cavity, Phys. Rev. Lett. **55**, 2790 (1985).
[33] H. Flayac and V. Savona, Input-output theory of the unconventional photon blockade, Phys. Rev. A **88**, 033836 (2013).
[34] H. Z. Shen, C. Shang, Y. H. Zhou, and X. X. Yi, Unconventional single-photon blockade in non-Markovian systems, Phys. Rev. A **98**, 023856 (2018).
[35] B. Sarma and A. K. Sarma, Quantum-interference-assisted photon blockade in a cavity via parametric interactions, Phys. Rev. A **96**, 053827 (2017).
[36] H. Flayac and V. Savona, Unconventional photon blockade, Phys. Rev. A **96**, 053810 (2017).
[37] B. Sarma and A. K. Sarma, Unconventional photon blockade in three-mode optomechanics, Phys. Rev. A **98**, 013826 (2018).
[38] M.-A. Lemonde, N. Didier, and A. A. Clerk, Antibunching and unconventional photon blockade with Gaussian squeezed states, Phys. Rev. A **90**, 063824 (2014).
[39] X.-W. Xu and Y. Li, Strong photon antibunching of symmetric and antisymmetric modes in weakly nonlinear photonic molecules, Phys. Rev. A **90**, 033809 (2014).
[40] O. Kyriienko, I. A. Shelykh, and T. C. H. Liew, Tunable single-photon emission from dipolaritons, Phys. Rev. A **90**, 033807 (2014).
[41] Y. H. Zhou, H. Z. Shen, and X. X. Yi, Unconventional photon blockade with second-order nonlinearity, Phys. Rev. A **92**, 023838 (2015).
[42] Y. H. Zhou, H. Z. Shen, X. Q. Shao, and X. X. Yi, Strong photon antibunching with weak second-order nonlinearity under dissipation and coherent driving, Opt. Express **24**, 17332 (2016).
[43] H. J. Snijders, J. A. Frey, J. Norman, H. Flayac, V. Savona, A. C. Gossard, J. E. Bowers, M. P. van Exter, D. Bouwmeester, and W. Löffler, Observation of the Unconventional Photon Blockade, Phys. Rev. Lett. **121**, 043601 (2018).
[44] C. Vaneph, A. Morvan, G. Aiello, M. Féchant, M. Aprili, J. Gabelli, and J. Estève, Observation of the Unconventional Photon Blockade in the Microwave Domain, Phys. Rev. Lett. **121**, 043602 (2018).
[45] E. Zubizarreta Casalengua, J. C. López Carreño, F. P. Laussy, and E. d. Valle, Photon statistics: Conventional and unconventional photon statistics, Laser Photonics Rev. **14**, 2070032 (2020).
[46] E. Zubizarreta Casalengua, J. C. López Carreño, F. P. Laussy, and E. del Valle, Tuning photon statistics with coherent fields, Phys. Rev. A **101**, 063824 (2020).
[47] C. Noh, Emission of single photons in the weak coupling regime of the Jaynes Cummings model, Sci. Rep. **10**, 1 (2020).
[48] O. Kyriienko, D. N. Krizhanovskii, and I. A. Shelykh, Nonlinear Quantum Optics with Trion Polaritons in 2D Monolayers: Conventional and Unconventional Photon Blockade, Phys. Rev. Lett. **125**, 197402 (2020).
[49] A. Miranowicz, M. Paprzycka, Y.-x. Liu, J. c. v. Bajer, and F. Nori, Two-photon and three-photon blockades in driven nonlinear systems, Phys. Rev. A **87**, 023809 (2013).
[50] W.-W. Deng, G.-X. Li, and H. Qin, Enhancement of the two-photon blockade in a strong-coupling qubit-cavity system, Phys. Rev. A **91**, 043831 (2015).
[51] S. Shamailov, A. Parkins, M. Collett, and H. Carmichael, Multi-photon blockade and dressing of the dressed states, Opt. Commun. **283**, 766 (2010).
[52] Q. Bin, X.-Y. Lü, S.-W. Bin, and Y. Wu, Two-photon blockade in a cascaded cavity-quantum-electrodynamics system, Phys. Rev. A **98**, 043858 (2018).
[53] A. Kowalewska-Kudłaszyk, S. I. Abo, G. Chimczak, J. Peřina, F. Nori, and A. Miranowicz, Two-photon blockade and photon-induced tunneling generated by squeezing, Phys. Rev. A **100**, 053857 (2019).
[54] C. Hamsen, K. N. Tolazzi, T. Wilk, and G. Rempe, Two-Photon Blockade in an Atom-Driven Cavity QED System, Phys. Rev. Lett. **118**, 133604 (2017).
[55] C. J. Zhu, Y. P. Yang, and G. S. Agarwal, Collective multiphoton blockade in cavity quantum electrodynamics, Phys. Rev. A **95**, 063842 (2017).
[56] J. Z. Lin, K. Hou, C. J. Zhu, and Y. P. Yang, Manipulation and improvement of multiphoton blockade in a cavity-QED system with two cascade three-level atoms, Phys. Rev. A **99**, 053850 (2019).
[57] G. H. Hovsepyan, A. R. Shahinyan, and G. Y. Kryuchkyan, Multiphoton blockades in pulsed regimes beyond stationary limits, Phys. Rev. A **90**, 013839 (2014).
[58] R. Huang, A. Miranowicz, J.-Q. Liao, F. Nori, and H. Jing, Nonreciprocal Photon Blockade, Phys. Rev. Lett. **121**, 153601 (2018).
[59] A. Miranowicz, J. c. v. Bajer, N. Lambert, Y.-x. Liu, and F. Nori, Tunable multiphonon blockade in coupled nanomechanical resonators, Phys. Rev. A **93**, 013808 (2016).
[60] X. Wang, A. Miranowicz, H.-R. Li, and F. Nori, Method for observing robust and tunable phonon blockade in a nanomechanical resonator coupled to a charge qubit, Phys. Rev. A **93**, 063861 (2016).

