nature portfolio

Peer Review File

Topological phonon blockade and its transfer via dark-mode engineering

Corresponding Author: Dr Deng-Gao Lai

This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The authors proposed a novel approach to engineer dark modes via synthetic magnetism such that exceptional points and topological operations can be shielded from the dark-mode disturbances. The dark-mode engineering is achieved by introducing a phase-dependent phonon hopping interaction in the Hamiltonian of optomechanical networks, which subsequently induces a synthetic gauge field. The dark state can be controlled by both the strength and phase of the synthetic magnetic field. The switch between topological phonon transfer (TPT) and topological phonon blockade (TPB) can be controlled by tunning the driving laser frequency and power. Topological operations around exceptional points hold potential to achieve open-system dynamics in communications and signal processing systems. Mitigating the effects of dark modes on topological responses poses a huge challenge and have attracted great attention in community of optomechanical engineering. Therefore, the current work is timely and quite original. I would recommend it for publication in Nature Communications if the authors can address the following comments:

In Fig 3, since TPT quality factor is a function of P and Δ, what is Δ used in Fig. 3a and what is P used in Fig.3b?
It is not immediately clear to me from the main text why the directional dependence of TPT factor (Fig.4) behaves as it is, i.e. why the counterclockwise direction shows more oscillations than the clockwise direction. The SI discussed a little bit about it. It would be nice to provide some simple physical intuition to help the readers understand the nonreciprocal behavior.
The supplementary material is more pleasant to read than the main manuscript because it gives way more discussion on the key definitions. The authors might consider moving some of the discussion about the experimental realization from SI to the main text. The current experimental implementation section in the main section reads insufficient. When the authors mentioned validating their approach by conducting parameter analyses and numerical simulations, I was expecting more discussion on their results but it ended abruptly there.

Reviewer #2

(Remarks to the Author)
The manuscript by Lai et al. proposes a way of switching between topological phonon transfer (TPT) and topological phonon blockade (TPB). They employ a synthetic magnetism to control the dark mode (DM) and show that they can obtain TPT in the DM breaking case. The main novelty of the manuscript comes from the fact that they can suitably switch between the DM breaking and non-breaking region. This manuscript offers a way of “avoiding” the DM to be able to do TPT. The authors have made a detailed calculation, which they provide in the supplemental material, and the paper is well-written. However, this manuscript does not seem to satisfy the criteria to be published in such a high-impact multidisciplinary journal. However, this manuscript can still be interesting to some other specialized journals.

I am a bit curious about how their explanation is related to this paper: https://doi.org/10.1126/science.1228370, where it is stated that the dark mode can still mediate some effective coupling between optical modes. However, in the current manuscript, the authors say that the DMs forbade any mode conversion. It would be nice if the authors could add a discussion comparing their statement with the other paper.

Reviewer #4

(Remarks to the Author)
The authors describe a switching mechanism between topological phonon blockade (TPB) and topological phonon transfer (TPT) driven by dark mode engineering in their manuscript. Under the influence of a synthetic magnetic field, this mechanism leads to a transition between dark-mode nonbreaking (DMN) and dark-mode breaking (DMB) regimes, accompanied by the emergence of an exceptional point (EP). They further demonstrate that performing topological operations in the DMN regime results in TPB between dark and bright modes, whereas in the DMB regime, TPT is enabled. The manuscript also discusses potential experimental platforms for realizing this phenomenon.

The work is well-organized, and the theoretical model is robust enough to support the conclusions. The proposed mechanism has broad applicability and could extend to other dark-state effects in quantum systems. These results are suitable for publication in Nature Communications after addressing the following points:

1. In general topological systems, the appearance of an EP point signifies a transition in the topological state of the system as a function of parameter space. This transition is often accompanied by a change in the system's topological invariants, such as the conversion from a topologically trivial to a topologically nontrivial state (e.g., from w = 0 to w = 1). Could the author further discuss this aspect in the context of the current study?
2. The author mentions the Quality Factor (Q-factor) of TPT in the main text, but I am unclear about its physical meaning. Is this Q-factor concept similar to the quality factor used to characterize the loss in microcavities (such as the light resonance losses)?
3. Finally, in discussing possible experimental platforms for realizing this phenomenon, the author suggests that it could be implemented in optical-mechanical platforms like optomechanical systems. I would like to add that the synthetic magnetic field indeed provides great convenience for generating gauge potentials for photons. Recently, there has been considerable discussion regarding platforms for synthetic dimensions in photons [Optica 5, 1396 (2018)], which could easily provide photon-photon interactions and synthetic magnetic fields [Light: Science & Applications 14, 39 (2025)]. Could the author elaborate on the potential for implementing this theoretical work within a synthetic dimension platform for photons? If this could eventually be transferred to a chip-based platform for scalable quantum information processors, I believe it would be highly significant.

Version 2:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The authors have satisfactorily addressed all the reviewers' comments. I certainly appreciated that the authors added extensive discussion on the experimental implementations and I think the detailed discussion on the potential experimental implementations makes the manuscript more impactful and suitable for the publication in Nat. Comm.

Therefore, I recommend this work for publication in Nat. Comm.

Reviewer #2

(Remarks to the Author)
I appreciate the authors' hard work in improving the quality of the manuscript. In the rebuttal, the authors have explained elaborately why the current work is suitable for a high-impact journal and also made some appropriate revisions in the manuscript. The current version looks reasonable to be published in Nature Communications.

Reviewer #4

(Remarks to the Author)
The Authors have performed major revisions to the Main Text and to the Supplementary Material. They have written an extensive response letter explaining their changes. I deem that they have addressed suitably
most of my comments.In addition, the results meet publication standards of Nature Communications.

*** REPORT OF REFEREE A – NCOMMS-24-77485A-Z ***

COMMENT A1

The authors proposed a novel approach to engineer dark modes via synthetic magnetism such that exceptional points and topological operations can be shielded from the dark-mode disturbances. The dark-mode engineering is achieved by introducing a phase-dependent phonon hopping interaction in the Hamiltonian of optomechanical networks, which subsequently induces a synthetic gauge field. The dark state can be controlled by both the strength and phase of the synthetic magnetic field. The switch between topological phonon transfer (TPT) and topological phonon blockade (TPB) can be controlled by tunning the driving laser frequency and power. Topological operations around exceptional points hold potential to achieve open-system dynamics in communications and signal processing systems.

OUR REPLY TO COMMENT A1

We thank Referee A for presenting an excellent summary of our manuscript and for providing the important comments and suggestions mentioned below, which are very helpful in improving our manuscript.

COMMENT A2

Mitigating the effects of dark modes on topological responses poses a huge challenge and have attracted great attention in community of optomechanical engineering. Therefore, the current work is timely and quite original. I would recommend it for publication in Nature Communications if the authors can address the following comments:

OUR REPLY TO COMMENT A2

We thank Referee A for recommending our manuscript for publication in Nature Communications and for acknowledging the timeliness and originality of our work.

Below, we respond to Referee A's comments and suggestions one by one. These comments and questions were very helpful in improving our manuscript. By revising our manuscript in response to these comments and suggestions, we have significantly improved the presentation of our results, and we believe that it is now suitable for publication in Nature Communications.

COMMENT A3

In Fig 3, since TPT quality factor is a function of P and Δ, what is Δ used in Fig. 3a and what is P used in Fig.3b?

