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Experimental demonstration of one-shot coherence distillation: realizing *N*-dimensional strictly incoherent operations: supplement

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Experimental demonstration of one-shot coherence distillation: Realizing N-dimensional strictly incoherent operations:supplementary material

This document provides supplementary information to "Experimental demonstration of one-shot coherence distillation: Realizing N-dimensional strictly incoherent operations"

1. OPERATIONS TO IMPLEMENT THE PROPOSAL IN FIG. 1.

We propose a general way to achieve N-dimensional SIO ($N \geq 2$) in Fig. 1. In the module **S**, two components $|i\rangle$ and $|j\rangle$ of the primary system should be coupled to the ancillary modes $|\tilde{0}\rangle_0$ and $|\tilde{1}\rangle_0$, then some unitary operations are performed on the 2×2 space. Focusing on the two components of the primary system, 2D SIO can be realized. In the module **S**, one can implement a set of unitary operations driven by the following Hamiltonian:

(1) The interaction Hamiltonian

$$H_{\mathrm{I}} = g_{\mathrm{I}} \sigma_{ij}^{+} \tilde{\sigma}_0^{-} + g_{\mathrm{I}}^{*} \sigma_{ij}^{-} \tilde{\sigma}_0^{+}, \tag{S1}$$

with $\sigma_{ij}^{+} = |i\rangle\langle j|$, $\sigma_{ij}^{-} = |j\rangle\langle i|$ corresponding to the primary system, $\tilde{\sigma}_0^{+} = |\tilde{1}\rangle_0 \langle\tilde{0}|$, and $\tilde{\sigma}_0^{-} = |\tilde{0}\rangle_0 \langle\tilde{1}|$ corresponding to the first ancillary qubit. This Hamiltonian realizes the coupling between the two primary levels and ancillary qubit. The corresponding unitary operations $U_{\mathrm{I}} = \exp(-it_{\mathrm{I}} H_{\mathrm{I}}/\hbar)$ realizes the the following map:

$$|i\rangle |\tilde{0}\rangle_0 \rightarrow \cos\eta_1 |j\rangle |\tilde{1}\rangle_0 + \sin\eta_1 |i\rangle |\tilde{0}\rangle_0 . \tag{S2}$$

(2) A local operation

$$H_{\mathrm{f}} = \kappa \sigma_{ij}^{+} + \kappa^{*} \sigma_{ij}^{-}, \tag{S3}$$

and thus $U_{\mathrm{f}} = \exp(-it_{\mathrm{f}} H_{\mathrm{f}}/\hbar)$, should be introduced to flip the components $|i\rangle \leftrightarrow |j\rangle$, then U_{I} [in Eq. (S1)] is performed again. Finally we have

$$\begin{aligned} |i\rangle |\tilde{0}\rangle_0 &\rightarrow \cos\eta_1 |i\rangle |\tilde{1}\rangle_0 + \sin\eta_1 |j\rangle |\tilde{0}\rangle_0 , \\ |j\rangle |\tilde{0}\rangle_0 &\rightarrow \cos\eta_2 |j\rangle |\tilde{1}\rangle_0 + \sin\eta_2 |i\rangle |\tilde{0}\rangle_0 , \end{aligned} \tag{S4}$$

which actually realizes the 2D SIO working on the primary modes $|i\rangle$ and $|j\rangle$.

(3) Another local operation

$$H_{\mathrm{a}} = \omega \tilde{\sigma}_0^{+} + \omega^{*} \tilde{\sigma}_0^{-} \tag{S5}$$

and thus $U_{\mathrm{a}} = \exp(-it_{\mathrm{a}} H_{\mathrm{a}}/\hbar)$, acting on the ancillary states following the primary components, except $|i\rangle$ and $|j\rangle$, adjust the ancillary modes into the same superposition state. This is done in order to obtain a product form of the primary modes and the ancillary modes. The process can be described as:

$$\sum_i \psi_i |i\rangle \otimes |\tilde{0}\rangle_0 \rightarrow \left\{ \begin{array}{c} \Big(\psi_i |i\rangle + \psi_j |j\rangle\Big) \otimes |\tilde{0}\rangle_0 \xrightarrow{U_{\mathrm{I}}, U_{\mathrm{f}}} \Big(\phi_{1i} |i\rangle + \phi_{1j} |j\rangle\Big) \otimes (\tilde{\alpha} |\tilde{0}\rangle_0 + \tilde{\beta} |\tilde{1}\rangle_0) \\ \sum\limits_{l \neq i,j} \psi_l |l\rangle \otimes |\tilde{0}\rangle_0 \xrightarrow{U_{\mathrm{a}}} \sum\limits_{l \neq i,j} \psi_l |l\rangle \otimes (\tilde{\alpha} |\tilde{0}\rangle_0 + \tilde{\beta} |\tilde{1}\rangle_0) \end{array} \right. . \tag{S6}$$

