Tunable optomechanically induced transparency by controlling the dark-mode effect

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We study tunable optomechanically induced transparency by controlling the dark-mode effect induced by two mechanical modes coupled to a common cavity field. This is realized by introducing a phase-dependent phonon-exchange interaction, which is used to form a loop-coupled configuration. Combining this phase-dependent coupling with the optomechanical interactions, the dark-mode effect can be controlled by the quantum interference effect. In particular, the dark-mode effect in this two-mechanical-mode optomechanical system can lead to a double-amplified optomechanically induced transparency (OMIT) window and a higher efficiency of the second-order sideband in comparison with the standard optomechanical system. This is because the effective mechanical decay rate related to the linewidth of the OMIT window becomes a twofold increase in the weak-coupling limit. When the dark-mode effect is broken, controllable double transparency windows appear and the second-order sideband, as well as the light delay or advance, is significantly enhanced. For an *N*-mechanical-mode optomechanical system, we find that in the presence of the dark-mode effect, the amplification multiple of the linewidth of the OMIT window is nearly proportional to the number of mechanical modes, and that the OMIT with a single window becomes the one with *N* tunable windows by breaking the dark-mode effect. The study will be useful in optical information storage within a large-frequency bandwidth and multichannel optical communication based on optomechanical systems.

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I. INTRODUCTION

Cavity optomechanical systems [1-3] are an important platform for manipulating mechanical properties through optical means and studying cavity-field statistics by mechanically changing the cavity boundary [4-20]. Optomechanically induced transparency (OMIT) [21-23], as a prominent application closely relevant to this platform, is the result of destructive interference between the anti-Stokes field and the probe field. The underlying physical mechanism is analogous to electromagnetically induced transparency [24,25]. The performance of an OMIT process is mainly described by the optical transmission and delay, which will directly determine the information transfer efficiency and storage time, respectively. Recently, various schemes have been proposed to tune the transmission rate and group delay of the signal light, such as OMIT in hybridized optomechanical systems [26-35], OMIT with higher-order sidebands [36-38], OMIT in quadratically nanomechanical systems [39,40], nonreciprocal OMIT [41],

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OMIT with Bogoliubov mechanical modes [42], and OMIT at exceptional points [43].

Although great advances have been made to improve OMIT, it remains a long-standing challenge to significantly broaden the linewidth of the OMIT window determining the frequency bandwidth of the information transmission and realize the ultralong optical delay [21–23]. The physical origin behind this obstacle is that the effective mechanical decay rate relating to the linewidth of the OMIT window is governed by the pump laser power (i.e., the intracavity photon number) [21–23], while a high pump power will cause the bistability of the system [22,23,26,35,36].

In parallel, the optomechanical dark mode [44–46], proposed in a system of two optical cavities coupled to a common mechanical resonator, has attracted much attention in recent years. It can be employed for the realization of high efficient quantum state conversion [45,46] and the exploitation of circumventing the decay [11,45–47]. Meanwhile, two mechanical modes or magnon modes coupled to a common cavity field can also be hybridized into a bright mode and a dark mode decoupled from the system [48–53]. This physical mechanism has been broadly applied to gradient memory or information storage [49], and energy transfer [50]. In particular, the effective mechanical decay rate of the bright mode is nearly twice

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FIG. 1. (a) Schematic diagram of the three-mode loop-coupled optomechanical system formed by a cavity-field mode *a* (with a decay rate κ) optomechanically coupled to two mechanical modes b_1 and b_2 (with decay rates γ_1 and γ_2). Two mechanical modes are coupled to each other via a phase-dependent phonon-phonon interaction with the coupling strength η and the modulation phase θ . The system is driven by a strong pump field of frequency ω_L and a weak probe field of frequency ω_p . (b) Frequency spectrum for the system shown in (a). The first-order sidebands with frequencies $\omega_L \pm \Omega$ are referred to as the Stokes and anti-Stokes fields, respectively. The sidebands with frequencies $\omega_L \pm 2\Omega$ are the second-order upper or lower sidebands. (c) We extend the scheme in (a) to a net-coupled optomechanical system: a cavity mode is simultaneously coupled to $N \ge 3$ (*N* is an integer) mechanical modes via optomechanical couplings, and the nearest-neighboring mechanical modes are coupled to each other through the phase-dependent phonon-exchange couplings H_i for j = 1-(N-1).

that of the individual mechanical mode [48–53]. Thus, based on the characteristic that the narrow linewidth of the OMIT window results from the effective mechanical decay rate of the mechanical mode [21–23], it is natural to ask the question whether we can significantly widen the OMIT window or steer the switch from the OMIT with a single transparent window to the case of multiple transmission windows by controlling the dark-mode effect in a multimechanical-mode optomechanical system.

In this paper, we study the sensitive impacts of the darkmode effect on the optical properties, including the optical transmission, group delay, and higher-order sidebands. We consider a three-mode loop-coupled optomechanical system composed of a cavity-field mode optomechanically coupled to two mechanical modes, which are coupled to each other by a phase-dependent phonon-phonon interaction. Combining this phase-dependent phonon-phonon coupling with the optomechanical interactions, the quantum interference effect leads to the breaking of the dark mode. We find that in the presence of the dark mode, OMIT and second-order sidebands can be significantly enhanced compared to those in a standard optomechanical system, and the amplification multiple of the linewidth of the OMIT window is nearly proportional to the number of mechanical modes. This provides a route to achieve the optical transmission within a large-frequency bandwidth. When breaking the dark-mode effect, the OMIT with a single window becomes one with multiple windows, and the transmission windows can be controlled and switched periodically by tuning the modulation phase. These results provide the possibility to enhance or steer the performance of OMIT, and realize the switch from single-channel to multichannel optical communication. In particular, the significantly enhanced second-order sideband and light delay or advance, which are potentially useful for the precise sensing [37,54– 58] and optical storage or quantum communication [21-23], respectively, can be achieved by controlling the dark-mode effect.

The rest of this paper is organized as follows. In Sec. II, we describe the model and present the OMIT solution. In Sec. III, we analyze OMIT, second-order sideband, and group delay by using the dark-mode effect. In Sec. IV, we extend our studies to the case of the *N*-mechanical-mode optomechanical system. Finally, we provide a brief conclusion in Sec. V. For keeping the completeness of this work, we present four appendices, in which we show some algebra equations and sideband parameters (Appendix A), the detailed calculation of the effective mechanical decay rate (Appendix B), the detailed derivation of Eq. (36) (Appendix C), and a possible experimental realization and a derivation of a phase-dependent phonon-hopping interaction between two mechanical resonators (Appendix D).

II. MODEL AND SOLUTIONS

We consider a three-mode loop-coupled optomechanical system, which consists of an optical cavity mode coupled to two mechanical modes via radiation-pressure force, as shown in Fig. 1(a). The two mechanical modes are coupled to each other via a phase-dependent phonon-phonon interaction with coupling strength η and modulation phase θ . Note that this model can be implemented in either photonic crystal optomechanical setups [59] or circuit electromechanical systems [60,61]. In the photonic crystal optomechanical systems, the phase-dependent phonon-exchange interaction has been suggested by using two assistant cavity fields [59]. The phasedependent phonon-exchange coupling in the circuit electromechanical setups can be indirectly induced by coupling the two mechanical modes to a charge qubit (see Appendix D) [62]. A pump laser with frequency ω_L and field amplitude $\varepsilon_L = \sqrt{2\kappa P_L/\hbar\omega_L}$, and a probe laser with frequency ω_p and amplitude $\varepsilon_p = \sqrt{2\kappa P_p/\hbar\omega_p}$ are applied to this system. Here, P_L and P_p are the control and probe light powers, respectively, and κ is the cavity-field decay rate. In a rotating frame defined by the unitary transformation operator $\exp(-i\omega_L a^{\dagger} a t)$, the

system Hamiltonian reads as (with $\hbar = 1$)

$$H = H_0 + H_{\rm int} + H_{\rm dr},\tag{1}$$

with

$$H_0 = \Delta_c a^{\dagger} a + \sum_{l=1}^2 \omega_l b_l^{\dagger} b_l, \qquad (2a)$$

$$H_{\text{int}} = \sum_{l=1}^{2} g_l a^{\dagger} a(b_l + b_l^{\dagger}) + \eta(e^{i\theta} b_1^{\dagger} b_2 + e^{-i\theta} b_2^{\dagger} b_1),$$
(2b)

$$H_{\rm dr} = i(\varepsilon_L a^{\dagger} + \varepsilon_p a^{\dagger} e^{-i\Omega t} - {\rm H.c.}), \qquad (2c)$$

where $a(a^{\dagger})$ and $b_l(b_l^{\dagger})$ are, respectively, the annihilation (creation) operators of the cavity mode and the *l*th mechanical mode, with the corresponding resonance frequencies ω_c and ω_l . The optomechanical and the phonon-phonon interaction strengths are denoted by g_l and η , respectively. The cavity-pump and probe-pump detunings are given by

$$\Delta_c = \omega_c - \omega_L, \quad \Omega = \omega_p - \omega_L. \tag{3}$$

When the control and probe lasers are simultaneously injected into the cavity, we consider the high-order sidebands with frequencies $\omega_L \pm n\Omega$ (*n* represents the order of the sidebands) induced by the nonlinear optomechanical interaction. The frequency spectrum of the optomechanical system is shown in Fig. 1(b), in which the first-order sidebands with frequencies $\omega_L \pm \Omega$ correspond to the anti-Stokes and Stokes fields and the sidebands with frequencies $\omega_L \pm 2\Omega$ denote the secondorder upper or lower sideband, respectively. The Langevin equations for the operators of the optical and mechanical modes can be written as

$$\dot{a} = -(\kappa + i\Delta_c)a - i\sum_{l=1,2} g_l a(b_l + b_l^{\dagger}) + \varepsilon_L + \varepsilon_p e^{-i\Omega t} + \sqrt{2\kappa} a_{\rm in},$$
(4a)

$$\dot{b}_1 = -(\gamma_1 + i\omega_1)b_1 - ig_1a^{\dagger}a - i\eta e^{i\theta}b_2 + \sqrt{2\gamma_1}b_{1,\text{in}},$$
 (4b)

$$\dot{b}_2 = -(\gamma_2 + i\omega_2)b_2 - ig_2a^{\dagger}a - i\eta e^{-i\theta}b_1 + \sqrt{2\gamma_2}b_{2,\text{in}},$$
 (4c)

where κ and γ_l are the decay rates of the cavity mode and the *l*th mechanical mode, respectively. The operators a_{in} and

$$b_{l,\text{in}}$$
 are, respectively, the noise operators of the cavity-field
mode and the *l*th mechanical mode. We express the variables
in Eq. (4) as the sum of their steady-state values and quantum
fluctuations, namely, $a = \alpha + \delta a$ and $b_{l=1,2} = \beta_l + \delta b_l$. As
the control field is assumed to be sufficiently strong, all
the operators can be identified with their expectation values
[22,36]. Then, the equations for the expectation values of
these fluctuations can be obtained as [22,36]

$$\delta \dot{a} = -(\kappa + i\Delta)\delta a - \sum_{l=1,2} ig_l \alpha (\delta b_l + \delta b_l^{\dagger}) - \sum_{l=1,2} ig_l \delta a (\delta b_l + \delta b_l^{\dagger}) + \varepsilon_p e^{-i\Omega t}, \qquad (5a)$$