OUR REPLY TO COMMENT A3

We apologize for the confusion caused by the unclear phrasing in our original manuscript. Inspired by Referee A's insightful comment, we have corrected this oversight in Figs. 3(a) and 3(b).

Specifically, Fig. 3(a) displays $\mathbf{F}_+$ versus Δ_{Max} for a fixed $P_{\mathrm{Max}} = 750\ \mu$W, and Fig. 3(b) exhibits $\mathbf{F}_+$ versus P_{Max} for a fixed $\Delta_{\mathrm{Max}} = -290$ kHz.

Please see the caption of Fig. 3 on page 4 of the revised main text.

COMMENT A4

It is not immediately clear to me from the main text why the directional dependence of TPT factor (Fig.4) behaves as it is, i.e. why the counterclockwise direction shows more oscillations than the clockwise direction. The SI discussed a little bit about it. It would be nice to provide some simple physical intuition to help the readers understand the nonreciprocal behavior.

OUR REPLY TO COMMENT A4

We are sorry for causing confusion about the lack of description on why the counterclockwise direction shows more oscillations than the clockwise direction. Inspired by Referee A's insightful comments, we proceed to elaborate in detail the underlying physical mechanisms.

For example, when the dark mode is initially excited and the system operates in the $\mathcal{DMB}$ regime, we reveal that adiabatically encircling an EP in the counterclockwise direction gives rise to an excellent TPT [see the red dashed curve in Fig. 4(b), $\mathbf{F}_+ \to 1$], but not in the clockwise sense [see the red solid curve and symbols in Fig. 4(a), $\mathbf{F}_+ \to 0$]. Physically, adiabatic behavior occurs when the system operates only in the less-damped eigenmode. This is because when working in the more-damped mode, the fierce competition between the differential-loss effect (that is exponentially large in τ) and the nonadiabatic transfer (which is exponentially small in τ) results in a breakdown of adiabaticity, which causes the system to eventually relax to the less-damped mode. Such a process can also be interpreted as a manifestation of the Stokes phenomenon of asymptotics.

Please see the last paragraph on page 4 and the first paragraph on page 5 of the revised main text.

COMMENT A5

The supplementary material is more pleasant to read than the main manuscript because it gives way more discussion on the key definitions. The authors might consider moving some of the discussion about the experimental realization from SI to the main text. The current experimental implementation section in the main section reads insufficient. When the authors mentioned validating their approach by conducting parameter analyses and numerical simulations, I was expecting more discussion on their results but it ended abruptly there.

OUR REPLY TO COMMENT A5

We sincerely appreciate Referee A for recognizing the readability and richness of our supplementary material. Following Referee A's insightful suggestion, we have moved the detailed discussions on the experimental realizations and the essential definitions to the revised main text from supplemental material, for enhancing readability and accessibility of the main text. Specifically, we have incorporated the following additions into the updated main text.

The proposed physical model is general and, in principle, can be implemented using standard optomechanical platforms. Realizing the TPT while maintaining immunity against dark modes requires two key ingredients: (i) in-parallel optomechanical couplings between multiple phonon modes and a shared photon mode, and (ii) phase-dependent phonon-hopping interactions between the nearest-neighbor phonon modes. Both types of interactions must be accessible within viable experimental platforms. While each type of coupling has been demonstrated independently in previous experiments, their simultaneous implementation within the same experimental setup has not yet been reported. Nevertheless, under current state-of-the-art experimental conditions, integrating both kinds of interactions into a unified system appears fully feasible.

Recent advances have enabled the experimental realization of in-parallel optomechanical couplings between multiple phonon modes and a shared photon mode in both optical [R1–R3] and microwave [R4–R8] domains. In the optical regime, these in-parallel optomechanical couplings are implemented via "membrane-in-the-middle" optomechanical architectures [R1–R3]; whereas in the microwave domain, they are realized using circuit electromechanical platforms [R4–R8]. Simultaneously, the phase-dependent phonon-hopping interaction between the nearest-neighbor phonon modes can be implemented using photonic-crystal optomechanical platforms [R9] or circuit electromechanical systems [R4–R8, R10]. In photonic-crystal-based implementations, this phase-dependent coupling arises from the mediation of two auxiliary cavity fields [R9]; whereas in circuit electromechanical systems, it emerges indirectly through the coupling of two mechanical resonators to a charge qubit.

(i) Building on these state-of-the-art experimental advances in both parallel optomechanical couplings and phase-dependent phonon-hopping interactions, the proposed model can be easily realized using photonic-crystal optomechanical architectures featuring optical and mechanical couplings between two optomechanical cavities [R9], as shown in Fig. R1(a). In this scheme, each cavity is driven by a distinct phase-correlated laser field, and the effective