(4) In the module **RS**, one should introduce a new ancillary mode $|\tilde{0}\rangle_k$ ($k \neq 0$) to reset the superposed ancillary modes $\tilde{\alpha} |\tilde{0}\rangle_0 + \tilde{\beta} |\tilde{1}\rangle_0$ to the initial situations, i.e., $|\tilde{0}\rangle_0$, by performing the coupling

$$H_{\mathrm{r}} = g_{\mathrm{r}} \tilde{\sigma}_0^{+} \tilde{\sigma}_k^{-} + g_{\mathrm{r}}^{*} \tilde{\sigma}_0^{-} \tilde{\sigma}_k^{+}, \tag{S7}$$

as well as the unitary operation $U_r = \exp(-it_r H_r/\hbar)$, which can reset the superposed ancillary modes $\alpha_n |\tilde{0}\rangle_0 + \beta_n |\tilde{1}\rangle_0$ to $|\tilde{0}\rangle_0$ by setting special times. This is the reset process mentioned in module **RS** of Fig. 1.

At present, it has become a mature technology to couple qubit systems and manipulate two-level states in various controllable quantum devices. Therefore, the operations needed in our proposal are implementable. In the optical setups shown in Fig. 2(a), one can find that the combination of the HWPs (i.e., $P_1^{(i,j)}$, $P_2^{(i,j)}$, and P_x) and the BDs realizes H_I, H_f and the map in Eq. (S4). HWP $P_3^{(i,j)}$ in Fig. 2(a) realizes the operation of H_a in Eq. (S5), and the PBS in Fig. 2(b) realizes the reset operation of H_r in Eq. (S7).

2. KRAUS REPRESENTATION OF THE PROPOSED INCOHERENT OPERATIONS.

Let us introduce the proposed SIO in detail. For the state transformation in a 2D space $|\psi\rangle = \sum_{i=0}^{1} \psi_i |i\rangle \rightarrow |\phi\rangle = \sum_{i=0}^{1} \phi_i |i\rangle$, the Kraus-operator representation is $|\phi\rangle\langle\phi| = \sum_{i=1}^{2} K_i |\psi\rangle\langle\psi| K_i^\dagger$, and the Kraus operators can be described as [1]

$$
\begin{aligned}
K_1 &= \sqrt{a}\,\frac{\phi_0}{\psi_0}|0\rangle\langle 0| + \sqrt{a}\,\frac{\phi_1}{\psi_1}|1\rangle\langle 1|, \\
K_2 &= \sqrt{1-a}\,\frac{\phi_0}{\psi_1}|0\rangle\langle 1| + \sqrt{1-a}\,\frac{\phi_1}{\psi_0}|1\rangle\langle 0|, \qquad \text{(S8)}
\end{aligned}
$$

where displayed $a = \frac{|\psi_0|^2 - |\phi_1|^2}{|\phi_0|^2 - |\phi_1|^2}$ and $0 \leq a \leq 1$, which is equivalent to ensure the majorization relation. A pure state $|\psi\rangle = \sum_{i=0}^{N} \psi_i |i\rangle$, majorized by another state $|\phi\rangle = \sum_{i=0}^{N} \phi_i |i\rangle$, should satisfy $\sum_{i=0}^{k} |\psi_i|^{2\downarrow} \leq \sum_{i=0}^{k} |\phi_i|^{2\downarrow}$, where $k \in [0, N]$ and the superscript "$\downarrow$" indicates the descending order of the elements. The majorization relation is sufficient and necessary for SIO (or special IO)-dominated pure states conversions [1–3].