$$\delta \dot{b}_{1} = -(\gamma_{1} + i\omega_{1})\delta b_{1} - ig_{1}(\alpha^{*}\delta a + \alpha\delta a^{\dagger})$$
$$-i\eta e^{i\theta}\delta b_{2} - ig_{1}\delta a^{\dagger}\delta a, \qquad (5b)$$
$$\delta \dot{b}_{2} = -(\gamma_{2} + i\omega_{2})\delta b_{2} - ig_{2}(\alpha^{*}\delta a + \alpha\delta a^{\dagger})$$

$$-i\eta e^{-i\theta}\delta b_1 - ig_2\delta a^{\dagger}\delta a, \qquad (5c)$$

where the quantum noise terms $(a_{in} \text{ and } b_{l=1,2,in})$ can be ignored due to their zero mean values. The parameter $\Delta = \Delta_c + \sum_{l=1,2} g_l(\beta_l + \beta_l^*)$ denotes the effective detuning of the cavity field shifted by the optomechanical interactions, and the steady-state mean values are given by $\alpha = \varepsilon_L/(\kappa + i\Delta)$, $\beta_1 = -i(g_1|\alpha|^2 + \eta e^{i\theta}\beta_2)/(\gamma_1 + i\omega_1)$, and $\beta_2 = -i(g_2|\alpha|^2 + \eta e^{-i\theta}\beta_1)/(\gamma_2 + i\omega_2)$. In Eq. (5), the nonlinear terms, such as $\delta a^{\dagger} \delta a$ and $\delta a(\delta b_l + \delta b_l^{\dagger})$, are kept to generate the required second-order sidebands, and the higher-order sidebands are neglected. The response of the probe field for the first- and second-order sidebands to the system can be exhibited by using the following ansatz [22,36]:

$$\delta a = A_1^- e^{-i\Omega t} + A_1^+ e^{i\Omega t} + A_2^- e^{-2i\Omega t} + A_2^+ e^{2i\Omega t}, \qquad (6a)$$

$$\delta b_1 = B_{1,1}^- e^{-i\Omega t} + B_{1,1}^+ e^{i\Omega t} + B_{1,2}^- e^{-2i\Omega t} + B_{1,2}^+ e^{2i\Omega t}, \quad (6b)$$

$$\delta b_2 = B_{2,1}^- e^{-i\Omega t} + B_{2,1}^+ e^{i\Omega t} + B_{2,2}^- e^{-2i\Omega t} + B_{2,2}^+ e^{2i\Omega t}.$$
 (6c)

By substituting Eq. (6) into Eq. (5), we obtain 12 algebraic equations which are divided into two groups, as shown in Appendix A. After solving Eqs. (A1) and (A2), we obtain the coefficients of the first- and second-order upper sidebands, which can describe the linear and nonlinear responses of the system. The coefficient of the first-order upper sideband is

$$A_{1}^{-} = \frac{T_{2}^{(1)}[-\kappa + i(\Delta + \Omega)] - 2i|\alpha|^{2} \left(g_{2}^{2} T_{3,1}^{(1)} + g_{1}^{2} T_{3,2}^{(1)}\right) - 4iT_{1}^{(1)}g_{1}g_{2}\eta|\alpha|^{2}\cos\theta}{-T_{2}^{(1)}[\Delta^{2} + (\kappa - i\Omega)^{2}] + 4|\alpha|^{2}\Delta\left(g_{2}^{2} T_{3,1}^{(1)} + g_{1}^{2} T_{3,2}^{(1)}\right) + 8T_{1}^{(1)}g_{1}g_{2}\eta|\alpha|^{2}\Delta\cos\theta}\varepsilon_{p},$$
(7)

and that of the second-order upper sideband is

$$A_{2}^{-} = \frac{\chi_{1}(\Omega)(A_{1}^{+})^{*} + i[g_{1}(B_{1,1}^{-} + (B_{1,1}^{+})^{*}) + g_{2}(B_{2,1}^{-} + (B_{2,1}^{+})^{*})]A_{1}^{-}}{\chi_{2}(\Omega) - [\kappa + i(\Delta - 2\Omega)]},$$
(8)

with

χ1

$$(A_{1}^{+})^{*} = \frac{-2(\alpha^{*})^{2}[\kappa - i(\Delta + \Omega)] [2g_{1}g_{2}\eta T_{1}^{(1)}\cos\theta + (g_{1}^{2}T_{3,2}^{(1)} + g_{2}^{2}T_{3,1}^{(1)})]}{(i\kappa + \Delta + \Omega) (\{4|\alpha|^{2}\Delta(g_{2}^{2}T_{3,1}^{(1)} + g_{1}^{2}T_{3,2}^{(1)}) - T_{2}^{(1)}[\Delta^{2} + (\kappa - i\Omega)^{2}]\} + 8g_{1}g_{2}\eta|\alpha|^{2}T_{1}^{(1)}\Delta\cos\theta)}\varepsilon_{p},$$
(9)

$$(\Omega) = \frac{2i\alpha \{g_1\alpha(B_{1,1}) + B_{1,1}\} + g_2\alpha(B_{2,1}) + B_{2,1}\} - A_1(i\kappa + \Delta + 2\Omega) [2g_1g_2\eta I_1 - \cos \theta + (g_1I_{3,2} + g_2I_{3,1})]}{[(i\kappa + \Delta + 2\Omega)T_2^{(2)} - 2|\alpha|^2 (g_1^2T_{3,2}^{(2)} + g_2^2T_{3,1}^{(2)})] - 4g_1g_2\eta|\alpha|^2 T_1^{(2)}\cos\theta}, \quad (10)$$

$$\chi_2(\Omega) = \left[\left[\kappa - i(\Delta + 2\Omega) \right]^{-1} - iT_2^{(2)} \left[2|\alpha|^2 \left(g_1^2 T_{3,2}^{(2)} + g_2^2 T_{3,1}^{(2)} + 2g_1 g_2 \eta T_1^{(2)} \cos \theta \right) \right]^{-1} \right]^{-1}.$$
(11)

Other parameters used are displayed in Appendix A. In the following, by using the input-output relation [63], we obtain the transmission rate of the probe field and the efficiency of the second-order sideband [22,23,36,38]:

$$|t_p|^2 = \left|1 - \frac{\kappa}{\varepsilon_p} A_1^-\right|^2, \quad \Lambda_p = \left|-\frac{\kappa}{\varepsilon_p} A_2^-\right|. \tag{12}$$

The associated transmission group delay caused by the rapid phase dispersion is given by [23]

$$\tau_1 = \frac{d \arg(t_p)}{d\Omega}|_{\Omega = \omega_l},\tag{13}$$

where arg(x) takes the argument of the complex number *x*.

III. OMIT BY CONTROLLING THE DARK MODE

A. Dark-mode effect and its breaking

In this section, we analyze the dark-mode effect and its breaking in the three-mode optomechanical system. In the strong-driving regime, the linearized optomechanical Hamiltonian can be inferred from the equations of motion for the quantum fluctuations in Eq. (5). Under the rotating-wave approximation (RWA), the linearized optomechanical Hamiltonian takes the form

$$H_{\text{RWA}} = \Delta \delta a^{\dagger} \delta a + \omega_1 \delta b_1^{\dagger} \delta b_1 + \omega_2 \delta b_2^{\dagger} \delta b_2 + \eta (e^{i\theta} \delta b_1^{\dagger} \delta b_2 + e^{-i\theta} \delta b_2^{\dagger} \delta b_1) + G_1 (\delta a \delta b_1^{\dagger} + \delta b_1 \delta a^{\dagger}) + G_2 (\delta a \delta b_2^{\dagger} + \delta b_2 \delta a^{\dagger}),$$
(14)

where Δ is the effective detuning of the cavity field shifted by the optomechanical interactions, and $G_l = g_l |\alpha|$ is the linearized optomechanical coupling strength.

To investigate the dark-mode effect in this system, we first consider the case where the phase-modulated phononexchange interaction is absent, i.e., $\eta = 0$. In this case, due to an optically mediated coupling, the two mechanical modes are hybridized into a bright mode and a dark mode, which are expressed as

$$B_{+} = \frac{1}{\sqrt{G_{1}^{2} + G_{2}^{2}}} (G_{1}\delta b_{1} + G_{2}\delta b_{2}),$$
(15)

$$B_{-} = \frac{1}{\sqrt{G_1^2 + G_2^2}} (G_2 \delta b_1 - G_1 \delta b_2).$$
(16)

Then, the Hamiltonian in Eq. (14) can be rewritten with the two hybridization modes B_{\pm} as

$$H_{\text{Hyb}} = \Delta \delta a^{\dagger} \delta a + \omega_{+} B_{+}^{\dagger} B_{+} + \omega_{-} B_{-}^{\dagger} B_{-} + \zeta (B_{+}^{\dagger} B_{-} + B_{-}^{\dagger} B_{+})$$

$$+G_{+}(\delta aB_{+}^{\dagger}+B_{+}\delta a^{\dagger}), \qquad (17)$$

where we introduce these parameters

$$\omega_{+} = \frac{G_{1}^{2}\omega_{1} + G_{2}^{2}\omega_{2}}{G_{1}^{2} + G_{2}^{2}}, \qquad \omega_{-} = \frac{G_{2}^{2}\omega_{1} + G_{1}^{2}\omega_{2}}{G_{1}^{2} + G_{2}^{2}}, \quad (18)$$

$$\zeta = \frac{G_1 G_2 (\omega_1 - \omega_2)}{G_1^2 + G_2^2}, \qquad G_+ = \sqrt{G_1^2 + G_2^2}.$$
(19)

It can be seen from Eq. (19) that when $\omega_1 = \omega_2$, we have $\zeta = 0$, and thus the mode B_- is decoupled from both the cavity

To break the dark-mode effect, a phase-dependent phononexchange interaction between the two mechanical modes is considered. Then, by introducing two new bosonic modes \tilde{B}_+ and \tilde{B}_- defined as

$$\tilde{B}_{+} = f\delta b_1 - e^{i\theta}h\delta b_2, \quad \tilde{B}_{-} = e^{-i\theta}h\delta b_1 + f\delta b_2, \quad (20)$$

Hamiltonian (14) becomes

$$H_{\rm D} = \Delta \delta a^{\dagger} \delta a + \tilde{\omega}_{+} \tilde{B}^{\dagger}_{+} \tilde{B}_{+} + \tilde{\omega}_{-} \tilde{B}^{\dagger}_{-} \tilde{B}_{-} + (\tilde{G}^{*}_{+} \delta a \tilde{B}^{\dagger}_{+} + \tilde{G}_{+} \tilde{B}_{+} \delta a^{\dagger}) + (\tilde{G}^{*}_{-} \delta a \tilde{B}^{\dagger}_{-} + \tilde{G}_{-} \tilde{B}_{-} \delta a^{\dagger}), \qquad (21)$$

where the coupling strengths and resonance frequencies are given by

$$\tilde{G}_{+} = fG_{1} - e^{-i\theta}hG_{2}, \qquad \tilde{G}_{-} = e^{i\theta}hG_{1} + fG_{2},
\tilde{\omega}_{\pm} = \frac{1}{2}(\omega_{1} + \omega_{2} \pm \sqrt{(\omega_{1} - \omega_{2})^{2} + 4\eta^{2}}), \qquad (22)$$

with $f = |\tilde{\omega}_{-} - \omega_{1}|/\sqrt{(\tilde{\omega}_{-} - \omega_{1})^{2} + \eta^{2}}$ and $h = \eta f/(\tilde{\omega}_{-} - \omega_{1})$. When the two mechanical modes have the same frequencies, $\omega_{1} = \omega_{2} = \omega_{m}$, and coupling strengths, $G_{1} = G_{2} = G$, the coupling strengths in Eq. (22) can be simplified as

$$\tilde{G}_{+} = \sqrt{2}G(1 + e^{-i\theta})/2, \quad \tilde{G}_{-} = \sqrt{2}G(1 - e^{i\theta})/2.$$
 (23)

It follows from Eq. (23) that, when $\theta = n\pi$ for an integer *n*, the cavity mode *a* is decoupled from one of the two hybridized mechanical modes \tilde{B}_- (for an even number *n*) and \tilde{B}_+ (for an odd number *n*). These features mean that the dark-mode effect can be broken by tuning the modulation phase $\theta \neq n\pi$.