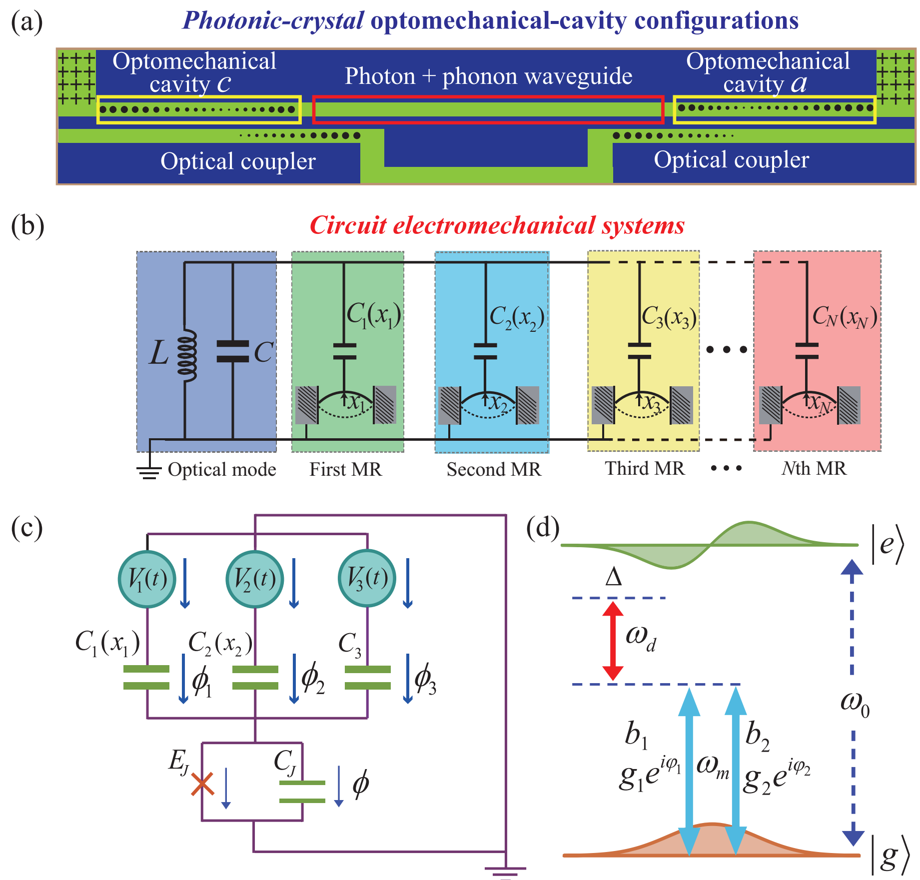

FIG. R1. (a) Realization of the model in a photonic-crystal optomechanical circuit. This system, fabricated on a silicon-on-insulator microchip, consists of two nanobeam optomechanical-crystal cavities connected by a central unpatterned nanobeam waveguide. Each of the left and right optical couplers is fed by an adiabatic fiber-to-chip coupler, facilitating evanescent coupling of light into either of two optical cavities. The model can be described as a four-mode optomechanical system, where the two photon (phonon) modes couple through a photon (phonon) hopping interaction. In the large-detuning regime, adiabatic elimination of any one auxiliary photon mode reduces the system into a three-mode loop-coupled optomechanical configuration, where two phonon modes are coupled to a common photon mode. Two different phase-correlated laser pumpings of the two optomechanical cavities induce an effective synthetic magnetic field. (b) Schematic of the electromechanical circuit, comprising a microwave cavity formed by $(N+1)$ capacitances [C and $C_{j=1\cdots N}(x_j)$] and an inductance (L). The jth capacitance, $C_{j=1\cdots N}(x_j)$, is modulated by the displacement x_j of the corresponding micromechanical resonator (MR) b_j, thereby altering the total capacitance and tuning the cavity frequency ω_c. A phase-dependent phonon-hopping interaction between nearest-neighboring MRs is realized via the superconducting circuit illustrated in panel (c). (c) Superconducting quantum circuit: A Josephson junction, characterized by Josephson energy E_J and capacitance C_J, is connected to three gate voltages $V_{j=1,2,3}(t)$ via the corresponding gate capacitances $C_{j=1,2}(x_j)$ and C_3. Two MRs couple to the superconducting charge qubit through the gate capacitances $C_{j=1,2}(x_j)$. By carefully designing the gate voltages, a phase-dependent phonon-hopping interaction between the two MRs is induced. The phase drops across these capacitors and the Josephson junction are denoted as ϕ_j and ϕ, respectively. (d) Energy-level structure and resonance frequencies of the coupled qubit-resonator system. Two MRs with frequency ω_m phase-dependently couple to the superconducting charge qubit with an energy separation ω_0. The ac gate voltages with frequency ω_d are applied to the qubit via the gate capacitors.

implementation of our model emerges through the adiabatic elimination of any one cavity-field mode under the large-detuning regime. Notably, we consider a multimode physical system featuring both optical and mechanical interactions between the two optomechanical cavities [Fig. R1(a)]. The corresponding system Hamiltonian reads:

$$\begin{aligned}\mathcal{H} \;=\; & \omega_c c^\dagger c + \omega_a a^\dagger a + \omega_1 b_1^\dagger b_1 + \omega_2 b_2^\dagger b_2 + g_c c^\dagger c(b_1^\dagger + b_1) + g_a a^\dagger a(b_2^\dagger + b_2) \\ & + J(c^\dagger a + a^\dagger c) + \xi(b_1^\dagger b_2 + b_2^\dagger b_1) + (\Omega_c c e^{i\omega_{L,c}t} + \Omega_c^* c^\dagger e^{-i\omega_{L,c}t}) \\ & + \left(\Omega_a a e^{i\omega_{L,a}t} + \Omega_a^* a^\dagger e^{-i\omega_{L,a}t}\right), \end{aligned} \tag{R1}$$

where a, c, and b_j denote the annihilation operators for the two photon modes (with frequencies ω_a and ω_c) and the jth phonon mode (with frequency ω_j), respectively. The optomechanical interactions between the photon and phonon modes are denoted by the g_a and g_c terms. The external drivings of the two cavities (a and c) are characterized by Ω_a and Ω_c, respectively. In the large-detuning regime of cavity mode a, this optical mode can be adiabatically eliminated, yielding an effective Hamiltonian:

$$\begin{aligned}\mathcal{H}_{\text{eff}} \;\approx\; & \tilde{\Delta}_c' \delta c^\dagger \delta c + \omega_1' \delta b_1^\dagger \delta b_1 + \omega_2' \delta b_2^\dagger \delta b_2 + \xi(e^{i\Theta}\delta b_1^\dagger \delta b_2 + e^{-i\Theta}\delta b_2^\dagger \delta b_1) \\ & + (G_{c,1}\delta c^\dagger \delta b_1 + G_{c,1}^* \delta c \delta b_1^\dagger) + (G_{c,2}\delta c^\dagger \delta b_2 + G_{c,2}^* \delta c \delta b_2^\dagger), \end{aligned} \tag{R2}$$

where $\tilde{\Delta}_c' = \Delta_c' + \delta\omega_c$ and $\omega_j' = \omega_j + \delta\omega_j$. As evident from Eq. (R2), both the proposed model and the resulting synthetic magnetism can be effectively realized under state-of-the-art experimental conditions using photonic-crystal optomechanical-cavity systems, as depicted in Fig. R1(a).

(ii) Moreover, the proposed model can be readily implemented using circuit electromechanical platforms [R4–R8, R10], which comprise N MRs (i.e., $b_{j=1\cdots N}$) coupled to a microwave cavity characterized by an equivalent inductance L and capacitance C, as illustrated in Fig. R1(b). In this quantum setup, the displacement $x_{j=1\cdots N}$ of each MR independently modulates the total capacitance via $C_{j=1\cdots N}(x_j)$, thereby tuning the cavity resonance frequency ω_c. This modulation interaction gives rise to an electromechanical coupling described by $g_{j=1\cdots N} = (\omega_c/2C)\partial C_j(x_j)/\partial x_j$, enabling the precise quantum control over the system's dynamics.

Meanwhile, an effective phase-dependent phonon-hopping interaction between the two nearest-neighbor MRs is induced by the coupling of the MRs to a superconducting charge qubit, as described in Fig. R1(c). To elucidate the underlying physical mechanism, we present the schematic energy levels and relevant resonance frequencies of this coupled qubit-resonator system in Fig. R1(d). Note that detailed analytical derivations and analyses of the proposed experimental implementation using circuit electromechanical systems are provided in the Supplemental Material.

By suppressing the first-order physical process, we derive an effective Hamiltonian that captures the second-order interaction:

$$\mathcal{H}_{\text{eff}}' \approx \sum_{j=1,2} \omega_m' b_j^\dagger b_j + \xi\left(b_1^\dagger b_2 e^{i\Theta} + b_2^\dagger b_1 e^{-i\Theta}\right), \tag{R3}$$

where $\omega_m' = (\omega_m + \frac{g_j^2}{\Delta}\tau_z)$, $\xi = \frac{g_1 g_2}{\Delta}\tau_z$, and $\Theta = \varphi_1 - \varphi_2$. The derived Hamiltonian in Eq. (R3) reveals that the phase-dependent interaction emerges between the two MRs. Assuming the qubit is initially in its ground state, we derive an effective phase-dependent phonon-hopping interaction, in which the modulation phase in the loop-coupling configuration induces synthetic magnetism. These findings highlight the direct relevance of the proposed phenomena to state-of-the-art experiments in circuit electromechanical systems, suggesting that current experimental capabilities are sufficient to realize the proposed scheme and that the predicted effects are observable with cutting-edge implementations.

Please see the section of "Experimental implementations" on page 7 of the revised main text.

In summary, we thank Referee A for carefully reading our manuscript and providing a highly valuable report, which has been extremely helpful in improving our manuscript. Thanks to these comments and suggestions, the quality of our manuscript has been significantly improved. We therefore hope that our paper is now suitable for publication in Nature Communications.