The Kraus operators above can be rewritten as

$$
\begin{aligned}
K_1 &= \cos 2\theta_1 |0\rangle\langle 0| + \sin 2\theta_2 |1\rangle\langle 1|, \\
K_2 &= \cos 2\theta_2 |0\rangle\langle 1| + \sin 2\theta_1 |1\rangle\langle 0|. \qquad \text{(S9)}
\end{aligned}
$$

In our optical setups, the polarization modes $|V\rangle$ and $|H\rangle$ correspond to the ancillary modes $|\tilde{0}\rangle_0$ and $|\tilde{1}\rangle_0$ in Eqs. (S1) and (S2). Then the map is realized in the optical setup,

$$
\begin{aligned}
|0V\rangle &\rightarrow \cos(2\theta_1)|0H\rangle + \sin(2\theta_1)|1V\rangle, \\
|1V\rangle &\rightarrow \cos(2\theta_2)|0V\rangle + \sin(2\theta_2)|1H\rangle, \qquad \text{(S10)}
\end{aligned}
$$

which can be translated into the Kraus-operator representation in Eq. (S9). For a pure input state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, if the angles $\theta_{1,2}$ of the HWP [denoted by $P_1^{(i,j)}$, $P_2^{(i,j)}$ in Fig. 2(a)] satisfy $|\alpha|^2 \sin(4\theta_1) = |\beta|^2 \sin(4\theta_2)$, one can obtain the pure state at the output, i.e.,

$$
\begin{aligned}
|\phi\rangle &= \frac{1}{Q}\left[\beta \sin(2\theta_2)|1\rangle + \alpha\cos(2\theta_1)|0\rangle\right], \\
Q &= \sqrt{|\beta|^2 \sin^2(2\theta_2) + |\alpha|^2 \cos^2(2\theta_1)}. \qquad \text{(S11)}
\end{aligned}
$$

Recall the definition of IO and SIO. For a chosen reference basis $\{|i\rangle\}$, the class of free states is denoted by $\mathcal{I}$. IO and SIO can be described by a set of Kraus operators $\{K_n\}$ satisfying $\sum_n K_n^\dagger K_n = 1$. For an IO, every Kraus operator should satisfy $K_n \mathcal{I} K_n^\dagger \subseteq \mathcal{I}$. While, for a SIO, an additional condition, i.e., $K_n^\dagger \mathcal{I} K_n \subseteq \mathcal{I}$, is needed. An operation is IO if and only if every column of K_n in the fixed basis $\{|i\rangle\}$ has at most one nonzero entry. More strictly, SIO requires that not only every column but also every line of K_n has at most one nonzero element [1, 2]. Therefore, the 2D operation proposed by us belongs to the SIO.

In order to consider the multi-step operations applicable to high-dimensional space, we introduce the SIO performed on the subspace spanned by the two modes $|i\rangle$ and $|j\rangle$, with its experiment setup in Fig. 2(a). The whole map, in the composite system of the spatial modes and the ancillary polarization modes ($|H\rangle$ and $|V\rangle$), reads

$$
\begin{aligned}
|iV\rangle &\rightarrow \cos 2\theta_1^{(i,j)}|iH\rangle + \sin 2\theta_1^{(i,j)}|jV\rangle, \\
|jV\rangle &\rightarrow \cos 2\theta_2^{(i,j)}|iV\rangle + \sin 2\theta_2^{(i,j)}|jH\rangle, \\
|kV\rangle &\rightarrow |k\rangle\left[\cos 2\theta_3^{(i,j)}|V\rangle + \sin 2\theta_3^{(i,j)}|H\rangle\right], \qquad \text{(S12)}
\end{aligned}
$$

where the parameters $\theta_1^{(i,j)}$ and $\theta_2^{(i,j)}$ correspond to the two HWPs $P_1^{(i,j)}$ and $P_2^{(i,j)}$ in the module **S.** The mode $|k\rangle$ ($k \neq i$ and j) denote the modes different from the i and j modes, and the angle $\theta_3^{(i,j)}$ corresponding to the HWP $P_3^{(i,j)}$ should be adjusted to prepare the same superposition structure of the polarization modes as those following the spatial modes of $|i\rangle$ and $|j\rangle$.

