nature portfolio

Peer Review File

Versatile Control of Nonlinear Topological States in Non-Hermitian Systems

Corresponding Author: Professor Tao Liu

This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 0:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The manuscript presents a theoretical investigation into the interplay of nonlinearity and non-Hermiticity in topological systems, focusing on achieving fully delocalized and reconfigurable topological modes (TMs). The results are interesting, with implications for designing compact and reconfigurable topological devices. The manuscript is generally well-structured. However, before the manuscript can be considered for publication, I recommend the authors address the following points:
1. The spectral localizer used in this work follows a real-space approach extended to non-Hermitian systems. However, prior works adopt slightly different forms. The authors should clearly compare and clarify the distinctions between the spectral localizer employed in this work and those used in Refs. [69] and [71].
2. Equation (9) appears to have an error in the definition of the similarity transformation operator S. The second term may not be a direct product. Please revise and clarify.
3. The authors adopt an intensity-dependent Kerr-type nonlinearity that modifies the intercell coupling coefficients. While this is an interesting and powerful theoretical model, it is important to address whether such coupling-dependent nonlinearity can be realized in experiments. Can the authors provide a realistic implementation scheme, where nonlinear modulation of the coupling strength (rather than the on-site potential) is feasible?
4. In Fig. 5(a), the on-site loss terms for different lattice sites are not clearly distinguished. For example, it is difficult to visually identify the sites with differing κa and κb. I suggest the authors improve the graphical presentation by using different arrow styles, marker sizes.
5. In the 2D stacked model, the stacking appears to preserve trivial topology along y. It would strengthen the paper to briefly discuss whether it is possible to engineer a similar SSH-type configuration along the y-direction and whether extended TMs with nontrivial topology in both directions can be realized. This may provide new insights into higher-dimensional generalizations.
In conclusion, this is a promising work that significantly contributes to the study of nonlinear and non-Hermitian topological systems. I recommend acceptance after revision, provided the authors give satisfactory responses to the points above.

Reviewer #2

(Remarks to the Author)
In their manuscript "Versatile Control of Nonlinear Topological States in Non-Hermitian Systems," Cai et al present a theoretical and numerical analysis of the interplay between non-Hermitian behaviors that yield the skin effect, localized Hermitian non-linearities, and material topology in a 1D system. They show that it's possible to develop robust systems and input intensities with completely delocalized nonlinear modes, as well as modes whose spatial profiles can be tailored to specific applications. Moreover, as these effects are the result of nonlinearities, they can be dynamically tailored after a device is fabricated. A key strength of this study is its use of the spectral localizer framework, which allows the authors to rigorously demonstrate that the modes they find are, in fact, topological, and can be connected to a provably integer-valued invariant. Overall, I believe that a suitably revised version of this manuscript could be publishable in Comm Phys.

First, there are a pair of papers by Fulga and co-authors that I believe are highly relevant to the authors' study, as these prior works consider, both theoretically and experimentally the application of the spectral localizer framework to systems exhibiting the non-Hermitian skin effect.

In particular, see
-- H. Liu and I. C. Fulga, Phys. Rev. B 108, 035107 (2023).
-- K. Ochkan, R. Chaturvedi, V. Könye, L. Veyrat, R. Giraud, D. Mailly, A. Cavanna, U. Gennser, E. M. Hankiewicz, B. Büchner, J. van den Brink, J. Dufouleur, and I. C. Fulga, Nat. Phys. 20, 395 (2024).
-- N. Chadha, A. G. Moghaddam, J. van den Brink, and C. Fulga, Phys. Rev. B 109, 035425 (2024).

On line 232, the authors state that nonlinear effects are inherently local. I think this is overly broad. This is generally true in photonics, but I'm not sure this is true in all coupled spin systems, for example. (And, there are even some photonic systems that can emulate such non-local interactions, such as those systems mimicking Ising models.)