*** REPORT OF REFEREE B – NCOMMS-24-77485A-Z ***

COMMENT B1

The manuscript by Lai et al. proposes a way of switching between topological phonon transfer (TPT) and topological phonon blockade (TPB). They employ a synthetic magnetism to control the dark mode (DM) and show that they can obtain TPT in the DM breaking case. The main novelty of the manuscript comes from the fact that they can suitably switch between the DM breaking and non-breaking region. This manuscript offers a way of "avoiding" the DM to be able to do TPT.

OUR REPLY TO COMMENT B1

We thank Referee B for the thorough and careful reading of our manuscript and for providing an excellent summary of our work. Incorporating Referee B's insightful comments and suggestions, we have made every effort to improve the manuscript. We address each point in detail below.

COMMENT B2

The authors have made a detailed calculation, which they provide in the supplemental material, and the paper is well-written. However, this manuscript does not seem to satisfy the criteria to be published in such a high-impact multidisciplinary journal. However, this manuscript can still be interesting to some other specialized journals.

OUR REPLY TO COMMENT B2

We sincerely appreciate Referee B's careful evaluation of our work and the acknowledgment of the detailed calculations and clarity of our manuscript. To meet the publication standards of Nature Communications, we have substantially revised our manuscript with our utmost effort, in light of Referee B's insightful comments and helpful suggestions.

Below, we present three key reasons why our manuscript fulfills the publication criteria of Nature Communications:

(1) Originality and Novelty: Our work introduces a new conceptual approach at the intersection of quantum optomechanics and topological physics.

A very recent Science article [R11][T. J. Kippenberg, Science **386**, 1383 (Dec. 2024)] emphasized that all dark modes, being decoupled from the system, remain inaccessible to quantum ground-state preparation, allowing only the bright mode, coupled to the system, to be cooled. Building on this insight, we are the first to harness the dark-right mode framework to explore topological effects, enabling both topological phonon transfer and topological phonon blockade between dark and bright modes. This innovation opens a novel direction in both topological and quantum optomechanical research.

(2) Significance and Impact: Our study addresses a long-standing and fundamental challenge in the field of topological phononics.

Two back-to-back Nature articles [R1, R12] [i.e., Nature **537**, 80 (2016); Nature **537**, 76 (2016)] implicitly pointed out a critical limitation: dark modes obstruct topological dynamics, preventing both mode conversion and phonon transfer. While previous efforts focused on circumventing dark modes, our work directly confronts and resolves this problem originating from dark-mode contaminations by engineering interactions that selectively activate dark modes for topological transport.

Specifically, unlike prior schemes [R1] [e.g., Nature **537**, 80 (2016)], which require non-degenerate mechanical resonators to achieve topological encircling around an exceptional point, our approach reveals that even degenerate mechanical resonators can support equivalent topological behavior. This result is both counterintuitive and unprecedented, marking a key conceptual advance in the field.

(3) Broad Applicability and Relevance: Our findings introduce a versatile strategy with wide-ranging implications.

The framework developed here can be broadly applied to control collective quantum motion in macroscopic mechanical systems. As highlighted in Ref. [R11] [Tobias J. Kippenberg, Science **386**, 1383 (Dec. 2024)], the quantum control of collective phonon modes in large-scale mechanical resonators represents an emerging research frontier. Our method for engineering nonreciprocal and topological phonon transfer via dark-mode manipulation contributes a powerful and widely applicable tool for this growing field.

A recent breakthrough experiment [R11] [Tobias J. Kippenberg, Science **386**, 1383 (Dec. 2024)] highlights a key limitation in observing quantum collective motion in macroscopic mechanical resonators: Only a single bright mode, coupled to the system, can be cooled to its quantum ground state, while all $N-1$ dark modes remain entirely decoupled from the system. This inherent decoupling renders the dark modes inaccessible to conventional quantum control and ground-state preparation.

Building on this pioneering work [R11] [Tobias J. Kippenberg, Science **386**, 1383 (Dec. 2024)], our study fundamentally overcomes this constraint by introducing a mechanism that enables simultaneous ground-state preparation of both bright and dark modes. By simply activating synthetic magnetism, our scheme unlocks the full quantum potential of collective phononic dynamics. In doing so, it establishes a new paradigm in quantum optomechanics, which is no longer bound by the limitations imposed by dark modes.

It is evident from Fig. R2(c) that in the $\mathcal{DMN}$ regime, all dark modes remain *uncooled* ($n_{\mathcal{D}} = 10^3$, see upper blue horizontal solid lines and discs); whereas in the $\mathcal{DMB}$ regime, they are simultaneously *cooled* to their quantum ground states ($n_{\mathcal{D}} < 1$, lower red solid curves and discs). These counterintuitive phenomena arise from the underlying physical mechanism: dark modes are naturally decoupled from the system and then, thermal noise trapped in dark modes cannot be extracted through sideband cooling [R13–R16], making the ground-