After this step, we realize the transformation

$$\sum_i \psi_i|i\rangle \rightarrow \phi_i|i\rangle + \phi_j|j\rangle + \sum_k \psi_k|k\rangle. \tag{S13}$$

The Kraus operators can be expressed as

$$\begin{aligned} K_1^{(i,j)} &= \cos 2\theta_1^{(i,j)}|i\rangle\langle i| + \sin 2\theta_2^{(i,j)}|j\rangle\langle j| + \sin 2\theta_3^{(i,j)} I_k, \\ K_2^{(i,j)} &= \cos 2\theta_2^{(i,j)}|i\rangle\langle j| + \sin 2\theta_1^{(i,j)}|j\rangle\langle i| + \cos 2\theta_3^{(i,j)} I_k, \\ I_k &= \sum_{k\neq i,j} |k\rangle\langle k|, \end{aligned} \tag{S14}$$

where the parameters

$$\begin{aligned} 2\theta_1^{(i,j)} &= \arccos(\sqrt{a_{ij}}\frac{\phi_i}{\psi_i}), \\ 2\theta_2^{(i,j)} &= \arcsin(\sqrt{a_{ij}}\frac{\phi_j}{\psi_j}), \\ 2\theta_3^{(i,j)} &= \arcsin\sqrt{a_{ij}}, \end{aligned} \tag{S15}$$

with $a_{ij} = \frac{|\psi_i|^2 - |\phi_j|^2}{|\phi_i|^2 - |\phi_j|^2}$. As a consequence, in the multi-step proposal, the Kraus operators are implemented on different two-dimensional subspaces. One of the total Kraus operators can be described as $\mathbb{K}_l = \prod_{(i,j)} K_q^{(i,j)}$, with $q = 1, 2$ and the superscript (i, j) going through all the component modes necessary to complete the state transformation. Obviously, the class of $\{\mathbb{K}_l\}$ still belongs to the SIO, and the index of the operators is $l = 2, 4, \ldots, 2^n$ (n is the number of the steps in Fig. 1). Therefore, we can finally obtain $\sum_l^{2^n} \mathbb{K}_l|\psi\rangle\langle\psi|\mathbb{K}_l^\dagger = |\phi\rangle\langle\phi|$.

Here, we can discuss the extensions of the experimental proposal. If we change the operations at the outputs to different local operations, other incoherent operations will be realized. For example, one can adjust the angle of P_x at path $|1\rangle$ to different angles ξ, and perform another HWP with angle $\pi/4 + \xi$ at the path $|0\rangle$, which ensures the polarization states of the two paths to be orthogonal with each other. There will be another type of incoherent operations with different Kraus operators, i.e.,

$$\begin{aligned} K_1' &= \cos 2\theta_1 \cos 2\xi|0\rangle\langle 0| + \sin 2\theta_2 \cos 2\xi|1\rangle\langle 1| \\ &\quad + \sin 2\theta_1 \sin 2\xi|1\rangle\langle 0| - \cos 2\theta_2 \sin 2\xi|0\rangle\langle 1|, \\ K_2' &= \cos 2\theta_1 \sin 2\xi|0\rangle\langle 0| + \sin 2\theta_2 \sin 2\xi|1\rangle\langle 1| \\ &\quad - \sin 2\theta_1 \cos 2\xi|1\rangle\langle 0| + \cos 2\theta_2 \cos 2\xi|0\rangle\langle 1|. \end{aligned} \tag{S16}$$

The incoherent operations above are different from the SIO proposed in this paper.

3. ANALYTICAL RESULTS OF THE DISTILLATION FIDELITY.

For any pure state and the incoherent operation $\mathcal{O} \in \{\text{MIO, DIO, SIO, IO}\}$, the fidelity can also be described by the m-distillation norm [4]:

$$F_{\mathcal{O}}(|\psi\rangle, |\Phi_m\rangle) = \frac{1}{m}\||\psi\rangle\|_{[m]}^2, \tag{S17}$$

where $||\psi\rangle\|_{[m]}$ is the m-distillation norm

$$||\psi\rangle\|_{[m]} = \min_{|\psi\rangle=|x\rangle+|y\rangle} \||x\rangle\|_{l_1} + \sqrt{m}\||y\rangle\|_{l_2}, \tag{S18}$$

where $||\bullet||_{l_1}$ and $||\bullet||_{l_3}$ are the l_1 norm and l_2 norm. For a d-dimensional pure state, the m-distillation norm can be described as

$$\| |\psi\rangle \|_{[m]} = \|\psi^{\downarrow}_{1:m-k^\star}\|_{l_1} + \sqrt{k^\star}\|\psi^{\downarrow}_{m-k^\star+1:d}\|_{l_2}, \tag{S19}$$

where $\psi^{\downarrow}_{1:k}$ denotes the vector consisting of the k largest (by magnitude) coefficients of $|\psi\rangle$, and analogously $\psi^{\downarrow}_{k+1:d}$ denotes the rest of the coefficients. The special number of $k^\star$ is defined by

$$k^\star = \arg\min_{1\leq k\leq m}(\|\psi^{\downarrow}_{m-k^\star+1:d}\|^2_{l_2}/k). \tag{S20}$$