B. OMIT and second-order sidebands in the presence of the dark mode

In the above sections, we have derived the transmission rate of the probe field and the efficiency of the second-order sideband, and have analyzed how to control the dark-mode effect in the two-mechanical-mode optomechanical system. Now, we study OMIT and its second-order sidebands in the presence of the dark mode ($\eta = 0$). To make the results feasible in experiments, we use the parameters realized in recent experiments [64,65], i.e., $\lambda = 1064$ nm, L = 25 mm, $\kappa = 2\pi \times 215$ kHz, $\omega_{l=1,2} = \omega_m = 2\pi \times 947$ kHz, $m_{l=1,2} =$ 145 ng, $Q_l = 6700$ ($\gamma_l = \omega_l/Q_l$), and $\varepsilon_p = 0.05\varepsilon_L$.

A single OMIT is due to the destructive interference between the probe field and the anti-Stokes scattering stimulated by the red-sideband driving [21–23]. Correspondingly, in the case of the two mechanical modes, there exist two routes of destructive interferences which lead to two transparency windows in the OMIT spectrum. However, we find some counterintuitive phenomena in the presence of the dark mode $(\eta = 0)$ shown in Fig. 2. There exists only one transparency window and one second-order sideband window in the twodegenerate-mechanical-mode optomechanical system [see the red dashed lines in Figs. 2(b) and 2(c)]. Physically, this is due to the fact that the bright mode plays an effective role in the destructive interference and the dark mode is decoupled from the system. Moreover, compared to a standard optomechanical system consisting of a cavity mode and a single mechanical mode, it exhibits not only a multifold amplified OMIT window, but also a significantly enhanced second-order sideband



FIG. 2. In the presence of the dark-mode effect ($\eta = 0$), (a) the linewidth Γ_{eff} of the OMIT window as a function of the pump power *P* for a standard optomechanical system ($g_2 = 0$, blue solid lines) and a two-degenerate-mechanical-mode optomechanical system ($g_1 = g_2$, red dashed lines). (b) The transmission rate $|t_p|^2$ of the probe light, and (c) the efficiency $\Lambda_p(\%)$ of the second-order sideband as a function of the probe-pump detuning Ω when $P_L = 1.5$ mW for a standard optomechanical system (blue solid lines) and a two-degenerate-mechanical-mode optomechanical system (red dashed lines). Here we take the normalized cavity-pump detuning $\Delta/\omega_m = 1$.

in the two-degenerate-mechanical-mode optomechanical system. In a standard optomechanical system, the width of the OMIT window is related to the effective mechanical decay rate given by [21-23]

$$\Gamma_{\rm eff} = \gamma_m + \gamma_{\rm opt}, \qquad (24)$$

where $\gamma_{\text{opt}} \approx G^2/\kappa$ stands for the optomechanically induced mechanical decay rate for a single mechanical mode (with the mechanical decay rate γ_m), and $G = g|\alpha|$ ($|\alpha|^2$ is the intracavity photon number) denotes the linearized optomechanical coupling strength.

For the two-degenerate-mechanical-mode optomechanical system, we can also obtain the effective mechanical damping rate of the bright mode by adiabatically eliminating the cavity field in the large-cavity-decay regime (see Appendix B), which is given by

$$\Gamma_{\rm eff} = \gamma_m + 2\gamma_{\rm opt}.$$
 (25)

Because of the parameter relation $\gamma_{opt} \gg \gamma_m$ [66–69] in the weak-coupling regimes, we can see from Eq. (25) that the linewidth of the OMIT window is approximately twice amplified in comparison with the case of the standard optomechanical system [see Eq. (24)]. Correspondingly, we plot the linewidth Γ_{eff} of the OMIT window as a function of the pump power P, as shown in Fig. 2(a). On the one hand, the linewidth Γ_{eff} of the OMIT window is significantly widened with the increase of the pump power P in both the degenerate-mechanical-mode and prototype cases. On the other hand, the OMIT window in the degenerate case [see the red dashed line in Fig. 2(a)] is approximately twofold amplified in comparison with the prototype case [see the blue solid line in Fig. 2(a) due to the existence of the dark-mode effect. This physical mechanism can be potentially used in the optical information storage within a large-frequency bandwidth.

C. OMIT and the second-order sidebands by breaking the dark mode

Since the above counterintuitive results are due to the dark-mode effect, it is natural to ask the question whether we can break the dark-mode effect to further explore the OMIT and the second-order sidebands. Thus, we compare the

dark-mode-breaking case ($\eta = 0.05\omega_m$ and $\theta = \pi/2$) with the dark-mode-unbreaking case ($\eta = 0$), as shown in Fig. 3. In the dark-mode-unbreaking case, a single transparency window emerges around $\Omega = \omega_m$, when $\Delta = \omega_{l=1,2} = \omega_m$ [see Fig. 3(a)].

In contrast, when switching to the dark-mode-breaking case, a single OMIT window with linewidth $(\gamma_m + 2\gamma_{opt})$ is divided into two symmetrical narrow OMIT windows around $\Omega \approx 0.95\omega_m$ and $\Omega \approx 1.05\omega_m$ with the linewidth $(\gamma_m + \gamma_{opt})$. Physically, the two degenerate mechanical modes are dressed



FIG. 3. (a), (b) The transmission rate $|t_p|^2$ of the probe light and (c), (d) the efficiency $\Lambda_p(\%)$ of the second-order sideband as a function of the probe-pump detuning Ω and the pump light power P_L in both the dark-mode-unbreaking ($\eta = 0$) and -breaking ($\eta =$ $0.05\omega_m$ and $\theta = \pi/2$) cases. (e) $|t_p|^2$ and (f) $\Lambda_p(\%)$ as functions of Ω when $P_L = 1.5$ mW. Other parameters used are $g_1 = g_2$ and $\Delta/\omega_m = 1$.



FIG. 4. (a) The transmission rate $|t_p|^2$ of the probe light as a function of the phonon-phonon coupling η and the probe-pump detuning Ω when $\theta = \pi/2$. (b) $|t_p|^2$ versus θ and Ω when $\eta/\omega_m = 0.05$. The white dashed lines correspond to $\theta = 0, \pi, \text{ and } 2\pi$. (c) $|t_p|^2$ as a function of Ω when η takes different values: $\eta/\omega_m = 0$ (black solid line), 0.02 (green dashed-dotted line), and 0.1 (red dotted line), under $\theta = \pi/2$. (d) $|t_p|^2$ versus θ at the two transparency windows $\Omega/\omega_m = 0.95$ (left) and $\Omega/\omega_m = 1.05$ (right), when $\eta/\omega_m = 0.05$. Here, the switch points (SP) stand for the cross between the left and right transmission rates. Here we choose $P_L = 1.5$ mW, $g_1 = g_2$, and $\Delta/\omega_m = 1$.

into two well-separated dressed modes by breaking the darkmode effect (see Sec. III A), and there exist the two routes of destructive interference which result in the two transparency windows even in the degenerate-mechanical-mode optomechanical system. Meanwhile, the second-order sideband shows that a single local minimum value is tuned to two symmetrical minimum values around $\Omega \approx 0.95\omega_m$ and $\Omega \approx$ $1.05\omega_m$. This is because the anti-Stokes field is resonantly enhanced, which promotes OMIT and leads to the suppression of the second-order sidebands.

Now, we investigate the sensitive effects of the phononphonon coupling strength η and the modulation phase θ on the transmission rates of the probe light and second-order sidebands. It is shown in Figs. 4(a) and 4(c) that the switching between a single wide transparency window and two narrow transparency windows can be realized by tuning η for $\theta = \pi/2$. The distance of the two OMIT windows is mainly determined by the coupling strength between the two phonon modes [27]. Meanwhile, the transmission rate of the probe light depends on θ , as shown in Figs. 4(b) and 4(d). In the region $0 < \theta < \pi$ ($\pi < \theta < 2\pi$), the left (right) OMIT window always becomes much deeper and broader while the right (left) one becomes weaker with the modulation phase θ , e.g., the right (left) transparency even becomes completely absorbed by decreasing the phase down near $\theta = \pi$ (2π).

Furthermore, it is shown in Fig. 4(d) how the OMIT at the two transparency points $\Omega/\omega_m = 0.95$ (left) and $\Omega/\omega_m = 1.05$ (right) is modulated by the phase θ . There are two switch points (SP) (i.e., the symmetrical transparency points $|t_p|_{\text{left}}^2 =$



FIG. 5. The efficiency $\Lambda_p(\%)$ of the second-order sideband as a function of the phonon-phonon coupling η and the probe-pump detuning Ω when the modulation phase takes different values (a) $\theta =$ 0 and (b) $\theta = \pi/2$. (c) $\Lambda_p(\%)$ versus θ and Ω when $\eta/\omega_m = 0.05$; the white dashed lines correspond to $\theta = 0$, π , and 2π . (d) $\Lambda_p(\%)$ versus Ω when η takes different values: $\eta/\omega_m = 0.1$ and 0.2, for $\theta =$ π . Here, we consider $P_L = 1.5$ mW, $g_1 = g_2$, and $\Delta/\omega_m = 1$.

 $|t_p|_{\text{right}}^2$ when the modulation phase is given by $\theta = \pi/2$ and $3\pi/2$. It obviously shows that the OMIT performance of the left window (the blue solid line) is better than that of the right one (the red dashed line) between the two SP (i.e., $\pi/2 < \theta < 3\pi/2$), while the opposite situation that the OMIT performance of the right one (the red dashed line) is better than that of the left window (the blue solid line) appears in the rest region (i.e., $0 < \theta < \pi/2$ and $3\pi/2 < \theta < 2\pi$). Moreover, one transparency window is lost in the presence of the dark mode ($\theta = n\pi$ for an integer *n*). Thus, the periodically controllable and switchable OMIT can be performed by tuning the modulation phase θ . Physically, the combination of the θ -dependent phonon-phonon interaction with the optomechanical couplings breaks the dark-mode effect and splits the OMIT spectrum with a single window.