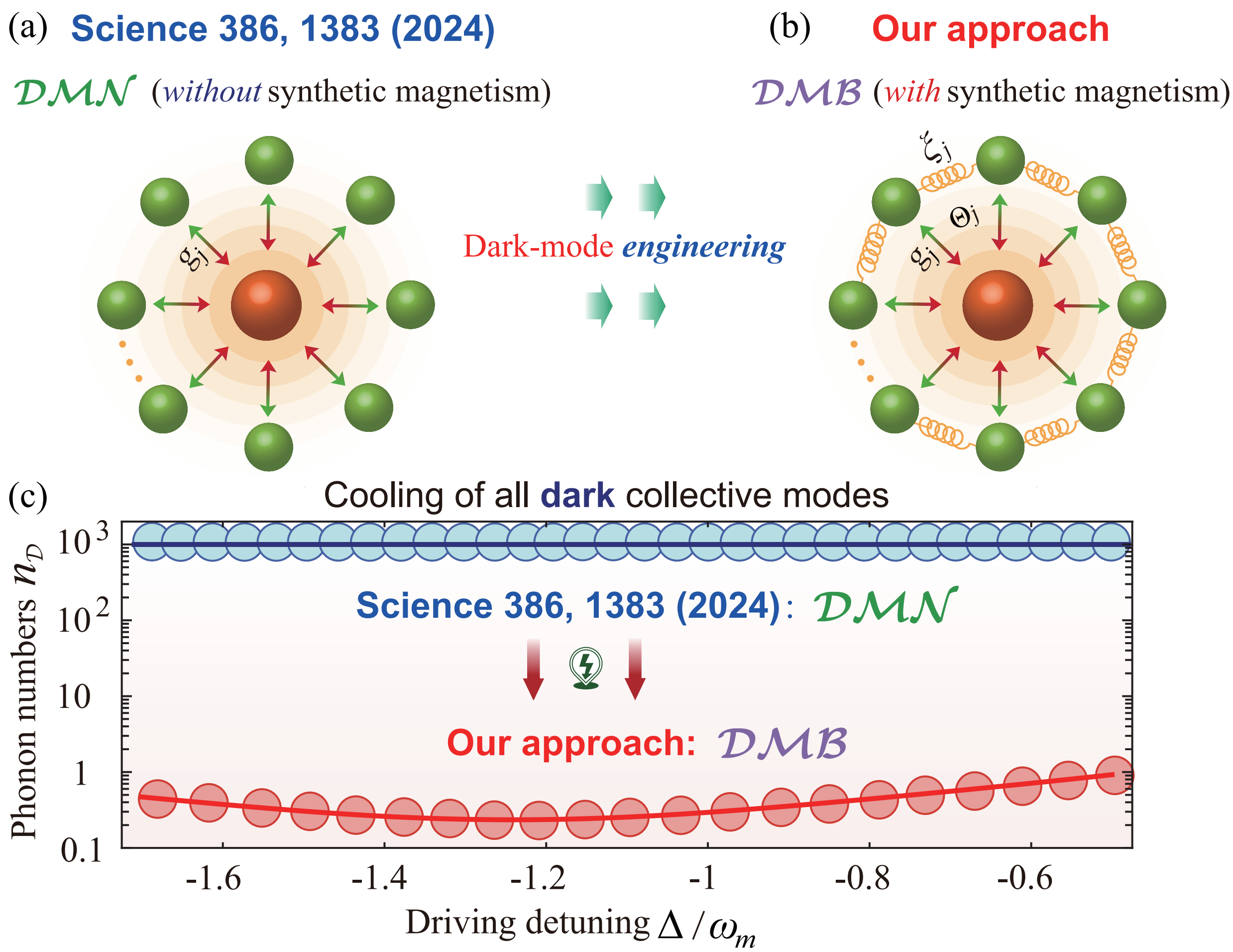

FIG. R2. (a) Schematic of quantum optomechanical networks consisting of N mechanical modes b_j (with decay rates γ_j) coupled to a shared cavity-field mode a (with damping rate κ) via optomechanical coupling g_j. The nearest-neighbor vibrational modes are coupled to each other via phase-dependent phonon-hopping interactions (η_j, Θ_j). Effective mean phonon number $n_{\mathcal{D}}$ of all $N-1$ dark modes versus the driving detuning Δ in the $\mathcal{DMN}$ ($\xi_j = 0$, horizontal solid lines) [R11] and $\mathcal{DMB}$ ($\xi_j/\omega_m = 0.05$, $\Theta_1/\pi = 1/2$, $\Theta_{j\in[2,N-1]} = 0$, dashed curves) regimes. Here we consider the case of $N = 4$ and set $\omega_j/\omega_m = 1$, $G_j/\omega_m = 0.1$, $\kappa/\omega_m = 0.2$, $\gamma_j/\omega_m = 10^{-6}$, and $\bar{n}_{\rm th} = 10^3$.

state preparation of these dark modes unattainable [R11]. In contrast, upon transitioning into the $\mathcal{DMB}$ regime, simultaneous quantum ground-state preparation of both dark and bright modes becomes achievable near the red-sideband resonance, i.e., $\Delta/\omega_m = -1$. These findings demonstrate that the proposed dark-mode engineering offers flexible control and effective protection of fragile collective quantum ground states.

Please see Fig. 5 (a) and the second and third paragraphs of the right column on page 5 of the revised main text. Note that the detailed analytical derivations of the final phonon occupations of the dark and bright modes are presented in Sec. V of the revised Supplementary Material.

Specifically, in this groundbreaking experiment [R11], the coupling of N phonon modes to a shared photon mode results in a single bright collective mode coupled to the system and $N-1$ dark collective modes are decoupled from this system. Consequently, although the quantum ground state can be prepared only for the bright collective mode, it remains intrinsically unattainable for all dark collective modes [R11]. Building upon this pioneering work [R11], our method can directly address this outstanding challenge posed by dark collective modes, introducing a novel paradigm for the quantum ground-state cooling of both dark and bright collective modes.

We will provide a detailed elaboration of this approach below.

Dark and bright collective modes and their engineering

(i) To elucidate the dark and bright collective modes, we first consider the conventional scenario [R11] in which phase-dependent phonon-hopping interactions are absent ($\xi_j = 0$) [see Fig. R2(a)], and all phonon modes exhibit degeneracy ($\omega_j = \omega_m$) and symmetric coupling ($g_j = g$). Upon performing the linearization procedure, the system Hamiltonian assumes the form of an effective beam-splitter interaction:

$$\mathcal{H} = -\Delta \delta a^\dagger \delta a + \omega_m \mathcal{B}^\dagger \mathcal{B} + \omega_m \sum_{l=1}^{N-1} \mathcal{D}_l^\dagger \mathcal{D}_l + G\sqrt{N}(\delta a^\dagger \mathcal{B} + \mathcal{B}^\dagger \delta a), \tag{R4}$$

where B represents a single bright collective mode coupled to the system, while D_l denotes the lth dark collective mode decoupled from this system [R11]:

$$\mathcal{B} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \delta b_j, \quad \textit{Bright} \text{ collective mode}, \tag{R5a}$$

$$\mathcal{D}_{l\in[1,N-1]} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \delta b_j e^{2\pi i (j-\frac{N+1}{2}) l/N}, \quad \textit{Dark} \text{ collective modes}, \tag{R5b}$$

with G denoting the effective optomechanical coupling strength. *These dark collective modes are decoupled from the system, thereby preventing their sideband cooling.*

(ii) Surprisingly, all dark collective modes can be controlled at will simply by employing synthetic magnetism ($\xi_j \neq 0$, $\Theta_j \neq 2n\pi$), as illustrated in Fig. R2(b). To illustrate the underlying physics behind this counterintuitive phenomenon, we assume, without loss of generality, that $\xi_j = \xi$. The system Hamiltonian can then be rewritten in terms of the mechanical normal modes associated with synthetic magnetism:

$$\mathcal{H} = -\Delta \delta a^\dagger \delta a + \sum_{k=1}^{N} \Omega_k B_k^\dagger B_k + \sum_{k=1}^{N} \left\{ \frac{G}{A} \left[\sin\left(\frac{k\pi}{N+1}\right) + \sum_{j=2}^{N} e^{i\sum_{\nu=1}^{j-1}\Theta_\nu} \sin\left(\frac{jk\pi}{N+1}\right) \right] a B_k^\dagger + \text{H.c.} \right\}, \tag{R6}$$

where B_k denotes the kth mechanical normal mode with resonance frequency $\Omega_k = \omega_m + 2\xi \cos\left(\frac{k\pi}{N+1}\right)$ for $(k = 1, ..., N)$. Note that the operators δb_j and B_k obey the following relationship:

$$\delta b_j = \begin{cases} \frac{1}{A} \sum_{k=1}^{N} \sin\left(\frac{k\pi}{N+1}\right) B_k, & j = 1, \\ \frac{1}{A} e^{-i\sum_{\nu=1}^{j-1}\Theta_\nu} \sum_{k=1}^{N} \sin\left(\frac{jk\pi}{N+1}\right) B_k, & j \geq 2, \end{cases} \tag{R7}$$

with $A = \sqrt{(N+1)/2}$. Clearly, Eq. (R6) elucidates the pivotal role of the modulation phases in shaping optomechanical interactions, as governed by the summation term $\sum_{\nu=1}^{j-1}\Theta_\nu$ ($j\in[2,N]$). This observation suggests that the implementation of a single modulation phase suffices for dark-mode engineering. To streamline our analytical analysis, we thus adopt the simplifying assumption $\Theta_{j\in[2,N-1]} = 0$ in the subsequent discussion. Furthermore, our findings establish that for odd values of k, the coupling strength between the photon mode a and the kth mechanical normal mode B_k remains nonzero. Conversely, when k is even, the coupling strength between a and B_k takes the following form:

$$G_k = \frac{G}{A}\left(1-e^{i\Theta_1}\right)\sin\left(\frac{k\pi}{N+1}\right). \tag{R8}$$

Evidently, Eq. (R8) reveals that when $\Theta_1 = 2n\pi$, the emergence of dark collective modes that completely decouple from the system (i.e., $G_k = 0$) gives rise to the $\mathcal{DMN}$ regime. Surprisingly, however, by simply tuning $\Theta_1 \neq 2n\pi$, all dark collective modes can acquire finite effective couplings ($G_k \neq 0$) to the system, thereby transitioning into the $\mathcal{DMB}$ regime. Consequently, while achieving the quantum ground state for all dark collective modes is inherently unattainable in the $\mathcal{DMN}$ regime, it is, counterintuitively, rendered feasible in the $\mathcal{DMB}$ regime.