To consider the conversion from the 3D input state

$$|\psi^3\rangle = \sqrt{\alpha}|2\rangle + \sqrt{(1-\alpha)/2}(|0\rangle + |1\rangle) \tag{S21}$$

into a 2D target state, for $0 \leq \alpha \leq 1/2$, the distillation fidelity can be easily verified to be 1 by using the m-distillation norm presentation. It implies that the state $|\psi^3\rangle$ can be successfully converted to the 2D maximal coherence state $|\Phi_2\rangle = \sqrt{1/2}\,(|0\rangle + |1\rangle)$ by choosing a proper incoherent operation. While, for $1/2 < \alpha \leq 1$, it can be calculated that

$$\begin{aligned} &\|\psi^{3\downarrow}_{2-k+1:3}\|^2_{l_2}/k = 1-\alpha, \text{ for } \ k=1, \\ &\|\psi^{3\downarrow}_{2-k+1:3}\|^2_{l_2}/k = 1/2, \quad \text{for } \ k=2. \end{aligned} \tag{S22}$$

Thus we have $k^\star = 1$, and

$$\begin{aligned} \| |\psi^3\rangle \|_{[2]} &= \|\psi^{\downarrow}_{1:1}\|_{l_1} + \|\psi^{\downarrow}_{1+1:3}\|_{l_2} \\ &= \sqrt{\alpha} + \sqrt{1-\alpha}. \end{aligned} \tag{S23}$$

Finally, the distillation fidelity becomes

$$\begin{aligned} F_{\mathcal{O}}(|\psi^3\rangle, |\Phi_2\rangle) &= \frac{1}{2}\| |\psi^3\rangle \|^2_{[2]} \\ &= \frac{1}{2}(\sqrt{\alpha} + \sqrt{1-\alpha})^2. \end{aligned} \tag{S24}$$

Obviously, a reasonable target state is $|\phi_{3\to 2}\rangle = \sqrt{\alpha}|0\rangle + \sqrt{1-\alpha}|1\rangle$, which can reach the distillation fidelity above.

In an analogous way, for the transformation from the 4D input state ($\alpha \in [0, 1/2]$)

$$|\psi^4\rangle = \sqrt{\alpha}(|0\rangle + |1\rangle) + \sqrt{1/2-\alpha}(|2\rangle + |3\rangle) \tag{S25}$$

into a 3D target state, we can obtain the fidelity

$$\begin{aligned} F_{\mathcal{O}}(|\psi^4\rangle, |\Phi_3\rangle) &= \frac{1}{3}\| |\psi^4\rangle \|^2_{[3]} \\ &= \left[\sqrt{\frac{2\alpha}{3}} + \sqrt{\frac{2-4\alpha}{3}}\right]^2 \end{aligned} \tag{S26}$$

for $\alpha \in [0, 1/6] \cup [1/3, 1/2]$. Thus a possible target state is

$$|\phi_{4\to 3}\rangle = \sqrt{2\alpha}|1\rangle + \sqrt{(1/2-\alpha)}(|2\rangle + |3\rangle). \tag{S27}$$

While, for $\alpha \in [1/6, 1/3]$, the optimal value $F_{\mathcal{O}}(|\psi^4\rangle, |\Phi_3\rangle) = 1$ can be reached, which means that in this region the achievable target state is the maximally coherent state $|\Phi_3\rangle = \sqrt{1/3}\,(|0\rangle + |1\rangle + |2\rangle)$.

4. STATE TRANSFORMATIONS FROM 3D STATE INTO 2D STATE.

In this section, we show more details of the map and the operations corresponding to the conversion $\psi_1|0\rangle + \psi_2|1\rangle + \psi_3|2\rangle \to \phi_1|0\rangle + \phi_2|1\rangle$. With the help of the ancillary modes, the devices in Fig. 3(b) realize the map as follows:

$$\begin{aligned}
|0V\rangle &\to \cos 2\theta_1(\cos 2\theta_4|0H\tilde{0}\rangle - \sin 2\theta_4|0V\tilde{0}\rangle) + \sin 2\theta_1|1H\tilde{1}\rangle, \\
|1V\rangle &\to \cos 2\theta_2(\cos 2\theta_5|0H\tilde{1}\rangle - \sin 2\theta_5|0V\tilde{1}\rangle) + \sin 2\theta_2|1H\tilde{0}\rangle, \\
|2V\rangle &\to \cos 2\theta_3|1V\tilde{0}\rangle - \sin 2\theta_3|1V\tilde{1}\rangle, \qquad \text{(S28)}
\end{aligned}$$

where, $\tilde{0}$ and $\tilde{1}$ distinguish the two groups of the spatial modes split by the PBS, and both with the polarization modes $|H\rangle$ and $|V\rangle$ acting as the ancillary modes. Then the Kraus operators can be derived from the above map,

$$\begin{aligned}
K_1 &= -\cos 2\theta_1 \sin 2\theta_4|0\rangle\langle 0| + \cos 2\theta_3|1\rangle\langle 2|, \\
K_2 &= \cos 2\theta_1 \cos 2\theta_4|0\rangle\langle 0| + \sin 2\theta_2|1\rangle\langle 1|, \\
K_3 &= -\cos 2\theta_2 \sin 2\theta_5|0\rangle\langle 1| - \sin 2\theta_3|1\rangle\langle 2|, \\
K_4 &= \sin 2\theta_1|1\rangle\langle 0| + \cos 2\theta_2 \cos 2\theta_5|0\rangle\langle 1|. \qquad \text{(S29)}
\end{aligned}$$

These Kraus operators provide a general conversion process from 3D states into 2D states. In the experiment, we consider a special case, i.e., the input state is

$$|\psi^3\rangle = \sqrt{\alpha}|2\rangle + \sqrt{(1-\alpha)/2}(|0\rangle + |1\rangle), \qquad \text{(S30)}$$

with $\alpha \in [0, 1/2]$ and the target state is $|\phi\rangle = \frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)$. The angles of the HWPs in Fig. 3(b) are set as

$$\begin{aligned}
2\theta_1 &= 2\theta_2 = \arccos\left[\frac{1}{\sqrt{2(1-\alpha)}}\right], \\
2\theta_3 &= -\pi/4, \\
2\theta_4 &= 2\theta_5 = -\arccos(\sqrt{1-2\alpha}). \qquad \text{(S31)}
\end{aligned}$$

The output state of the whole system is

$$\frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)\left[\sqrt{\frac{1-\alpha}{2}}(|H\tilde{1}\rangle + |H\tilde{0}\rangle) + \sqrt{\alpha}(|V\tilde{0}\rangle + |V\tilde{1}\rangle)\right], \qquad \text{(S32)}$$

from which we will obtain the target state by performing spatial tomography on the modes of $|0\rangle$ and $|1\rangle$. Note that the tomography has been done on both of the two groups of the path.

5. EXPERIMENTAL IMPLEMENTATION OF THE STATE PREPARATION, SPATIAL TOMOGRAPHY, AND SOME STATE TRANSFORMATIONS.

In Fig. S1(a), we show the setup for the state preparation, where a pure state of the spatial modes can be prepared. Figure S1(b) shows the device for the spatial tomography measurement. In this stage, each of the spatial modes is appended with a polarization mode, which makes the manipulation of the spatial modes convenient. By adjusting the HWPs and QWPs, the photons in three modes can enter the same single-photon detector where we can analyze all the information of the three spatial modes. With the help of BDs, the spatial modes can be coherently combined, i.e., the superpositions of the spatial modes can be mapped into the superpositions of polarizations. Finally, the tomography of the spatial modes can be realized by polarization measurements.

For the input state

$$|\psi^3\rangle = \sqrt{\alpha}|2\rangle + \sqrt{(1-\alpha)/2}(|0\rangle + |1\rangle), \qquad \text{(S33)}$$

in the region $\alpha \in (1/2, 1]$, the distillation fidelity becomes $F_{\mathcal{O}}(|\psi^3\rangle, |\Phi_2\rangle) = \frac{1}{2}(\sqrt{\alpha} + \sqrt{1-\alpha})^2$, which implies that the maximally coherent state $|\Phi_2\rangle$ cannot be reached. Instead, a possible target state becomes $|\phi_{3\to 2}\rangle = \sqrt{\alpha}|0\rangle + \sqrt{1-\alpha}|1\rangle$. Figure S1(c) can realize the state transformation $|\psi^3\rangle \to |\phi_{3\to 2}\rangle$, where only three input paths ($|0\rangle$, $|1\rangle$, $|2\rangle$) are needed to accomplish the conversion. A beam displacer is employed to combine the spatial modes $|0\rangle$ and $|1\rangle$. The angle of the HWP in the path of $|2\rangle$ is adjusted according to the initial superposition coefficients of $|0\rangle$ and $|1\rangle$.