Correspondingly, the dependence of the second-order sideband on the θ -dependent phonon-phonon interaction is shown in Fig. 5. We can see from Fig. 5(a) that due to the existence of the dark-mode effect for $\theta = 0$ (see Sec. III A), there always exists only one local minimum window even with the increase of the phonon-phonon coupling η . When $\theta =$ $\pi/2$ in Fig. 5(b), the local minimum window is split into two windows in the second-order sideband by tuning η , in which the second-order sideband is significantly enhanced. For example, the maximum efficiency of the second-order sidebands is about 9% [Fig. 5(b)], which is much higher than that of the case $\theta = 0$ [Fig. 5(a)]. The θ -dependent secondorder sideband is exhibited in Fig. 5(c) when $\eta/\omega_m = 0.05$. We can see from Fig. 5(c) that in the region $0 < \theta < \pi$ $(\pi < \theta < 2\pi)$, the maximum efficiency becomes much larger (weaker). Especially, the highest efficiency emerges when the modulation phase takes $\theta = \pi$. This indicates that the second-order sidebands shown in Fig. 5(d) can be significantly enhanced by increasing η when $\theta = \pi$ [e.g., Λ_p is about



FIG. 6. (a) Optical group delay τ_1 as a function of the applied pump power P_L for the dark-mode-unbreaking case ($\eta = 0$) at its transmission window $\Omega/\omega_m = 1$ (the black solid line) and the dark-mode-breaking case ($\eta = 0.05\omega_m$ and $\theta = \pi/2$) at the left OMIT window $\Omega = 0.95\omega_m$ (the blue dashed-dotted line) and the right OMIT window $\Omega = 1.05\omega_m$ (the red dashed line), respectively. (b) At the left OMIT window $\Omega = 0.95\omega_m$ and (c) the right OMIT window $\Omega = 1.05\omega_m$, τ_1 versus θ when $\eta = 0.05\omega_m$ and $P_L = 1.5$ mW. The insets in (b) and (c) are the zoomed-in plots of τ_1 as a function of the modulation phase θ . Here, we consider $g_1 = g_2$ and $\Delta/\omega_m = 1$.

12.5%, which is three times larger than that of $\theta = 0$ shown in Fig. 5(a)]. The significant enhancement of the second-order sidebands is important for its potential applications in weaksignal sensing [37,54–58], e.g., precise sensing of weak forces [54] and charges [37,55].

D. Controllable group delay

In general, the optical group delay can be created due to the fact that the dispersion curve varies drastically with the frequency within the OMIT window [23], which can be useful in optical information storage without absorption. In the dark-mode-unbreaking case ($\eta = 0$), the slow-light effect emerges at its transmission window $\Delta = \omega_m$ [the black solid line in Fig. 6(a)]. When breaking the dark-mode effect (e.g., $\eta = 0.05\omega_m$ and $\theta = \pi/2$), the group delay time significantly increases at both the left transmission window $\Delta = 0.95\omega_m$ [the blue dashed-dotted line in Fig. 6(a)] and the right transmission window $\Delta = 1.05\omega_m$ [the red dashed line in Fig. 6(a)] in comparison with the dark-mode-unbreaking case. This implies that we can significantly develop the storage of the signal light by breaking the dark-mode effect.

In the dark-mode-breaking case, we also plot the optical group delay τ_1 as a function of the modulation phase θ at the left transmission window $\Delta = 0.95\omega_m$ [the blue solid line in Fig. 6(b)] and the right transmission window $\Delta = 1.05\omega_m$ [the red solid line in Fig. 6(c)]. The slow-to-fast and fast-to-slow light effects can emerge at the two OMIT windows by tuning the modulation phase θ [see the insets in Figs. 6(b)] and 6(c)]. The delay time of the signal light can even reach 520 μ s for the left window [see Fig. 6(b)] and 620 μ s for the right window [see Fig. 6(c)], which is approximatively 100 times longer for the group delay in comparison with the dark-mode-unbreaking case. These results can lead to achieving ultraslowing or ultra-advancing signals, which can be used in optical storage or quantum communication [21–23].

IV. TUNABLE OMIT IN AN N-MECHANICAL-MODE OPTOMECHANICAL SYSTEM

In this section, we extend our scheme to investigate the OMIT in an *N*-mechanical-mode optomechanical system, in

which a cavity mode is optomechanically coupled to $N \ge 3$ (for an integer *N*) mechanical modes. And the neighboring mechanical modes are coupled to each other through the phase-dependent phonon-phonon interactions H_j for j = 1-(N-1) with the coupling strength η_j and modulation phase θ_j [see Fig. 1(c)]. Thus, the Hamiltonian of the net-coupled optomechanical system can be written, in a frame rotating at the driving frequency, as

$$H_{I} = \Delta_{c}a^{\dagger}a + \sum_{j=1}^{N} \omega_{j}b_{j}^{\dagger}b_{j} + \sum_{j=1}^{N} g_{j}a^{\dagger}a(b_{j} + b_{j}^{\dagger}) + i(\varepsilon_{L}a^{\dagger} + \varepsilon_{p}a^{\dagger}e^{-i\Omega t} - \text{H.c.}) + \sum_{j=1}^{N-1}H_{j}, \quad (26)$$

with

$$H_{j} = \eta_{j} (e^{i\theta_{j}} b_{j}^{\dagger} b_{j+1} + e^{-i\theta_{j}} b_{j+1}^{\dagger} b_{j}), \qquad (27)$$

where $\Delta_c = \omega_c - \omega_L$ is the detuning of the cavity frequency ω_c with respect to the driving frequency ω_L . We consider the strong driving case of the cavity and then perform the linearization procedure in this system. Meanwhile, the couplings between the cavity field and these mechanical modes are much smaller than the mechanical frequency and hence the RWA is justified. In this case, the linearized optomechanical Hamiltonian under the RWA is given by

$$H_{\text{RWA}} = \Delta \delta a^{\dagger} \delta a + \sum_{j=1}^{N} \omega_j \delta b_j^{\dagger} \delta b_j + \sum_{j=1}^{N} G_j (\delta a^{\dagger} \delta b_j + \delta b_j^{\dagger} \delta a) + \sum_{j=1}^{N-1} \eta_j (e^{i\theta_j} \delta b_j^{\dagger} \delta b_{j+1} + e^{-i\theta_j} \delta b_{j+1}^{\dagger} \delta b_j), \qquad (28)$$

where $\Delta = \Delta_c + \sum_{j=1}^N g_j(\beta_j + \beta_j^*)$ is the normalized driving detuning after the linearization, and $G_j = g_j |\alpha|$ is the linearized optomechanical coupling strength between the *j*th mechanical mode and the cavity-field mode. For convenience, we consider the case where all the mechanical modes have the same resonance frequencies ($\omega_j = \omega_m$), optomechanical



FIG. 7. (a) In the presence of the dark-mode effect ($\eta = 0$), the transmission rate $|t_p|^2$ of the probe light as a function of the probe-pump detuning Ω when N = 1 (the black solid curve), N = 2 (the blue dashed curve), N = 3 (the red dashed-dotted curve), and N = 4 (the dotted olive curve). The transmission rate $|t_p|^2$ of the probe light versus Ω when (b) N = 3 and (c) N = 4, for the dark-mode-unbreaking ($\eta = 0$, the solid lines) and -breaking ($\eta/\omega_m = 0.05$ and $\theta_1 = \pi/2$, the dashed-dotted lines) cases. Other parameters used are $P_L = 1.5$ mW, $g_1 = g_2$, and $\Delta/\omega_m = 1$.

coupling strengths ($G_j = G$), and phonon-exchange coupling strengths ($\eta_j = \eta$).

In the absence of the phase-dependent phonon-phonon interaction between the adjacent mechanical modes $(H_j = 0)$, the system exists a bright mode $B_+ = \sum_{j=1}^{N} \delta b_j / \sqrt{N}$ and (N-1) dark modes decoupled from the cavity-field mode *a*. As a result, there exists only one route of destructive interference in the *N*-degenerate-mechanical-mode optomechanical system. To see this, we plot the transmission rate of the probe light $|t_p|^2$ as a function of the probe-pump detuning Ω with the increase of the number of mechanical modes, as shown in Fig. 7(a). It is obviously shown that the OMIT window can be amplified by increasing the number of mechanical modes. Physically, this can be explained by the enhanced effective mechanical decay rate of the bright mode, which is given by

$$\Gamma_{\text{effect}} = \gamma_m + N\gamma_{\text{opt}}.$$
(29)

Equation (29) shows that, in comparison with the typical optomechanical system, approximately N times amplification can be observed for the OMIT linewidth [see Fig. 7(a)]. This study provides a route to realize the optical transmission within a large-frequency bandwidth.

To control the dark-mode effect in this *N*-mechanicalmode optomechanical system, we introduce the phasedependent phonon-phonon couplings $(H_j \neq 0)$ between the neighboring mechanical modes. We can diagonalize the Hamiltonian of these coupled mechanical modes as [70,71]

$$H_{\text{mrt}} = \omega_m \sum_{j=1}^N \delta b_j^{\dagger} \delta b_j + \eta \sum_{j=1}^{N-1} (e^{i\theta_j} \delta b_j^{\dagger} \delta b_{j+1} + e^{-i\theta_j} \delta b_{j+1}^{\dagger} \delta b_j)$$
$$= \sum_{k=1}^N \Omega_k B_k^{\dagger} B_k, \qquad (30)$$

where B_k describes the *k*th diagonalized mechanical mode with the resonance frequency

$$\Omega_k = \omega_m + 2\eta \, \cos\left(\frac{k\pi}{N+1}\right). \tag{31}$$

The relationship between δb_i and B_k is given by

$$\delta b_j = \begin{cases} \frac{1}{A} \sum_{k=1}^N \sin\left(\frac{k\pi}{N+1}\right) B_k, & j = 1\\ \frac{1}{A} e^{-i \sum_{\nu=1}^{j-1} \theta_\nu} \sum_{k=1}^N \sin\left(\frac{jk\pi}{N+1}\right) B_k, & j \ge 2 \end{cases}$$
(32)

where $A = \sqrt{(N+1)/2}$. The Hamiltonian in Eq. (28) can be rewritten with these diagonalized mechanical normal modes as

$$H_{\rm RWA} = \Delta \delta a^{\dagger} \delta a + \sum_{k=1}^{N} \Omega_k B_k^{\dagger} B_k + H_{\rm om}, \qquad (33)$$

where the Hamiltonian $H_{\rm om}$ with the optomechanical coupling reads as

$$H_{\rm om} = \frac{G}{A} \sum_{k=1}^{N} \left[\sin\left(\frac{k\pi}{N+1}\right) + \sum_{j=2}^{N} e^{i\varphi} \sin\left(\frac{jk\pi}{N+1}\right) \right] aB_k^{\dagger} + \text{H.c.}, \qquad (34)$$

with

$$\varphi = \sum_{\nu=1}^{j-1} \theta_{\nu}.$$
 (35)

We can find from Eq. (34) that the function of these modulation phases is dominated by Eq. (35). Hence, the dark-modebroken effect can be realized by modulating a single phase. For simplicity, below we consider $\theta_j = 0$ for $j = 2 \cdot (N - 1)$. For an even k, the interaction between the cavity mode and the mechanical normal mode B_k can be expressed as (see Appendix C)

$$H_{\rm ck} = \frac{G}{A} \left[\left(1 - e^{i\theta_1} \right) \sin\left(\frac{k\pi}{N+1}\right) \right] aB_k^{\dagger} + \text{H.c.}, \quad k = \text{even.}$$
(36)

We see from Eq. (36) that when $\theta_1 = 2n\pi$, the coupling strength between the *k*th mechanical normal mode $B_{k=\text{even}}$ and the cavity mode *a* is equal to zero. In this case, all the even normal modes are decoupled from the cavity field. Thus, the dark modes can be broken by choosing a proper parameter $(\theta_1 \neq 2n\pi)$. When switching from the dark-mode-unbreaking to breaking cases, a single transparency window is divided into multiple transparency windows, and the number of the transparency windows depends on the number of mechanical modes [see Figs. 7(b) and 7(c)]. Physically, this is the result of the destructive interferences between the weak probe field and the multiple anti-Stokes fields which are generated by the interactions of the strong-coupling field and the dressed mechanical modes. These results indicate that we can steer the switch from single-channel to multichannel optical communications by breaking the dark-mode effect.