On line 275, the authors discuss how the topological zero modes they find are robust against disorder. I would urge the authors to be a bit more specific about exactly what is robust. In Fig. 4g, the index change at x=60 is robust because the local gap mu is non-zero and large on both sides of the index shift. In Fig. 4i, the index change near x=1 is similarly protected, due to the bump in mu for slightly larger x. For Fig. 4h, the situation is, I think, more subtle – you know that the index has to shift somewhere in the range of x \in [3,57], but where it shifts isn't protected. But, the entire range of \mu ~ 0 corresponds to a single delocalized state that is still protected from moving into the x > 60 region of the system. All of this need to be clarified – a single, simple remark about topological protection I don't believe does justice to these varied phenomena.

On lines 317-319, the authors state that their method enables the creation of arbitrary extended waveforms. But, this method requires significant overhead in terms of needing both nonlinearities and non-Hermiticity. It seems like one could create a similar arbitrary extended waveform by simply adding on-site perturbations to a lattice until a single state had the desired shape. This would require similar lattice tuning. The authors should discuss the advantages of their proposed method over this naïve solution.

On lines 400-404, I do not find this contrast to be obvious. E.g., Figs. 6e3 and 6e4 look very similar to me. I do believe an analysis of a 2D system is important for the manuscript, but I'm not sure I yet see the value in the current 2D system the authors are studying.

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The authors have addressed all my comments point by point, and I am satisfied with their responses. I recommend acceptance.

Reviewer #2

(Remarks to the Author)
I greatly appreciate the authors' detailed responses to my questions and comments. Overall, I believe this manuscript can proceed to publication at Comm Phys.

But, there is one minor correction I'd encourage the authors to make: In Eq. 5, the equality condition corresponds to a ``perfect'' perturbation driving the smallest singular value of L exactly to 0. Thus, exactly at this point, the system's local topological invariant (eg Eq. 4) is undefined, and so it's not clear if any zero mode is topological under such a perturbation. I would strongly recommend that the authors alter Eq. 5 to be strictly less than.

List of main changes

(The main changes in the additional manuscript are highlighted in red font solely for the reviewer's convenience)

1. We have revised the equation (9) in the original manuscript. Now, it is equation (12) in the revised manuscript.

2. The discussion of a possible experimental setup has been added to the revised Supplementary Information.

3. We have revised the 2D model in the original manuscript, and the discussion in the section *Delocalization of TZMs in a 2D non-Hermitian nonlinear lattice* has been completely rewritten in the revised manuscript.

4. More details about robustness, including equation (5), are added in the main text in the revised manuscript.

5. Some references are added. Some equations are shown explicitly.

Response to the Reviewer #1

We sincerely thank Reviewer #1 for the thorough and constructive review, as well as for the positive evaluation of our work and the recommendation for acceptance after revision. We greatly appreciate the insightful comments, which have helped us improve the clarity and quality of the manuscript. We have carefully addressed all points raised, and our detailed responses are provided below.

Comment 1:

The manuscript presents a theoretical investigation into the interplay of nonlinearity and non-Hermiticity in topological systems, focusing on achieving fully delocalized and reconfigurable topological modes (TMs). The results are interesting, with implications for designing compact and reconfigurable topological devices. The manuscript is generally well-structured.

Our reply :

We sincerely thank the Reviewer for their thoughtful and encouraging feedback. We greatly appreciate the recognition that "the results are interesting, with implications for designing compact and reconfigurable topological devices," and that "the manuscript is generally well-structured."

In response to the Reviewer's comments, we have carefully revised the manuscript to enhance the clarity of the underlying physical concepts and improve the overall presentation. The key revisions are clearly marked in red in the updated version.

Comment 2:

The spectral localizer used in this work follows a real-space approach extended to non-Hermitian systems. However, prior works adopt slightly different forms. The authors should clearly compare and clarify the distinctions between the spectral localizer employed in this work and those used in Refs. [69] and [71].