Occupations of dark and bright collective modes

By utilizing the covariance matrix representation in the basis of mechanical modes, we can systematically quantify the phonon occupation in all dark and bright collective modes. Given that the initial state is Gaussian, the Gaussian nature of the state is preserved throughout its evolution. As a result, the covariance matrix provides a complete description of the system and, in the vibrational-mode basis, takes the form

$$\mathrm{Cov} = \begin{pmatrix} \left\langle \delta b_1^\dagger \delta b_1 \right\rangle & \left\langle \delta b_1^\dagger \delta b_2 \right\rangle & \cdots & \left\langle \delta b_1^\dagger \delta b_N \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle \delta b_N^\dagger \delta b_1 \right\rangle & \left\langle \delta b_N^\dagger \delta b_2 \right\rangle & \cdots & \left\langle \delta b_N^\dagger \delta b_N \right\rangle \end{pmatrix}, \tag{R9}$$

where

$$\left\langle \delta b_i^\dagger \delta b_j \right\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{b_i^\dagger b_j}(\omega)\, d\omega. \tag{R10}$$

The occupations of the dark and bright collective modes can be directly determined by diagonalizing the covariance matrix Eq. (R9). The eigenvalues of Eq. (R9) encapsulate the occupations of these collective modes: the lowest eigenvalue corresponds to the phonon population in a single bright collective mode (i.e., $n_{\mathcal{B}}$), while the remaining $N-1$ eigenvalues represent the occupations of $N-1$ dark collective modes (i.e., $n_{\mathcal{D}_l}$), respectively. Given the presence of $N-1$ dark collective modes, we define the mean phonon number of these dark modes as

$$n_{\mathcal{D}} = \frac{1}{N-1}\sum_{l=1}^{N-1} n_{\mathcal{D}_l}, \tag{R11}$$

which explicitly reveals that the condition $n_{\mathcal{D}} < 1$ signifies the simultaneous preparation of all dark collective modes in their quantum ground states.

Quantum collective cooling using dark mode engineering

The engineering of dark collective modes provides a viable pathway for realizing and protecting fragile quantum collective motion from various disturbances associated with dark modes in practical devices. Moreover, it facilitates the implementation of noise-free quantum optomechanical networks. As shown in Figs. R2(c), in the $\mathcal{DMN}$ regime, all dark collective modes remain *uncooled* ($n_{\mathcal{D}} = 10^3$), whereas in the $\mathcal{DMB}$ regime, they are effectively *cooled* to their quantum motional ground states ($n_{\mathcal{D}} < 1$). Physically, in the $\mathcal{DMN}$ regime, dark collective modes are intrinsically decoupled from the system, preventing the extraction of thermal phonons trapped within these dark modes via sideband cooling [R13–R16], thus rendering quantum ground-state preparation unattainable. In contrast, in the $\mathcal{DMB}$ regime, all dark and bright collective modes can be simultaneously driven into their quantum ground states near the red-sideband resonance ($\Delta \approx -\omega_m$). These findings underscore that the preparation of fragile quantum collective ground states can be not only flexibly engineered but also robustly protected through synthetic magnetism.

COMMENT B3

I am a bit curious about how their explanation is related to this paper: https://doi.org/10.1126/science.1228370, where it is stated that the dark mode can still mediate some effective coupling between optical modes. However, in the current manuscript, the authors say that the DMs forbade any mode conversion. It would be nice if the authors could add a discussion comparing their statement with the other paper.

OUR REPLY TO COMMENT B3

We appreciate Referee B for bringing these important references [R17–R19] [**Science 338, 1609 (2012)**] to our attention. We have cited these significant articles in the revised main text. Please see Refs. [65-67] in the updated version.

Optomechanical dark modes, analogous to the coherent-population trapped state or dark state in atomic physics [R20, R21], can be broadly classified into two categories: **optical** dark modes and **mechanical** dark modes. Specifically, optical dark modes arise from the coupling of two optical modes to a shared mechanical mode [R17–R19] [**Science 338, 1609 (2012)**], while mechanical dark modes emerge from the coupling of two mechanical modes to a common optical mode [R11] [**Science 386, 1383 (Dec. 2024)**]. **Notably, the paper [R17] highlighted by Referee B focuses on optical dark modes that enable optical mode conversion, whereas our work focuses on mechanical dark modes that suppress mechanical mode conversion.**

The phonon-mediated and photon-mediated processes, which respectively correspond to the optical and mechanical mode conversions, can be pursued using multimode optomechanical systems. However, the influence of optomechanical dark modes on these two conversions differs fundamentally. Below, we present a comprehensive exposition of these aspects.

(i) In the previous References [R17–R19] [**Science 338, 1609 (2012)**], an optical dark (bright) mode, formed by a special coherent superposition of two optical modes, is decoupled from (coupled to) the mechanical mode. Consequently, the formation of the optical dark mode serves as a shield, protecting the system from mechanical dissipation. Despite being decoupled, the optical dark mode can still mediate an effective interaction between the two optical modes, thereby facilitating the optical-field conversion between them. Moreover, the presence of the optical dark mode enables a phonon-mediated coupling (which is immune to thermal mechanical motion) for various quantum applications, thereby eliminating the need for quantum ground-state cooling of the mechanical resonator.

(ii) In our work, the mechanical dark mode emerges as a distinct coherent superposition of two mechanical modes coupled to a common cavity mode. The interference-induced cancellation in coupling effectively decouples the mechanical dark mode from the system. While this decoupling still permits an effective interaction between the two mechanical modes, both mechanical mode conversion and topological phonon transfer between mechanical dark and bright modes become fundamentally suppressed, regardless of adiabatic trajectory design or system parameter tuning. The underlying physical mechanism of this counterintuitive phenomenon is due to dark-mode-induced disruption of both topological operations and EPs.

In this work, we address this long-standing challenge posed by mechanical dark modes and demonstrate a fundamentally distinct one-way topological phonon transfer, revealing an unexpected immunity to dark-mode-induced obstruction. This robustness arises from the interplay between topological operations and synthetic magnetism, enabling a controlled transition between the $\mathcal{DMN}$ and $\mathcal{DMB}$ regimes. In the following, we offer a comprehensive analysis of this phenomenon.