V. CONCLUSION

In summary, we presented a theoretical proposal to enhance and steer the OMIT, second-order sidebands, and group delay by controlling the dark-mode effect. In the three-mode loop-coupled configuration, the combination of the phasedependent phonon-phonon interaction between the two mechanical modes with the optomechanical couplings controls the dark-mode effect via the quantum interference effect. In the presence of the dark mode, there exists only one OMIT window in the two-mechanical-mode optomechanical system, and this transmission window is nearly twofold amplified compared to that in the typical optomechanical system. We also studied the switching from the OMIT with a single window to that of the tunable double windows by breaking the dark-mode effect. Moreover, the controllable second-order sidebands and the slow-to-fast or fast-to-slow light switching can be achieved by controlling the dark-mode effect. Finally, we extended this method to investigate the optical transmission in an *N*-mechanical-mode optomechanical system.

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APPENDIX A: ALGEBRA EQUATIONS AND SIDEBAND PARAMETERS

In this Appendix, we present 12 algebra equations divided into two groups and the sideband parameters used in Eqs. (7) and (8). The linear response of the probe field is described by the first group,

$$\begin{aligned} \varepsilon_{p} &= [\kappa + i(\Delta - \Omega)]A_{1}^{-} + i[g_{1}\alpha(B_{1,1}^{-} + (B_{1,1}^{+})^{*}) + g_{2}\alpha(B_{2,1}^{-} + (B_{2,1}^{+})^{*})], \\ 0 &= [\gamma_{1} + i(\omega_{1} - \Omega)]B_{1,1}^{-} + i[g_{1}(\alpha^{*}A_{1}^{-} + \alpha(A_{1}^{+})^{*}) + \eta e^{i\theta}B_{2,1}^{-}], \\ 0 &= [\gamma_{2} + i(\omega_{2} - \Omega)]B_{2,1}^{-} + i[g_{2}(\alpha^{*}A_{1}^{-} + \alpha(A_{1}^{+})^{*}) + \eta e^{-i\theta}B_{1,1}^{-}], \\ 0 &= [\kappa - i(\Delta + \Omega)](A_{1}^{+})^{*} - i[g_{1}\alpha^{*}(B_{1,1}^{-} + (B_{1,1}^{+})^{*}) + g_{2}\alpha^{*}(B_{2,1}^{-} + (B_{2,1}^{+})^{*})], \\ 0 &= [\gamma_{1} - i(\omega_{1} + \Omega)](B_{1,1}^{+})^{*} - i[g_{1}(\alpha^{*}A_{1}^{-} + \alpha(A_{1}^{+})^{*}) + \eta e^{-i\theta}(B_{2,1}^{+})^{*}], \\ 0 &= [\gamma_{2} - i(\omega_{2} + \Omega)](B_{2,1}^{+})^{*} - i[g_{2}(\alpha^{*}A_{1}^{-} + \alpha(A_{1}^{+})^{*}) + \eta e^{i\theta}(B_{1,1}^{+})^{*}], \end{aligned}$$

while the second-order sideband is exhibited by the second group,

$$\begin{aligned} 0 &= [\kappa + i(\Delta - 2\Omega)]A_2^- + i\{g_1[\alpha B_{1,2}^- + \alpha(B_{1,2}^+)^* + A_1^-(B_{1,1}^- + (B_{1,1}^+)^*)] + g_2[\alpha(B_{2,2}^- + (B_{2,2}^+)^*) + A_1^-(B_{2,1}^- + (B_{2,1}^+)^*)]\}, \\ 0 &= [\gamma_1 + i(\omega_1 - 2\Omega)]B_{1,2}^- + i[g_1(\alpha^* A_2^- + \alpha(A_2^+)^* + (A_1^+)^* A_1^-) + \eta e^{i\theta}B_{2,2}^-], \\ 0 &= [\gamma_2 + i(\omega_2 - 2\Omega)]B_{2,2}^- + i[g_2(\alpha^* A_2^- + \alpha(A_2^+)^* + (A_1^+)^* A_1^-) + \eta e^{-i\theta}B_{1,2}^-], \\ 0 &= (\kappa - i\Delta - 2i\Omega)(A_2^+)^* - i\{g_1[\alpha^* B_{1,2}^- + \alpha^*(B_{1,2}^+)^* + (A_1^+)^* B_{1,1}^- + (A_1^+)^*(B_{1,1}^+)^*] + g_2[\alpha^* B_{2,2}^- + \alpha^*(B_{2,2}^+)^* \\ &+ (A_1^+)^* B_{2,1}^- + (A_1^+)^*(B_{2,1}^+)^*]\}, \end{aligned}$$
(A2)
$$&= (\gamma_1 - i\omega_1 - 2i\Omega)(B_{1,2}^+)^* - i[g_1(\alpha^* A_2^- + \alpha(A_2^+)^* + (A_1^+)^* A_1^-) + \eta e^{-i\theta}(B_{2,2}^+)^*], \\ 0 &= (\gamma_2 - i\omega_2 - 2i\Omega)(B_{1,2}^+)^* - i[g_2(\alpha^* A_2^- + \alpha(A_2^+)^* + (A_1^+)^* A_1^-) + \eta e^{i\theta}(B_{1,2}^+)^*]. \end{aligned}$$

The other parameters used in Eqs. (7) and (8) are

$$B_{1,1}^{-} = \frac{g_1 \alpha^* V_1 [\gamma_2 + i(\omega_2 - \Omega)] - ig_2 \alpha^* V_1 \eta e^{i\theta}}{(i\kappa + \Delta + \Omega) (\{4|\alpha|^2 \Delta (g_2^2 T_{3,1}^{(1)} + g_1^2 T_{3,2}^{(1)}) - T_2^{(1)} [\Delta^2 + (\kappa - i\Omega)^2]\} + 8g_1 g_2 \eta |\alpha|^2 T_1^{(1)} \Delta \cos \theta)} \varepsilon_p,$$
(A3)

$$B_{2,1}^{-} = \frac{g_2 \alpha^* V_1 [\gamma_1 + i(\omega_1 - \Omega)] - ig_1 \alpha^* V_1 \eta e^{-i\theta}}{(i\kappa + \Delta + \Omega) (\{4|\alpha|^2 \Delta (g_2^2 T_{3,1}^{(1)} + g_1^2 T_{3,2}^{(1)}) - T_2^{(1)} [\Delta^2 + (\kappa - i\Omega)^2]\} + 8g_1 g_2 \eta |\alpha|^2 T_1^{(1)} \Delta \cos \theta)} \varepsilon_p,$$
(A4)

$$(B_{1,1}^{+})^{*} = \frac{-iV_{2}[g_{1}\alpha^{*}(i\gamma_{2}+\omega_{2}+\Omega) - g_{2}\alpha^{*}\eta e^{-i\sigma}]}{(i\kappa + \Delta + \Omega)(\{4|\alpha|^{2}\Delta(g_{2}^{2}T_{3,1}^{(1)} + g_{1}^{2}T_{3,2}^{(1)}) - T_{2}^{(1)}[\Delta^{2} + (\kappa - i\Omega)^{2}]\} + 8g_{1}g_{2}\eta|\alpha|^{2}T_{1}^{(1)}\Delta\cos\theta)}\varepsilon_{p},$$

$$(A5)$$

$$(B_{2,1}^{+})^{*} = \frac{c g_{1\alpha} \eta v_{3} + g_{2\alpha} v_{3}(v_{1} + \omega_{1} + \omega_{2})}{-\left(\left\{4|\alpha|^{2}\Delta\left(g_{2}^{2}T_{3,1}^{(1)} + g_{1}^{2}T_{3,2}^{(1)}\right) - T_{2}^{(1)}[\Delta^{2} + (\kappa - i\Omega)^{2}]\right\} + 8g_{1}g_{2}\eta|\alpha|^{2}T_{1}^{(1)}\Delta\cos\theta\right)}\varepsilon_{p},\tag{A6}$$

where

$$V_1 = \left[-\eta^2 + (i\gamma_1 + \omega_1 + \Omega)(i\gamma_2 + \omega_2 + \Omega)\right] \left[\kappa - i(\Delta + \Omega)\right]^2,\tag{A7}$$

$$V_2 = [\eta^2 + (-i\gamma_1 + \omega_1 - \Omega)(i\gamma_2 - \omega_2 + \Omega)][\kappa - i(\Delta + \Omega)]^2,$$
(A8)

$$V_3 = [-\eta^2 + (i\gamma_1 - \omega_1 + \Omega)(i\gamma_2 - \omega_2 + \Omega)][\kappa - i(\Delta + \Omega)],$$
(A9)

$$T_1^{(1)} = -\omega_1 \omega_2 + \eta^2 + (\gamma_1 - i\Omega)(\gamma_2 - i\Omega),$$
(A10)

$$T_2^{(1)} = [\eta^2 + (-i\gamma_1 + \omega_1 - \Omega)(i\gamma_2 - \omega_2 + \Omega)]\{\eta^2 + [\gamma_1 - i(\omega_1 + \Omega)][\gamma_2 - i(\omega_2 + \Omega)]\},$$
(A11)

$$T_{3,1}^{(1)} = \left(\gamma_1^2 + \omega_1^2 - \Omega^2 - 2i\gamma_1\Omega\right)\omega_2 - \omega_1\eta^2,$$
(A12)

$$T_{3,2}^{(1)} = \left(\gamma_2^2 + \omega_2^2 - \Omega^2 - 2i\gamma_2\Omega\right)\omega_1 - \omega_2\eta^2,$$
(A13)

and $T_1^{(2)}$, $T_2^{(2)}$, $T_{3,1}^{(2)}$, and $T_{3,2}^{(2)}$ can be obtained with replacing the Ω as 2Ω in the $T_1^{(1)}$, $T_2^{(1)}$, $T_{3,1}^{(1)}$, and $T_{3,2}^{(1)}$.

APPENDIX B: DERIVATION OF THE EFFECTIVE MECHANICAL DECAY RATE OF THE BRIGHT MODE

In this Appendix, we derive the effective mechanical decay rate of the bright mode in the two-mechanical-mode optomechanical system. We consider the case where the phasedependent phonon-exchange interaction and the probe field are absent (i.e., $\eta = 0$ and $\varepsilon_p = 0$). Based on Eq. (4), the linearized Langevin equations for quantum fluctuations are given by

$$\delta \dot{a} = -(\kappa + i\Delta)\delta a - i\sum_{l=1,2} G_l(\delta b_l + \delta b_l^{\dagger}) + \sqrt{2\kappa}a_{\rm in},$$
(B1a)

$$\delta \dot{b}_{l=1,2} = -iG_l^*\delta a - (\gamma_l + i\omega_l)\delta b_l - iG_l\delta a^{\dagger} + \sqrt{2\gamma_l}b_{l,\rm in},$$
(B1b)

where $\Delta = \Delta_c + \sum_{l=1,2} g_l(\beta_l + \beta_l^*)$ is the normalized driving detuning, and $G_l = g_l |\alpha|$ denotes the linearized optomechanical coupling strength between the cavity-field mode and the *l*th mechanical mode.