Our reply :

We thank the Reviewer for raising this important point regarding the spectral localizer. The spectral localizer offers a key advantage in that it incorporates the system's spatial configuration, enabling a real-space description of material topology. This makes it particularly suitable for capturing the local nature of nonlinear effects. *The applied spectral localizer is the same as that in Ref. [71], which was used for a 1D Hermitian system. In contrast, the formula in Ref. [69] was applied to a 2D system.*

The detailed demonstration is as follows:

(1). In our work, to apply the spectral localizer to a one-dimensional (1D) non-Hermitian system, we first map the non-Hermitian Hamiltonian $\hat{\mathcal{H}}$ to a Hermitian one $\hat{\mathcal{H}}_{\mathrm{S}}$ via a similarity transformation $\hat{S}$, such that $\hat{\mathcal{H}}_{\rm S} = \hat{S}\hat{\mathcal{H}}\hat{S}^{-1}$, with corresponding eigenvectors related by $|\bar{\psi}\rangle = \hat{S}\,|\psi\rangle$. This allows us to leverage the information from the Hermitian Hamiltonian $\hat{\mathcal{H}}_{\rm S}$ and its eigenmodes $|\bar{\psi}\rangle$ to characterize the topological properties of the non-Hermitian nonlinear system.

(2). The nonlinear spectral localizer is a composite operator that incorporates the system's Hamiltonian $\hat{\mathcal{H}}_{\rm S}$ accounting for its current occupations $|\bar{\psi}\rangle$ and position operators $\hat{X}$, using a nontrivial Clifford representation. The 1D nonlinear spectral localizer is explicitly written as

$$L_{\zeta\equiv(x,\bar{\omega})}(\hat{X},\hat{\mathcal{H}}_{\rm S}) = \begin{pmatrix} 0 & \eta(\hat{X}-x\mathbf{I}) - i(\hat{\mathcal{H}}_{\rm S}-\bar{\omega}\mathbf{I}) \\ \eta(\hat{X}-x\mathbf{I}) + i(\hat{\mathcal{H}}_{\rm S}-\bar{\omega}\mathbf{I}) & 0 \end{pmatrix}, \tag{R1}$$

which is consistent with the 1D formulation used in Reference [71], but differs from the 2D formulation presented in Reference [69], owing to the dimensional and symmetry distinctions between the models.

(3). Furthermore, when the system respects chiral symmetry, i.e., $\hat{\mathcal{H}}_{\rm S}\hat{\Pi} = -\hat{\Pi}\hat{\mathcal{H}}_{\rm S}$ (We have corrected the typo in the Methods section), the spectral localizer $L_\zeta(\hat{X},\hat{\mathcal{H}}_{\rm S})$ in Eq. (R1) can be further written in a reduced form as

$$\tilde{L}_{\zeta\equiv(x,\bar{\omega})}(\hat{X},\hat{\mathcal{H}}_{\rm S}) = \eta(\hat{X}-x\mathbf{I})\hat{\Pi} + \hat{\mathcal{H}}_{\rm S} - i\bar{\omega}\hat{\Pi}, \tag{R2}$$

which matches the form employed in Equation (1) in Reference [71].

We have revised the main text accordingly and now present this chiral-symmetric form in Eq. (R2) as Equation (3) in the revised manuscript, to clarify the connection and eliminate any potential confusion.

Comment 3:

Equation (9) appears to have an error in the definition of the similarity transformation operator $\hat{S}$. The second term may not be a direct product. Please revise and clarify.

Our reply :

We thank the reviewer for pointing it out. We apologize for the mistakes and typos in the definition of the similarity transformation operator $\hat{S}$, and we have corrected this confusion according to your suggestion in the revised manuscript. Below, we provide the revised version of Equation (9) in the original manuscript (Now, it is Equation (12) in the revised manuscript).

The similarity transformation operator $\hat{S}$ should be written as

$$\hat{S} = \begin{pmatrix} \mathbf{I}_{2N} & \\ & 0_{L-2N} \end{pmatrix} + \begin{pmatrix} 0_{2N} & \\ & \mathcal{R}_{L-2N} \end{pmatrix}, \tag{R3}$$

where $\mathbf{I}$ is identity matrix, and $\mathcal{R}_{L-2N}$ is diagonal matrix, with dimensions $L-2N$, whose diagonal elements are $\{1, r, r, r^2, \cdots, r^{\frac{L-1}{2}-N-1}, r^{\frac{L-1}{2}-N}, r^{\frac{L-1}{2}-N}\}$.