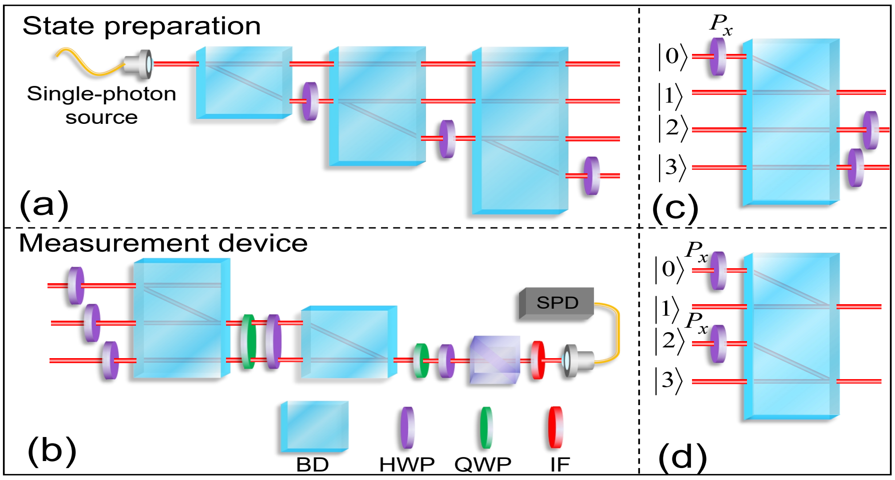

Fig. S1. Experimental implementation of the state preparation, spatial tomography, and some state transformations. (a) Device for preparing the input pure states. (b) Setup for the spatial tomography measurement on 3D states. (c) and (d) show the operations combining two paths into one path, which accomplish the dimension reductions. The devices are quarter-wave plates (QWP), phase compensators (PC), and interference filters (IF).

All the operations, performed on the spatial modes belong to SIO, which can be verified by their Kraus-operator representation in the Eq. (S29).

For the 4D input state (for $\alpha \in [0, 1/2]$)

$$|\psi^4\rangle = \sqrt{\alpha}(|0\rangle + |1\rangle) + \sqrt{1/2 - \alpha}(|2\rangle + |3\rangle), \tag{S34}$$

to extract the 2D maximally coherent resource $|\Phi_2\rangle$, the distillation fidelity $F_{\mathcal{O}}(|\psi^4\rangle, |\Phi_2\rangle)$ is proven to be 1 over the entire range $\alpha \in [0, 1/2]$. Thus a reasonable target state is $|\Phi_2\rangle = (|2\rangle + |3\rangle)/\sqrt{2}$. The general proposal (shown in the Figure 1 in the main text) can be employed to reach the target state. While, due to the special structure of $|\psi^4\rangle$, a simplified device in Fig. S1(d) is designed by using a BD to combine the paths $|0\rangle$ and $|1\rangle$ into the paths $|2\rangle$ and $|3\rangle$, respectively. This completes the state transformation.

To extract the 3D coherent resource state $|\Phi_3\rangle$, the distillation fidelity is $F_{\mathcal{O}}(|\psi^4\rangle, |\Phi_3\rangle) = \frac{2}{3}(\sqrt{\alpha} + \sqrt{1-2\alpha})^2$ in the region of $\alpha \in [0, 1/6] \cup [1/3, 1/2]$. Therefore, a reasonable target state is $|\phi_{4\to3}\rangle = \sqrt{2\alpha}|1\rangle + \sqrt{(1/2-\alpha)}(|2\rangle + |3\rangle)$. While, in the region $\alpha \in [1/6, 1/3]$, the fidelity is $F_{\mathcal{O}}(|\psi^4\rangle, |\Phi_3\rangle) = 1$. Thus, the maximally coherent state $|\Phi_3\rangle$ is the target state.

The device in Fig. S1(c) with four input paths can realize the conversion $|\psi^4\rangle \to |\phi_{4\to3}\rangle$ (in the region $\alpha \in [0, 1/6] \cup [1/3, 1/2]$), where the path $|0\rangle$ and $|1\rangle$ are combined to realize the dimension reduction.

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