Below, we consider the case where the linearized optomechanical coupling strengths $G_{1,2}$ are real and the system works in the parameter regime $\omega_{1,2} \gg \kappa \gg G_{1,2} \gg \gamma_{1,2}$. In this case, the cavity-field mode can be eliminated adiabatically, and then the solution of the cavity-field fluctuation operator $\delta a(t)$ at the timescale $t \gg 1/\kappa$ can be obtained as

$$\delta a(t) \approx -\frac{iG_1}{\kappa + i(\Delta + \omega_1)} \delta b_1^{\dagger}(t) - \frac{iG_1}{\kappa + i(\Delta - \omega_1)} \delta b_1(t) - \frac{iG_2}{\kappa + i(\Delta + \omega_2)} \delta b_2^{\dagger}(t) - \frac{iG_2}{\kappa + i(\Delta - \omega_2)} \delta b_2(t) + F_{a,in}(t),$$
(B2)

where we introduce the noise operator

$$F_{a,\text{in}}(t) = \sqrt{2\kappa} e^{-(\kappa + i\Delta)t} \int_0^t a_{\text{in}}(s) e^{(\kappa + i\Delta)s} ds.$$
(B3)

Substitution of Eq. (B2) into Eqs. (B1 b) and (B1) leads to the equations of motion

$$\delta \dot{b}_{1}(t) = -(\Gamma_{1} + i\Omega_{1})\delta b_{1}(t) + \xi_{1}\delta b_{2}(t) - iG_{1}F_{a,\text{in}}(t) - iG_{1}F_{a,\text{in}}^{\dagger}(t) + \sqrt{2\gamma_{1}}b_{1,\text{in}}(t), \delta \dot{b}_{2}(t) = \xi_{2}\delta b_{1}(t) - (\Gamma_{2} + i\Omega_{2})\delta b_{2}(t) - iG_{2}F_{a,\text{in}}(t) - iG_{2}F_{a,\text{in}}^{\dagger}(t) + \sqrt{2\gamma_{2}}b_{2,\text{in}}(t),$$
(B4)

where we introduce the parameters

$$\xi_{1} = \frac{G_{1}G_{2}[\kappa + i(\Delta + \omega_{2})]}{\kappa^{2} + (\Delta + \omega_{2})^{2}} - \frac{G_{1}G_{2}[\kappa - i(\Delta - \omega_{2})]}{\kappa^{2} + (\Delta - \omega_{2})^{2}},$$

$$\xi_{2} = \frac{G_{1}G_{2}[\kappa + i(\Delta + \omega_{1})]}{\kappa^{2} + (\Delta + \omega_{1})^{2}} - \frac{G_{1}G_{2}[\kappa - i(\Delta - \omega_{1})]}{\kappa^{2} + (\Delta - \omega_{1})^{2}},$$
(B5)

and

$$\Gamma_l = \gamma_l + \gamma_{l,\text{opt}}, \qquad \Omega_l = \omega_l - \omega_{l,\text{opt}}, \qquad (B6)$$

with

$$\gamma_{l,\text{opt}} = \frac{G_l^2 \kappa}{\kappa^2 + (\Delta - \omega_l)^2} - \frac{G_l^2 \kappa}{\kappa^2 + (\Delta + \omega_l)^2},$$
$$\omega_{l,\text{opt}} = \frac{G_l^2 (\Delta + \omega_l)}{\kappa^2 + (\Delta + \omega_l)^2} + \frac{G_l^2 (\Delta - \omega_l)}{\kappa^2 + (\Delta - \omega_l)^2}, \quad l = 1, 2.$$
(B7)

Under the parameter condition $\omega_{1,2} \gg \kappa \gg G_{1,2}$ and at resonance $\Delta = \omega_1 = \omega_2$, we have

$$\xi_1 \approx -[G_1 G_2 / \kappa - i(G_1 G_2 / 2\omega_2)],$$

$$\xi_2 \approx -[G_1 G_2 / \kappa - i(G_1 G_2 / 2\omega_1)],$$
(B8)

and

$$\gamma_{l,\text{opt}} \approx G_l^2 / \kappa, \quad \omega_{l,\text{opt}} \approx G_l^2 / (2\omega_l), \qquad l = 1, 2.$$
 (B9)

Here, we assume that the two mechanical modes have the identical resonance frequencies ($\omega_l = \omega_m$), decay rates ($\gamma_l = \gamma_m$), and optomechanical couplings ($G_l = G$). Thus, we can obtain the optomechanically induced mechanical decay rate $\gamma_{l,\text{opt}} = \gamma_{\text{opt}}$ and the resonant frequency $\omega_{l,\text{opt}} = \omega_{\text{opt}}$. Then, the equations of motion for the bright mode B_+ can be

written as

$$\dot{B}_{+} = \frac{\dot{b}_{1} + \dot{b}_{2}}{\sqrt{2}} = -(\Gamma_{\rm eff} + i\Omega_{\rm eff})B_{+} + 2iG_{1}F_{\rm in} + \sqrt{2\gamma_{m}}B_{+,\rm in}, \qquad (B10)$$

where

$$\Gamma_{\rm eff} = \gamma_m + 2\gamma_{\rm opt},\tag{B11a}$$

$$\Omega_{\rm eff} = \omega_m - 2\omega_{\rm opt},\tag{B11b}$$

$$B_{+,\text{in}} = [b_{1,\text{in}}(t) + b_{2,\text{in}}(t)]/\sqrt{2},$$
 (B11c)

$$F_{\rm in} = [F_{a,\rm in}(t) + F_{a,\rm in}^{\dagger}(t)]/\sqrt{2}.$$
 (B11d)

Here, Γ_{eff} and Ω_{eff} are, respectively, the effective mechanical decay rate and resonant frequency of the bright mode B_+ .

APPENDIX C: DERIVATION OF EQ. (36)

In this Appendix, we show a detailed derivation of Eq. (36). Based on Eqs. (34) and (35), for $N \ge 3$, we obtain the effective coupling coefficient between the cavity *a* and the mode B_k as

$$\frac{G}{A}\left[\sin\left(\frac{k\pi}{N+1}\right) + \sum_{j=2}^{N} e^{i\sum_{\nu=1}^{j-1}\theta_{\nu}} \sin\left(\frac{jk\pi}{N+1}\right)\right] = \frac{G}{A}\left\{\left[\sin\left(\frac{1}{N+1}k\pi\right) + e^{i\sum_{\nu=1}^{N-1}\theta_{\nu}} \sin\left(\frac{N}{N+1}k\pi\right)\right] + \left[e^{i\theta_{1}}\sin\left(\frac{2}{N+1}k\pi\right) + e^{i\sum_{\nu=1}^{N-2}\theta_{\nu}} \sin\left(\frac{N-1}{N+1}k\pi\right)\right] + \left[e^{i(\theta_{1}+\theta_{2})}\sin\left(\frac{3}{N+1}k\pi\right) + e^{i\sum_{\nu=1}^{N-3}\theta_{\nu}}\sin\left(\frac{N-2}{N+1}k\pi\right)\right] + \cdots\right\}. \quad (C1)$$

Below, we consider two cases corresponding to odd and even numbers N, respectively.

(i) We first consider the case where N is an odd number and $\theta_j = 0$ for $j = 2 \cdot (N - 1)$, and the coefficient becomes

$$\frac{G}{A}\left[\sin\left(\frac{k\pi}{N+1}\right) + \sum_{j=2}^{N} e^{i\sum_{\nu=1}^{j-1}\theta_{\nu}} \sin\left(\frac{jk\pi}{N+1}\right)\right] = \frac{G}{A}\left\{\left[\sin\left(\frac{k\pi}{N+1}\right) + e^{i\theta_{1}}\sin\left(\frac{Nk\pi}{N+1}\right)\right] + e^{i\theta_{1}}\left[\sin\left(\frac{2k\pi}{N+1}\right) + \sin\left(\frac{N-1}{N+1}k\pi\right)\right] + e^{i\theta_{1}}\left[\sin\left(\frac{3k\pi}{N+1}\right) + \sin\left(\frac{N-2}{N+1}k\pi\right)\right] + \dots + e^{i\theta_{1}}\sin\left(\frac{k\pi}{2}\right)\right\}. \quad (C2)$$

When k is an odd number, we have

$$\frac{G}{A}\left[\sin\left(\frac{k\pi}{N+1}\right) + \sum_{j=2}^{N} e^{i\sum_{\nu=1}^{j-1}\theta_{\nu}} \sin\left(\frac{jk\pi}{N+1}\right)\right] = \frac{G}{A}\left[(1+e^{i\theta_{1}})\sin\left(\frac{k\pi}{N+1}\right) + 2e^{i\theta_{1}}\sin\left(\frac{2k\pi}{N+1}\right) + 2e^{i\theta_{1}}\sin\left(\frac{2k\pi}{N+1}\right) + 2e^{i\theta_{1}}\sin\left(\frac{k\pi}{N+1}\right)\right] + 2e^{i\theta_{1}}\sin\left(\frac{k\pi}{N+1}\right) + 2$$

When *k* is an even number, we obtain

$$\frac{G}{A}\left[\sin\left(\frac{k\pi}{N+1}\right) + \sum_{j=2}^{N} e^{i\sum_{\nu=1}^{j-1}\theta_{\nu}} \sin\left(\frac{jk\pi}{N+1}\right)\right] = \frac{G}{A}\left(1 - e^{i\theta_{1}}\right) \sin\left(\frac{k\pi}{N+1}\right).$$
(C4)



FIG. 8. (a) The circuit optomechanical system consists of a microwave cavity represented by the equivalent inductance L and capacitance C, and two micromechanical resonators $b_{j=1,2}$. The displacement $x_{j=1,2}$ of each mechanical resonator results in a time-varying capacitor $C_{j=1,2}(x_j)$ and independently modulates the total capacitance C and, hence, the cavity frequency ω_c . A phase-dependent phonon-hopping interaction $\eta(e^{i\theta}b_1^{\dagger}b_2 + e^{-i\theta}b_2^{\dagger}b_1)$ between the two micromechanical resonators is generated via a superconducting quantum circuit given in (b). (b) Schematic diagram of the superconducting quantum circuit: a Josephson junction with the Josephson energy E_J and the capacitance C_J is connected to three gate voltages $V_{j=1,2,3}(t)$ through the corresponding gate capacitance $C_{j=1,2}(x_j)$ and C_3 . Two mechanical resonators are coupled the superconducting charge qubit through the gate capacitances $C_{j=1,2}(x_j)$. The gate voltages are properly designed such that a phase-dependent phonon-hopping interaction between the two mechanical resonators can be induced. The phase drops across these capacitor $C_{j=1,2,3}$ and the Josephson junction are marked as ϕ_j and ϕ , respectively.