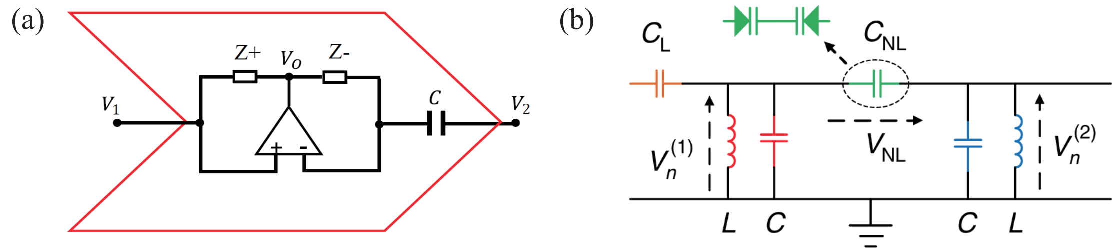

FIG. R1. (a) Experimental circuit realization of nonreciprocal hopping between two neighbor nodes via the negative impedance converters through current inversion (INIC), which consists of capacitor, resistor and operational amplifier. (b) Experimental circuit realization of nonlinear hopping, which consists of two shut LC resonators, a linear capacitor C_{L} and a nonlinear capacitor C_{NL}.

Comment 4:

The authors adopt an intensity-dependent Kerr-type nonlinearity that modifies the intercell coupling coefficients. While this is an interesting and powerful theoretical model, it is important to address whether such coupling-dependent nonlinearity can be realized in experiments. Can the authors provide a realistic implementation scheme, where nonlinear modulation of the coupling strength (rather than the on-site potential) is feasible?

Our reply :

We sincerely thank the Reviewer for this insightful comment. We fully agree that addressing the experimental feasibility of coupling-dependent nonlinearity is essential for connecting our theoretical developments with physical implementation.

In the revised supplementary information, we added the possible experimental setup, as shown below:

The non-Hermitian nonlinear lattices considered in this work can be feasibly implemented across a variety of experimental platforms, including photonic systems [PhysRevA.100.063830SM, Sone2025SM] and electronic circuits [Hadad2018SM, arXiv:2403.10590SM, arXiv:2505.09179SM]. Here, we focus on an electronic circuit platform that allows for both tunable nonreciprocal hopping [Helbig2020SM, Guo2024SM, 2025SM] and controllable nonlinearity [Hadad2018SM, arXiv:2403.10590SM, arXiv:2505.09179SM]. In particular, the key ingredients of our model, i.e., nonreciprocal hopping and amplitude-dependent nonlinear hopping, can be effectively realized within this platform. In the following, we provide a detailed description of the circuit implementation of each hopping mechanism.

(i) The nonreciprocal hopping between neighboring nodes is implemented using negative impedance converter (INIC) circuits, which introduce direction-dependent current inversion [Helbig2020SM, 2025SM], as illustrated in Fig. R1(a). In this configuration, each pair of adjacent nodes is connected via a capacitor C_1 and an INIC. The INIC efficiently introduces an asymmetric capacitive coupling, exhibiting an equivalent capacitance of $\pm C_2$ depending on the direction of signal flow.

(ii) The nonlinear hopping between neighboring nodes is realized by utilizing an nonlinear capacitor [Hadad2018SM], as shown in Fig. R1(b). Specifically, we consider an interaction between two resonators coupled via a linear capacitor C_{L} and an nonlinear capacitor C_{NL}. Following the derivation [Hadad2018SM], the dynamics of the system can be modeled using coupled-mode equations:

$$-j\frac{da_1}{dt} = \omega_0 a_1 + [\kappa + \nu(|V_{C_1} - V_{C_2}|)]a_2, \tag{R4}$$

$$-j\frac{da_2}{dt} = \omega_0 a_2 + [\kappa + \nu(|V_{C_1} - V_{C_2}|)]a_1, \tag{R5}$$

where ω_0 is the resonance frequency, and the couplings consist of a linear term $\kappa = C_{\mathrm{L}}/(C_1 + C_2)$ and a nonlinear term $\nu(V) = C_{\mathrm{NL}}(|V_{C_1} - V_{C_2}|)/(C_1 + C_2)$, which depends on the voltage difference across the nonlinear capacitor. This structure allows the coupling strength to be modulated dynamically by the local voltage amplitude, thus realizing an effective nonlinear hopping mechanism. Note that different forms of amplitude-dependent nonlinear coupling between two nodes have also been realized in circuit platforms [arXiv:2403.10590SM,arXiv:2505.09179SM]. In addition, amplitude-dependent coupling can be implemented in optical systems using nonlinear fibers [PhysRevA.100.063830SM,Sone2025SM], where the coupling strength varies with the light intensity.