In the $\mathcal{DMN}$ regime (i.e., in the absence of synthetic magnetism), we find that regardless of laser power, the mechanical spectrum of the dark mode remains unchanged under parameter evolution. However, the eigenvalue of the bright mode traces a closed trajectory, returning to its initial point, signifying the absence of the EP due to the influence of the dark mode [see Fig. 1(c) of the main text]. Strikingly, in the $\mathcal{DMB}$ regime (i.e., in the presence of synthetic magnetism), a starkly different behavior emerges: for lower laser power, each eigenvalue follows a closed path, returning to its origin, whereas for higher laser power, the eigenvalues evolve along open trajectories, with each terminating at the starting point of the other, indicating the formation of the EP via dark-mode breaking [see Fig. 1(d) of the main text]. By tuning Δ and P, the EP [marked by the yellow star in Fig. 1(d)] emerges exclusively in the $\mathcal{DMB}$ regime but is entirely absent in the $\mathcal{DMN}$ regime, underscoring the crucial role of synthetic magnetism in overcoming dark-mode constraints.

Furthermore, in the $\mathcal{DMN}$ regime (i.e., without synthetic magnetism), only the bright mode evolves with P, while the dark mode remains completely invariant, irrespective of system parameter tuning. This leads to the full deactivation of both the EP and topological operations [see Fig. 2(a) of the main text], thereby blocking both mode conversion and phonon transfer between the dark and bright modes. In contrast, in the $\mathcal{DMB}$ regime (i.e., with synthetic magnetism), both the bright and dark modes evolve simultaneously with P, facilitating the emergence of the EP characteristics and thereby activating both EPs and topological operations [see Fig. 2(b) of the main text]. This transition enables the controlled revival of both topological mode conversion and phonon transfer.

As a consequence, in the $\mathcal{DMN}$ regime, thermal phonons confined in the dark mode remain completely isolated from the system, regardless of system parameter adjustments, leading to topological phonon blockade. Strikingly, in the $\mathcal{DMB}$ regime, synthetic magnetism enables an efficient extraction of thermal phonons from the dark mode to the bright mode, realizing topological phonon transfer between the dark and bright modes. These findings demonstrate that synthetic magnetism provides an unprecedented degree of control over dark modes, establishing an ultraflexible switch between topological phonon blockade and topological phonon transfer while also safeguarding topological functionalities against dark-mode-induced perturbations in practical devices.

Physically, in conventional schemes (i.e., without synthetic magnetism), topological operations are inherently fragile to dark modes, which remain decoupled from the system and ultimately lead to the breakdown of EPs and topological operations [R1, R12]. However, by leveraging synthetic magnetism, full topological robustness is achieved, guaranteeing the integrity of all topological operations. This strategy provides a groundbreaking avenue to overcome dark-mode-induced constraints, enabling the realization of topological quantum resources immune to dark-mode-induced constraints for practical applications.

We have added some discussions to further demonstrate the differences and relationship between the mechanical and optical dark modes. Please see the first and second paragraphs in the section of "Discussions" on page 8 of the revised main text.

We sincerely appreciate Referee B for the thorough review of our manuscript and for providing insightful comments and constructive suggestions. Incorporating this valuable feedback has greatly enhanced the clarity and rigor of our work. We now believe that our manuscript meets the high standards necessary for publication in Nature Communications.

*** REPORT OF REFEREE C – NCOMMS-24-77485A-Z ***

COMMENT C1

The authors describe a switching mechanism between topological phonon blockade (TPB) and topological phonon transfer (TPT) driven by dark mode engineering in their manuscript. Under the influence of a synthetic magnetic field, this mechanism leads to a transition between dark-mode nonbreaking (DMN) and dark-mode breaking (DMB) regimes, accompanied by the emergence of an exceptional point (EP). They further demonstrate that performing topological operations in the DMN regime results in TPB between dark and bright modes, whereas in the DMB regime, TPT is enabled. The manuscript also discusses potential experimental platforms for realizing this phenomenon.

OUR REPLY TO COMMENT C1

We thank Referee C for providing an excellent summary of our manuscript and for offering insightful comments and helpful suggestions, which have significantly improved the manuscript.

COMMENT C2

The work is well-organized, and the theoretical model is robust enough to support the conclusions. The proposed mechanism has broad applicability and could extend to other dark-state effects in quantum systems. These results are suitable for publication in Nature Communications after addressing the following points:

OUR REPLY TO COMMENT C2

We sincerely thank Referee C for the positive feedback on the organization of our manuscript and the robustness of our theoretical model, and especially for recommending our work for publication in Nature Communications. We also appreciate the recognition of the broad applicability of our proposed mechanism and its potential to extend to other dark-state effects in quantum systems.

Referee C's insightful comments and suggestions have been very helpful in improving the manuscript. We have carefully addressed all the points raised and revised the manuscript accordingly. Detailed point-by-point responses are provided below.

COMMENT C3

1. In general topological systems, the appearance of an EP point signifies a transition in the topological state of the system as a function of parameter space. This transition is often accompanied by a change in the system's topological invariants, such as the conversion from a topologically trivial to a topologically nontrivial state (e.g., from $w = 0$ to $w = 1$). Could the author further discuss this aspect in the context of the current study?

OUR REPLY TO COMMENT C3

Yes, indeed. In a general topological system, the emergence of an EP signals a transition in the system's topological phase as a function of the parameter space. This transition is often accompanied by a change in the system's topological invariants, such as the conversion from a topologically trivial to a topologically nontrivial state [R22]. **However, these behaviors always occur in the "traditional energy-band topology", involving non-Hermitian *gap* structures [R22].**

In this work, we focus on a three-mode optomechanical system consisting of two mechanical modes coupled to a shared cavity-field mode. By adiabatically eliminating the cavity-field mode, we obtain an effective non-Hermitian Hamiltonian for the two mechanical modes. In this system, topological operations encircling an EP enable nonreciprocal phonon transfer between two mechanical normal modes. **Note that our system is a unique non-Hermitian *gapless* structure, and the traditional concept of "energy-band topology" is not directly applicable to our studied system. This indicates that the system mentioned by Referee C is fundamentally different from the one investigated in our work, as our focus is exclusively on the non-Hermitian gapless structure rather than on the gap structure. Therefore, the transition from a topologically trivial to a nontrivial state, which happens in the traditional energy-band topology, lies beyond the scope of our system.**

Specifically, our model is a few-mode optomechanical system not a solid lattice, which refers to the ordered arrangement of atoms, molecules, or ions within the structure of a solid material. Moreover, our three-mode optomechanical system studied here is not an optomechanical lattice or optomechanical crystal, which enable the synthesis of non-trivial band structures and have been actively studied in the field of topological phononics [R23, R24]. ***Note that yet nearly all previous schemes have used optomechanical lattices and/or crystals for studying the TPT. By contrast, new dynamics and applications in the TPT are expected when using few-mode (e.g, three-mode) optomechanics, i.e., systems that use a small number of optomechanical degrees of freedom.*** Our work on the TPT differs from what is known in the optomechanical lattices/crystals (with energy bands), primarily because we are not focused on a one-dimensional (1D) chain with a topological band structure exhibiting topologically protected edge states [R24], a two-dimensional (2D) honeycomb lattice realizing the strained graphene model with edge states [R24], or multiscale optomechanical crystals comprising over 800 cavity-optomechanical elements fabricated on the surface of a silicon microchip [R23], but on the few-mode cavity optomechanical system (*without* energy bands) consisting of the two vibrational modes coupled to a common optical mode [R1].