(ii) Then, we consider the case where N is an even number and $\theta_j = 0$ for $j = 2 \cdot (N - 1)$, and the coefficient can be simplified as

$$\frac{G}{A}\left[\sin\left(\frac{k\pi}{N+1}\right) + \sum_{j=2}^{N} e^{i\sum_{\nu=1}^{j-1}\theta_{\nu}}\sin\left(\frac{jk\pi}{N+1}\right)\right] = \frac{G}{A}\left\{\left[\sin\left(\frac{k\pi}{N+1}\right) + e^{i\theta_{1}}\sin\left(\frac{N}{N+1}k\pi\right)\right] + e^{i\theta_{1}}\left[\sin\left(\frac{2k\pi}{N+1}\right) + \sin\left(\frac{N-1}{N+1}k\pi\right)\right] + e^{i\theta_{1}}\left[\sin\left(\frac{3k\pi}{N+1}\right) + \sin\left(\frac{N-2}{N+1}k\pi\right)\right] + \cdots\right\}.$$
(C5)

If k is an odd number, we obtain

$$\frac{G}{A}\left[\sin\left(\frac{k\pi}{N+1}\right) + \sum_{j=2}^{N} e^{i\sum_{\nu=1}^{j-1}\theta_{\nu}} \sin\left(\frac{jk\pi}{N+1}\right)\right] = \frac{G}{A}\left[(1+e^{i\theta_{1}})\sin\left(\frac{k\pi}{N+1}\right) + 2e^{i\theta_{1}}\sin\left(\frac{2k\pi}{N+1}\right) + 2e^{i\theta_{1}}\sin\left(\frac{3k\pi}{N+1}\right) + 2e^{i\theta_{1}}\sin\left(\frac{3k\pi}{N+1}\right) + \cdots\right].$$
(C6)

If *k* is an even number, we have

$$\frac{G}{A}\left[\sin\left(\frac{k\pi}{N+1}\right) + \sum_{j=2}^{N} e^{i\sum_{\nu=1}^{j-1}\theta_{\nu}} \sin\left(\frac{jk\pi}{N+1}\right)\right] = \frac{G}{A}(1-e^{i\theta_1})\sin\left(\frac{k\pi}{N+1}\right).$$
(C7)

According to Eqs. (C2)–(C7), we can summarize that for an even number k, the optomechanical interaction between the mechanical mode $B_{k=\text{even}}$ and the cavity mode a is described by Eq. (36).

APPENDIX D: A POSSIBLE EXPERIMENTAL REALIZATION AND A DERIVATION OF A PHASE-DEPENDENT PHONON-HOPPING INTERACTION BETWEEN TWO MECHANICAL RESONATORS

1. A possible experimental realization

In this Appendix, we propose a possible experimental implementation of our scheme, as shown in Fig. 8(a). The

proposal can be implemented via the circuit optomechanical system [60,61] consisting of a microwave cavity described by the equivalent inductance L and capacitance C, and two micromechanical resonators $b_{j=1,2}$. The electromechanical coupling arises when the displacement $x_{j=1,2}$ of each mechanical resonator, which is expressed as time-varying capacitors $C_{j=1,2}(x_j)$, independently modulates the total capacitance C, and therefore the resonance frequency of the cavity ω_c .

This electromechanical coupling can be described by $g_j = (\omega_c/2C)\partial C_j/\partial x_j$. Meanwhile, an effective phase-dependent phonon-hopping interaction between the two mechanical resonators is introduced by coupling them to a superconducting charge qubit. Note that this effective phase-dependent phonon-hopping interaction can be generated in a superconducting quantum circuit, as shown in Fig. 8(b). The detailed derivation of the phase-dependent phonon-exchange interaction is presented in the next subsection.

2. Derivation of a phase-dependent phonon-hopping interaction between two mechanical resonators

Below, a detailed derivation of an effective phasedependent phonon-hopping interaction between two mechanical resonators will be presented in a superconducting quantum circuit, where the two mechanical resonators are coupled to a superconducting charge qubit, as shown in Fig. 8(b). In this circuit, a Josephson junction with the Josephson energy E_J and the capacitance C_J is connected to three gate voltages $V_{j=1,2,3}(t)$ through the corresponding gate capacitances $C_{j=1,2}(x_j)$ and C_3 . Here, the two gate capacitors with capacitances $C_{j=1,2}(x_j)$ are formed by one fixed plate and one mechanical resonator. The third capacitor has a constant capacitance. We denote the phase drops across these capacitor $C_{j=1,2,3}$ and the Josephson junction as ϕ_j and ϕ , respectively. In this circuit, the energy stored in these capacitors is the total kinetic energy [72], which can be written as

$$T = \frac{1}{2}C_1(x_1)\dot{\Phi}_1^2 + \frac{1}{2}C_2(x_2)\dot{\Phi}_2^2 + \frac{1}{2}C_3\dot{\Phi}_3^2 + \frac{1}{2}C_J\dot{\Phi}^2, \quad (D1)$$

where $\Phi_{j=1,2,3}$ and Φ are the generalized magnetic fluxes associated with the phase drops ϕ_j and ϕ across the capacitances C_j and the Josephson junction. The relation between the generalized magnetic flux and the phase drop is defined by $\phi_{j=1,2,3} = 2\pi \Phi_j / \Phi_0$, where Φ_0 is the magnetic flux quanta. The Josephson energy is identified as the potential energy, which takes the form as [72]

$$U = -E_J \cos\left(\frac{2\pi}{\Phi_0}\Phi\right),\tag{D2}$$

where E_J is the Josephson energy of this junction.

Based on these voltages relations in these loops, we have the relations

$$V_j(t) + \dot{\Phi}_j + \dot{\Phi} = 0, \quad j = 1, 2, 3$$
 (D3)

then the Lagrangian of this system can be expressed as

$$L = T - U$$

= $\frac{1}{2}C_1(x_1)V_1^2(t) + \frac{1}{2}C_2(x_2)V_2^2(t) + \frac{1}{2}C_3V_3^2(t)$
+ $\frac{1}{2}[C_1(x_1) + C_2(x_2) + C_3 + C_J]\dot{\Phi}^2$
+ $[C_1(x_1)V_1(t) + C_2(x_2)V_2(t) + C_3V_3(t)]\dot{\Phi}$
+ $E_J \cos\left(\frac{2\pi}{\Phi_0}\Phi\right).$ (D4)

We introduce the momentum canonically conjugate to Φ as

$$P = \frac{\partial L}{\partial \dot{\Phi}} = [C_1(x_1)V_1(t) + C_2(x_2)V_2(t) + C_3V_3(t)] + [C_1(x_1) + C_2(x_2) + C_3 + C_J]\dot{\Phi}.$$
 (D5)

Then, the Hamiltonian of this circuit can be derived as

$$H = \frac{1}{2} \frac{4e^2}{C_{\Sigma}(x_1, x_2)} [\hat{n} - n_g(x_1, x_2, t)]^2 - E_J \cos\left(\frac{2\pi}{\Phi_0}\Phi\right) - \frac{1}{2} [C_1(x_1)V_1^2(t) + C_2(x_2)V_2^2(t) + C_3V_3^2(t)], \quad (D6)$$

where we introduce the Cooper-pair number operator \hat{n} , the gate capacitance $C_{\Sigma}(x_1, x_1)$, and the gate Cooper-pair number n_g , which are defined by

$$P = 2e\hat{n}, C_{\Sigma}(x_1, x_2) = C_1(x_1) + C_2(x_2) + C_3 + C_J, \quad (D7)$$

and

$$n_g(x_1, x_2, t) = \frac{1}{2e} [C_1(x_1)V_1(t) + C_2(x_2)V_2(t) + C_3V_3(t)].$$
(D8)

The quantization of this circuit can be performed by introducing the commutative relation between the number operator \hat{n} and the phase operator $\hat{\phi}$ as $[\hat{\phi}, \hat{n}] = i$. Then, we can express the Hamiltonian in the eigenrepresentation of the number operator as

$$H = \frac{1}{2} \frac{4e^2}{C_{\Sigma}(x_1, x_2)} \sum_{n \in \mathbb{Z}} [n - n_g(x_1, x_2, t)]^2 |n\rangle \langle n|$$

$$- \frac{E_J}{2} \sum_{n \in \mathbb{Z}} (|n\rangle \langle n + 1| + |n + 1\rangle \langle n|)$$

$$- \frac{1}{2} [C_1(x_1)V_1^2(t) + C_2(x_2)V_2^2(t) + C_3V_3^2(t)].$$
(D9)

In this work, we consider the case where this circuit works in the charge qubit regime $E_C \gg E_J$, with $E_C = 4e^2/C_{\Sigma}$ being the Coulomb energy. In particular, we choose the gate charge in the vicinity of $\frac{1}{2}$, so that the states $|0\rangle$ and $|1\rangle$ have almost degenerate energies. In this case, other states have higher energies and can be ignored in the our discussions. Then, the Hamiltonian becomes

$$H \approx \frac{1}{2} \frac{4e^2}{C_{\Sigma}(x_1, x_2)} \Big[n_g(x_1, x_2, t)^2 |0\rangle \langle 0| \\ + \Big[1 - n_g(x_1, x_2, t) \Big]^2 |1\rangle \langle 1| \Big] - \frac{1}{2} \Big[C_1(x_1) V_1^2(t) \\ + C_2(x_2) V_2^2(t) + C_3 V_3^2(t) \Big] \\ - \frac{E_J}{2} (|0\rangle \langle 1| + |1\rangle \langle 0|).$$
(D10)

By introducing the Pauli operators $|0\rangle\langle 0| - |1\rangle\langle 1| = \sigma_z$ and $|0\rangle\langle 0| + |1\rangle\langle 1| = I$, we can express the Hamiltonian as

$$H = \frac{1}{2} \frac{4e^2}{C_{\Sigma}(x_1, x_2)} \bigg[n_g(x_1, x_2, t) - \frac{1}{2} \bigg] \sigma_z$$
$$-\frac{E_J}{2} \sigma_x + M, \tag{D11}$$

where the term M stands for the ac voltage driving term on these two mechanical resonators

$$M = \frac{1}{4} \frac{4e^2}{C_{\Sigma}(x_1, x_2)} [1 - 2n_g(x_1, x_2, t) + 2n_g^2(x_1, x_2, t)] -\frac{1}{2} [C_1(x_1)V_1^2(t) + C_2(x_2)V_2^2(t) + C_3V_3^2(t)].$$
(D12)

We consider the case in which the voltage drivings are far off resonance to these two mechanical resonators (namely, the driving frequencies of the two voltages are much smaller than the resonance frequencies of the two mechanical resonators) and then the term M will be discarded in our following discussions. When the vibration amplitudes of the mechanical resonators are much smaller than the distances between the fixed plate and the rest mechanical resonator of the capacitors, we can approximate the capacitances as

$$C_1(x_1) \approx C_{10} \left(1 - \frac{x_1}{l_1} \right), \quad C_2(x_2) \approx C_{20} \left(1 - \frac{x_2}{l_2} \right), \quad (D13)$$

where C_{j0} (for j = 1, 2) are the capacitances of the gate capacitors when the mechanical resonators are at rest, and $l_{j=1,2}$ are the rest distances between the fixed plate and the mechanical resonators in these gate capacitors. In addition, we choose the following gate voltages for our purpose:

$$V_{1}(t) = V_{10} \cos(\omega_{1}t + \varphi_{1}),$$

$$V_{2}(t) = V_{20} \cos(\omega_{2}t + \varphi_{2}),$$

$$V_{3}(t) = \frac{e - C_{10}V_{1}(t) - C_{20}V_{2}(t)}{C_{3}}.$$
 (D14)

In this case, we can obtain the relation

$$n_{g}(x_{1}, x_{2}, t) - \frac{1}{2} = -\left[\frac{C_{10}V_{10}}{2e}\frac{x_{1}}{l_{1}}\cos(\omega_{1}t + \varphi_{1}) + \frac{C_{20}V_{20}}{2e}\frac{x_{2}}{l_{2}}\cos(\omega_{2}t + \varphi_{2})\right].$$
 (D15)

By making the rotation for the qubit $-\sigma_x \rightarrow \tau_z$ and $\sigma_z \rightarrow \tau_x$, we can express the Hamiltonian up to the first order of the mechanical displacements x_1 and x_2 as

$$H_{I} \approx \frac{E_{J}}{2}\tau_{z} - \frac{E_{C}}{2} \left[\frac{C_{10}V_{10}}{2e} \frac{x_{1}}{l_{1}} \cos(\omega_{1}t + \varphi_{1}) + \frac{C_{20}V_{20}}{2e} \frac{x_{2}}{l_{2}} \cos(\omega_{2}t + \varphi_{2}) \right] \tau_{x}, \quad (D16)$$

where $E_C = 4e^2/C_{\Sigma 0}$ under the approximation $C_{\Sigma}(x_1, x_2) \approx (C_{10} + C_{20} + C_J) \equiv C_{\Sigma 0}$. We should point out that the mechanical displacement terms in $C_{\Sigma}(x_1, x_2)$ only introduce the second-order terms of $x_{j=1,2}/l_j$, which have been neglected in our considerations.