[61] C. W. S. Chang, C. Sabín, P. Forn-Díaz, F. Quijandría, A. M. Vadiraj, I. Nsanzineza, G. Johansson, and C. M. Wilson, Observation of Three-Photon Spontaneous Parametric Down-Conversion in a Superconducting Parametric Cavity, Phys. Rev. X **10**, 011011 (2020).

[62] C. J. Villas-Boas and D. Z. Rossatto, Multiphoton Jaynes-Cummings Model: Arbitrary Rotations in Fock Space and Quantum Filters, Phys. Rev. Lett. **122**, 123604 (2019).

[63] Y.-Z. Zhang, On the 2-mode and k-photon quantum Rabi models, Rev. Math. Phys. **29**, 1750013 (2017).

[64] X. Guo, C.-L. Zou, H. Jung, and H. X. Tang, On-Chip Strong Coupling and Efficient Frequency Conversion between Telecom and Visible Optical Modes, Phys. Rev. Lett. **117**, 123902 (2016).

[65] L. Garziano, V. Macrì, R. Stassi, O. Di Stefano, F. Nori, and S. Savasta, One Photon Can Simultaneously Excite Two or More Atoms, Phys. Rev. Lett. **117**, 043601 (2016).

[66] V. Macrì, F. Nori, S. Savasta, and D. Zueco, Spin squeezing by one-photon–two-atom excitation processes in atomic ensembles, Phys. Rev. A **101**, 053818 (2020).

[67] A. F. Kockum, A. Miranowicz, V. Macrì, S. Savasta, and F. Nori, Deterministic quantum nonlinear optics with single atoms and virtual photons, Phys. Rev. A **95**, 063849 (2017).

[68] A. F. Kockum, V. Macrì, L. Garziano, S. Savasta, and F. Nori, Frequency conversion in ultrastrong cavity QED, Sci. Rep. **7**, 1 (2017).