[PhysRevA.100.063830SM] A. Bisianov, M. Wimmer, U. Peschel, and O. A. Egorov, "Stability of topologically protected edge states in nonlinear fiber loops," Phys. Rev. A 100, 063830 (2019).

[Sone2025SM] K. Sone, M. Ezawa, Z. Gong, T. Sawada, N. Yoshioka, and T. Sagawa, "Transition from the topological to the chaotic in the nonlinear Su–Schrieffer–Heeger model," Nat. Commun. 16, 422 (2025).

[Hadad2018SM] Y. Hadad, J. C. Soric, A. B. Khanikaev, and A. Alù, "Self-induced topological protection in nonlinear circuit arrays," Nat. Electron. 1, 178 (2018).

[arXiv:2403.10590SM] M. Padlewski H. Lissek P. Delplace R. Fleury X. Guo, L. Jezequel, "Practical realization of chiral nonlinearity for strong topological protection," arXiv:2403.10590 (2024).

[arXiv:2505.09179SM] J. Wu, R.-C. Shen, L. Zhang, F. Chen, B. Wang, H. Chen, Y. Yang, and H. Xue, "Nonlinearity-induced reversal of electromagnetic non-Hermitian skin effect," arXiv:2505.09179 (2024).

[Helbig2020SM] T. Helbig, T. Hofmann, S. Imhof, M. Abdelghany, T. Kiessling, L. W. Molenkamp, C. H. Lee, A. Szameit, M. Greiter, and R. Thomale, "Generalized bulk–boundary correspondence in non-Hermitian topolectrical circuits," Nat. Phys. 16, 74 (2020).

[Guo2024SM] C.-X. Guo, L. Su, Y. Wang, L. Li, J.e Wang, X. Ruan, Y. Du, D. Zheng, S. Chen, and H. Hu, "Scale-tailored localization and its observation in non-Hermitian electrical circuits," Nat. Commun. 15, 9120 (2024).

[2025SM] T. Liu W. Ju H. Wang, J. Liu, "Observation of impurity-induced scale-free localization in a disordered non-Hermitian electrical circuit," Front. Phys. 20, 14203 (2025).

Comment 5:

In Fig. 5(a), the on-site loss terms for different lattice sites are not clearly distinguished. For example, it is difficult to visually identify the sites with differing κ_a and κ_b. I suggest the authors improve the graphical presentation by using different arrow styles, marker sizes.

Our reply :

We thank the Reviewer for this helpful suggestion. In the revised manuscript, we have improved the graphical presentation in Fig. 5(a) by using arrows of different lengths and colors to more clearly distinguish the on-site loss terms: shorter blue arrows represent κ_a, while longer red arrows represent κ_b. We hope that this revision clarifies the figure and enhances its overall readability.

Comment 6:

In the 2D stacked model, the stacking appears to preserve trivial topology along y. It would strengthen the paper to briefly discuss whether it is possible to engineer a similar SSH-type configuration along the y-direction and whether extended TMs with nontrivial topology in both directions can be realized. This may provide new insights into higher-dimensional generalizations.

Our reply :

We appreciate the insightful comment and suggestion regarding the 2D model. Indeed, in the original 2D model, the stacking of 1D topological chains primarily results in trivial topology along the y axis. Interestingly, in an SSH-type configuration along the y direction, we find that extended TZMs with nontrivial topology can emerge along both directions. We provide further discussion on this point below.