By including the free Hamiltonian of the two mechanical resonators and using the relations $x_{j=1,2} = \sqrt{\hbar/(2m\omega_m)}(b_j + b_j^{\dagger})$ and $p_{j=1,2} = -i\sqrt{\hbar m\omega_m/2}(b_j - b_j^{\dagger})$, the total Hamiltonian of this circuit system becomes

$$egin{aligned} H_I &pprox \omega_m b_1^{\dagger} b_1 + \omega_m b_2^{\dagger} b_2 + rac{\omega_0}{2} au_z \ &- [g_1 (b_1 + b_1^{\dagger}) (e^{i(\omega_d t + arphi_1)} + e^{-i(\omega_d t + arphi_1)}) \end{aligned}$$

$$+g_{2}(b_{2}+b_{2}^{\dagger})(e^{i(\omega_{d}t+\varphi_{2})}+e^{-i(\omega_{d}t+\varphi_{2})})] \times (\tau_{+}+\tau_{-}),$$
(D17)

where we consider the case $\omega_1 = \omega_2 = \omega_d$ and introduce these parameters

$$g_1 = \frac{E_C}{4} \frac{C_{10}V_{10}}{2e} \frac{x_{10}}{l_1}, \quad g_2 = \frac{E_C}{4} \frac{C_{20}V_{20}}{2e} \frac{x_{20}}{l_2},$$
 (D18)

 $\omega_0 = E_J$, with $x_{j0} = \sqrt{\hbar/(2m\omega_m)}$ being the zero-point fluctuation of these mechanical resonators.

To analyze the physical processes in this system, we now work in the rotating frame with respect to

$$H_0 = \omega_m b_1^{\dagger} b_1 + \omega_m b_2^{\dagger} b_2 + \frac{\omega_0}{2} \tau_z, \qquad (D19)$$

then the Hamiltonian becomes

$$V_{I}(t) = -g_{1}(\tau_{+}b_{1}^{\dagger}e^{i(\omega_{0}+\omega_{m}+\omega_{d})t}e^{i\varphi_{1}} + b_{1}\tau_{-}e^{-i(\omega_{0}+\omega_{m}+\omega_{d})t}e^{-i\varphi_{1}})$$

$$-g_{2}(\tau_{+}b_{2}^{\dagger}e^{i(\omega_{0}+\omega_{m}+\omega_{d})t}e^{i\varphi_{2}} + b_{2}\tau_{-}e^{-i(\omega_{0}+\omega_{m}+\omega_{d})t}e^{-i\varphi_{2}})$$

$$-g_{1}(\tau_{+}b_{1}^{\dagger}e^{i(\omega_{0}+\omega_{m}-\omega_{d})t}e^{-i\varphi_{1}} + b_{1}\tau_{-}e^{-i(\omega_{0}+\omega_{m}-\omega_{d})t}e^{i\varphi_{1}})$$

$$-g_{2}(\tau_{+}b_{2}^{\dagger}e^{i(\omega_{0}-\omega_{m}+\omega_{d})t}e^{i\varphi_{1}} + b_{1}^{\dagger}\tau_{-}e^{-i(\omega_{0}-\omega_{m}-\omega_{d})t}e^{i\varphi_{2}})$$

$$-g_{1}(\tau_{+}b_{1}e^{i(\omega_{0}-\omega_{m}+\omega_{d})t}e^{i\varphi_{2}} + b_{2}^{\dagger}\tau_{-}e^{-i(\omega_{0}-\omega_{m}+\omega_{d})t}e^{-i\varphi_{2}})$$

$$-g_{1}(\tau_{+}b_{2}e^{i(\omega_{0}-\omega_{m}+\omega_{d})t}e^{-i\varphi_{1}} + b_{1}^{\dagger}\tau_{-}e^{-i(\omega_{0}-\omega_{m}-\omega_{d})t}e^{-i\varphi_{2}})$$

$$-g_{2}(\tau_{+}b_{2}e^{i(\omega_{0}-\omega_{m}-\omega_{d})t}e^{-i\varphi_{2}}$$

$$+b_{2}^{\dagger}\tau_{-}e^{-i(\omega_{0}-\omega_{m}-\omega_{d})t}e^{i\varphi_{2}}).$$
(D20)

Here, we can see that in this system there are eight physical processes, which are determined by the four detunings $\omega_0 + \omega_m \pm \omega_d$ and $\omega_0 - \omega_m \pm \omega_d$. From the viewpoint of the qubit and the resonators, the terms including $\omega_0 + \omega_m \pm \omega_d$ and $\omega_0 - \omega_m \pm \omega_d$ are the counterrotating terms and the corotating terms, respectively. In this work, the motivation for introducing the ac voltages $V_1(t)$ and $V_2(t)$ is to pick up the phase-sensitive interactions between the mechanical resonators and the charge qubit. For this purpose, we choose the ac voltages with the frequency ω_d to pick up the terms with $\omega_0 - \omega_m - \omega_d$. Namely, we choose the parameters to satisfy the following parameter conditions:

$$\omega_0 + \omega_m \pm \omega_d \gg \omega_0 - \omega_m + \omega_d \gg \omega_0 - \omega_m - \omega_d.$$
(D21)

The terms with $\omega_0 + \omega_m \pm \omega_d$ and $\omega_0 - \omega_m + \omega_d$ are the far-off-resonance terms and the terms with $\omega_0 - \omega_m - \omega_d$ are the target terms which work in the large-detuning regime. The energy levels and these involved resonance frequencies of this coupled qubit-resonator system are shown in Fig. 9. In this case, the qubit-resonator interactions work in the large-detuning regime $\Delta \gg g_{j=1,2}\sqrt{n_j}$, where n_j is the maximal excitation number involved in the *j*th mechanical resonator, and then we can obtain a phase-dependent photon-hopping interaction between the two mechanical resonators. Here, the phase is the difference between the two phases φ_1 and φ_2 associated with the qubit-resonator couplings.



FIG. 9. Schematic diagram of the energy levels and these involved resonance frequencies of this coupled qubit-resonator system. Two mechanical resonators with resonance frequency ω_m are phase dependently coupled to the superconducting charge qubit with the energy separation ω_0 . The ac gate voltages with frequency ω_d are applied to the Josephson junction through the gate capacitors.

Based on the above analyses, we can obtain the approximate Hamiltonian as

$$V_{I}(t) \approx -[\tau_{+}(g_{1}b_{1}e^{-i\varphi_{1}} + g_{2}b_{2}e^{-i\varphi_{2}})e^{i\Delta t} + (g_{1}b_{1}^{\dagger}e^{i\varphi_{1}} + g_{2}b_{2}^{\dagger}e^{i\varphi_{2}})\tau_{-}e^{-i\Delta t}],$$
(D22)

where we introduce the detuning $\Delta = \omega_0 - \omega_m - \omega_d$. The time factor can be eliminated by going back to the Schrödinger representation, in which the Hamiltonian of the system can be written as

$$H_{\rm eff} = \omega_m b_1^{\dagger} b_1 + \omega_m b_2^{\dagger} b_2 + \frac{\omega_0 - \omega_d}{2} \tau_z - \tau_+ (g_1 b_1 e^{-i\varphi_1} + g_2 b_2 e^{-i\varphi_2}) - (g_1 b_1^{\dagger} e^{i\varphi_1} + g_2 b_2^{\dagger} e^{i\varphi_2}) \tau_-.$$
(D23)

In this work, we consider the physical process associated with the detuning Δ working in the large detuning case. Then, we can adiabatically eliminate the qubit coherence in the physical processes and an effective phonon-phonon interaction between the two mechanical modes can be induced by the second-order perturbation. In this case, we can derive an effective Hamiltonian to describe the interactions using the method of the Frohlich-Nakajima transformation [73,74]. To this end, we express the effective Hamiltonian H_{eff} as two parts

$$H_{0} = \omega_{m}b_{1}^{\dagger}b_{1} + \omega_{m}b_{2}^{\dagger}b_{2} + \frac{\omega_{0} - \omega_{d}}{2}\tau_{z},$$

$$H_{I} = -\tau_{+}(g_{1}b_{1}e^{-i\varphi_{1}} + g_{2}b_{2}e^{-i\varphi_{2}}) - \tau_{-}(g_{1}b_{1}^{\dagger}e^{i\varphi_{1}} + g_{2}b_{2}^{\dagger}e^{i\varphi_{2}}).$$
 (D24)

We also introduce the operator

$$S = \frac{1}{\Delta} \tau_{+} (g_{1}b_{1}e^{-i\varphi_{1}} + g_{2}b_{2}e^{-i\varphi_{2}}) - \frac{1}{\Delta} (g_{1}b_{1}^{\dagger}e^{i\varphi_{1}} + g_{2}b_{2}^{\dagger}e^{i\varphi_{2}})\tau_{-},$$
(D25)

which is determined by the equation

$$H_I + [H_0, S] = 0.$$
 (D26)

This equation means that the first-order physical process is eliminated. An effective Hamiltonian describing the secondorder physical interaction can then be obtained as

$$H'_{\text{eff}} = H_0 + \frac{1}{2}[H_I, S]$$

= $\omega_m b_1^{\dagger} b_1 + \omega_m b_2^{\dagger} b_2 + \frac{\omega_0 - \omega_d}{2} \tau_z + \frac{g_1^2}{\Delta} \tau_z b_1^{\dagger} b_1$
+ $\frac{g_2^2}{\Delta} \tau_z b_2^{\dagger} b_2 + \frac{(g_1^2 + g_2^2)}{\Delta} \tau_+ \tau_-$
+ $\frac{g_{1}g_2}{\Delta} \tau_z (b_1^{\dagger} b_2 e^{i(\varphi_1 - \varphi_2)} + b_2^{\dagger} b_1 e^{-i(\varphi_1 - \varphi_2)}).$ (D27)

The above Hamiltonian shows that there is no transition in the qubit states, and that a conditional phase-dependent interaction between the two mechanical resonators is introduced. We assume that the qubit is initial in its ground state $|g\rangle$ ($\tau_z |g\rangle = -|g\rangle$), then a phase-dependent phonon-hopping interaction is obtained.

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