So far, in cavity optomechanics, TPT has been realized through two approaches: (i) using optomechanical lattices with a mechanically mediated coupling, similar to studies of mechanical metamaterials, and (ii) using few-mode optomechanics, as studied here, with topological operations encircling an EP:

(i) Specifically, in optomechanical lattices [R23, R24], some theoretical works have predicted a significantly richer and new dynamics, including: topological phases of light and sound, collective and quench dynamics, quantum many-body dynamics and entanglement, as well as standard TPT. For example, the optomechanical lattices demonstrate topological microwave modes in one-dimensional circuit optomechanical chains that realize the Su-Schrieffer-Heeger model, as well as the strained graphene model in a two-dimensional optomechanical honeycomb lattice [R24]. To realize optomechanical lattices that support excitation transport, it is imperative that disorder in the optical (or microwave) cavity can be sufficiently small to allow the construction of lattice models. This allows for the reconstruction of the full underlying lattice Hamiltonian and the direct measurement of residual disorder. Such optomechanical lattices, coupled with measurement techniques, offer an avenue to explore collective dynamics, quantum many-body and quench dynamics, topological properties, and more broadly, emergent nonlinear dynamics in complex optomechanical systems with many degrees of freedom.

(ii) In few-mode cryogenic optomechanical systems [R1], imposing topological operations encircling an EP on this system can lead to nonreciprocal phonon transfer between two mechanical normal modes. These results open up new directions in system control and the exploration of other dynamical effects related to EPs, including the behavior of thermal and quantum fluctuations near them. Previous unidirectional topological behavior [R1], induced by imposing topological operations winding around an EP, is highly sensitive to dark modes. These dark modes can deactivate topological operations and EPs, resulting in a complete blockade of both mode conversion and phonon transfer between the dark and bright modes. In this work, we demonstrate how to overcome this outstanding challenge and achieve a versatile yet unique nonreciprocal TPT and TPB through dark-mode engineering. The proposed mechanism is generally valid and can be extended to manipulate various dark-state effects in quantum physics, advancing the development of scalable quantum information processors using photons and phonons. This study paves the way for generating a fundamentally different topological quantum resource with immunity against both dark modes and dark states.

To avoid misunderstandings and to clarify our explanations, we have added a section to show the difference and relationship between the traditional energy-band topology and our approach in the revised version. Please see the third paragraph in the section of "Discussions" on page 8 of the revised main text.

COMMENT C4

2. The author mentions the Quality Factor (Q-factor) of TPT in the main text, but I am unclear about its physical meaning. Is this Q-factor concept similar to the quality factor used *to characterize the loss in microcavities (such as the light resonance losses)?*

OUR REPLY TO COMMENT C4

We apologize for the misunderstanding and confusion caused by our unclear description of the quality factor of TPT in the original manuscript. The quality factor of TPT discussed in the main text differs from the conventional quality factor used to characterize losses in micro-cavities (e.g., light resonance losses).

In our manuscript, the "Quality Factor of TPT" refers to the efficiency of topological phonon transfer (TPT) and is used to quantify energy exchange between the two mechanical normal modes. To improve clarity and avoid any ambiguity, we have replaced the phrase "the quality factor of TPT" with "the efficiency of TPT" throughout the main text and Supplemental Material.

COMMENT C5

*3. Finally, in discussing possible experimental platforms for realizing this phenomenon, the author suggests that it could be implemented in optical-mechanical platforms like optomechanical systems. I would like to add that the synthetic magnetic field indeed provides great convenience for generating gauge potentials for photons. Recently, there has been considerable discussion regarding platforms for synthetic dimensions in photons [Optica **5**, 1396 (2018)], which could easily provide photon-photon interactions and synthetic magnetic fields [Light: Science & Applications **14**, 39 (2025)]. Could the author elaborate on the potential for implementing this theoretical work within a synthetic dimension platform for photons? If this could eventually be transferred to a chip-based platform for scalable quantum information processors, I believe it would be highly significant.*

OUR REPLY TO COMMENT C5

We thank Referee C for bringing these important references to our attention. We have cited these interesting References [R25–R28] in the revised manuscript. Please see Refs. [60-63] in the revised main text.

In our manuscript, we propose that cavity optomechanical platforms offer a versatile and promising framework for realizing the generating phenomena. The underlying physical model is highly adaptable, allowing its implementation across a broad range of optomechanical systems where the requisite interactions can be engineered. A particularly accessible realization uses the well-established "membrane-in-the-middle" configuration [R1, R2] or multi-membrane "Fabry-Pérot-cavity" architectures [R3]. In addition to these platforms, our scheme is also compatible with photonic-crystal optomechanical cavities [R9] and circuit electromechanical systems. In photonic crystals, synthetic magnetism arises from two auxiliary cavity fields, while in circuit platforms, it emerges from the coupling of two mechanical resonators to a superconducting charge qubit.

Building on Referee C's insightful observation that synthetic magnetic fields provide a powerful route to engineering gauge potentials for photons, we highlight the growing interest in photonic synthetic dimension platforms [R25]. These platforms not only enable strong photon-photon interactions but also naturally facilitate the realization of synthetic magnetic fields [R25–R28].

Specifically, the synthetic magnetic field significantly aids the generation of gauge potentials for photons. Recent advances have generated intense interest in photonic platforms with synthetic dimensions [R25], which naturally enable strong photon-photon interactions and synthetic magnetic fields [R25–R28]. These developments highlight the significant potential for realizing our theoretical proposal within synthetic-dimensional photonic systems. Importantly, the proposed dark-mode-engineering mechanism could be easily transferred to chip-based platforms, marking a significant step toward scalable quantum information processing. This prospect is particularly exciting, as it offers a practical pathway to integrate topological photonic or phononic functionalities into compact, chip-scale architectures.

Inspired by Referee C's insightful suggestions, we have added some discussion on the potential for implementing our theoretical work within a synthetic dimension platform for photons, demonstrating that it could eventually be transferred to a chip-based platform for scalable quantum information processors. Please see the last paragraph on page 7 and the first paragraph on page 8 of the revised main text.

We sincerely appreciate Referee C's thorough review of our manuscript and their valuable comments and suggestions. Incorporating these valuable feedback has significantly improved the clarity and impact of our work. We now believe our manuscript meets the standards for publication in Nature Communications.

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*** REPORT OF REFEREE A – NCOMMS-24-77485B ***

COMMENT A1

The authors have satisfactorily addressed all the reviewers' comments. I certainly appreciated that the authors added extensive discussion on the experimental implementations and I think the detailed discussion on the potential experimental implementations makes the manuscript more impactful and suitable for the publication in Nat. Comm.

Therefore, I recommend this work for publication in Nat. Comm.

OUR REPLY TO COMMENT A1

We sincerely thank Referee A for recommending the publication of our study in Nature Communications.

*** REPORT OF REFEREE B – NCOMMS-24-77485B ***

COMMENT B1

I appreciate the authors' hard work in improving the quality of the manuscript. In the rebuttal, the authors have explained elaborately why the current work is suitable for a high-impact journal and also made some appropriate revisions in the manuscript. The current version looks reasonable to be published in Nature Communications.

OUR REPLY TO COMMENT B1

We are grateful to Referee B for recommending our work for publication in Nature Communications.

*** REPORT OF REFEREE C – NCOMMS-24-77485B***

COMMENT C1

The Authors have performed major revisions to the Main Text and to the Supplementary Material. They have written an extensive response letter explaining their changes. I deem that they have addressed suitably most of my comments.In addition, the results meet publication standards of Nature Communications.

OUR REPLY TO COMMENT C1

We thank Referee C for recommending the publication of our work in Nature Communications.