[69] R. Stassi, V. Macrì, A. F. Kockum, O. Di Stefano, A. Miranowicz, S. Savasta, and F. Nori, Quantum nonlinear optics without photons, Phys. Rev. A **96**, 023818 (2017).

[70] D. Pelliccia, V. Schettini, F. Sciarrino, C. Sias, and F. De Martini, Contextual realization of the universal quantum cloning machine and of the universal-NOT gate by quantum-injected optical parametric amplification, Phys. Rev. A **68**, 042306 (2003).

[71] F. Hudelist, J. Kong, C. Liu, J. Jing, Z. Ou, and W. Zhang, Quantum metrology with parametric amplifier-based photon correlation interferometers, Nat. Commun. **5**, 1 (2014).

[72] S. Huang and G. S. Agarwal, Enhancement of cavity cooling of a micromechanical mirror using parametric interactions, Phys. Rev. A **79**, 013821 (2009).

[73] W. Qin, A. Miranowicz, H. Jing, and F. Nori, Generating Long-Lived Macroscopically Distinct Superposition States in Atomic Ensembles, Phys. Rev. Lett. **127**, 093602 (2021).

[74] W. Qin, Y.-H. Chen, X. Wang, A. Miranowicz, and F. Nori, Strong spin squeezing induced by weak squeezing of light inside a cavity, Nanophotonics **9**, 4853 (2020).

[75] C. W. Gardiner, *Quantum Noise* (Springer, New York, 2000).

[76] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University, Cambridge, England, 1997).

[77] G. S. Agarwal, *Quantum Optics* (Cambridge University, Cambridge, England, 2012).

[78] F. Zou, X.-Y. Zhang, X.-W. Xu, J.-F. Huang, and J.-Q. Liao, Multiphoton blockade in the two-photon Jaynes-Cummings model, Phys. Rev. A **102**, 053710 (2020).

[79] W. Qin, A. Miranowicz, P.-B. Li, X.-Y. Lü, J. Q. You, and F. Nori, Exponentially Enhanced Light-Matter Interaction, Cooperativities, and Steady-State Entanglement Using Parametric Amplification, Phys. Rev. Lett. **120**, 093601 (2018).

[80] C. Leroux, L. C. G. Govia, and A. A. Clerk, Enhancing Cavity Quantum Electrodynamics via Antisqueezing: Synthetic Ultrastrong Coupling, Phys. Rev. Lett. **120**, 093602 (2018).

[81] Y. Wang, C. Li, E. M. Sampuli, J. Song, Y. Jiang, and Y. Xia, Enhancement of coherent dipole coupling between two atoms via squeezing a cavity mode, Phys. Rev. A **99**, 023833 (2019).

[82] B. Wielinga and G. J. Milburn, Quantum tunneling in a Kerr medium with parametric pumping, Phys. Rev. A **48**, 2494 (1993).

[83] F. Minganti, N. Bartolo, J. Lolli, W. Casteels, and C. Ciuti, Exact results for Schrödinger cats in driven-dissipative systems and their feedback control, Sci. Rep. **6**, 26987 (2016).

[84] P. Zhao, Z. Jin, P. Xu, X. Tan, H. Yu, and Y. Yu, Two-Photon Driven Kerr Resonator for Quantum Annealing with Three-Dimensional Circuit QED, Phys. Rev. Appl. **10**, 024019 (2018).

[85] T. Gevorgyan and G. Kryuchkyan, Parametrically driven nonlinear oscillator at a few-photon level, J. Mod. Opt. **60**, 860 (2013).

[86] G. H. Hovsepyan, A. R. Shahinyan, L. Y. Chew, and G. Y. Kryuchkyan, Phase locking and quantum statistics in a parametrically driven nonlinear resonator, Phys. Rev. A **93**, 043856 (2016).

[87] N. Bartolo, F. Minganti, W. Casteels, and C. Ciuti, Exact steady state of a Kerr resonator with one- and two-photon driving and dissipation: Controllable Wigner-function multimodality and dissipative phase transitions, Phys. Rev. A **94**, 033841 (2016).