We can extend the 1D non-Hermitian nonlinear model to two dimensions (2D) by stacking 1D topological interface chains along the y direction, as shown in Fig. R2(a). The blue dashed box marks a unit cell containing four sublattices, with odd and even indices j distinguishing sites along y. Hopping amplitudes along the y direction are both nonlinear and site-dependent, where dashed purple lines indicate hopping terms with negative signs. The nonlinear intracell coupling is given by

$$u_{i,j} = u_0 + \gamma_1(|\eta_{i,j}|^2 + |\eta_{i,j+1}|^2),\ j \in \text{odd}, \tag{R6}$$

where $\eta_{i,j} \in \{a_{i,j}, b_{i,j}\}$ represents the state amplitudes of the two sites linked by the corresponding nonlinear coupling, and the intercell coupling, characterized by strength v_0, is linear.

Along the x direction, for each chain, the intercell couplings, $t_{i,j}$ and $\lambda_{i,j}$, are nonlinear and given

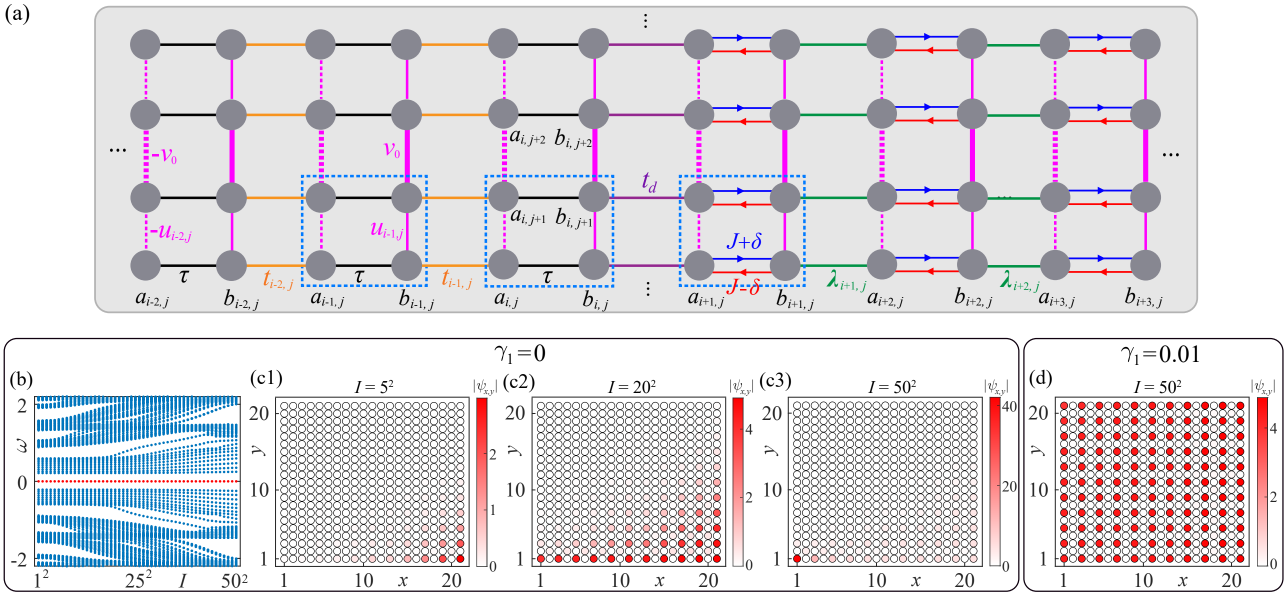

FIG. R2. **Delocalization of TZMs in a 2D non-Hermitian nonlinear lattice**. (a) Schematic of a 2D non-Hermitian nonlinear interface model, formed by stacking 1D topological interface chains with staggered nonlinear hopping along the y direction. Each unit cell, indicated by a blue dashed box, contains four sublatties. The nonlinear intracell hopping, along the y direction, is given by $u_{i,j} = u_0 + \gamma_1(|\eta_{i,j}|^2 + |\eta_{i,j+1}|^2)$ ($j \in$ odd), where $\eta_{i,j} \in \{a_{i,j}, b_{i,j}\}$ are the state amplitudes of the two sites linked by the corresponding nonlinear coupling. Dashed purple lines indicate hopping terms with negative signs along the y direction. (b) ω versus I for $\gamma_1 = 0$, where red dots mark TZMs. The corresponding spatial distributions $|\psi_{x,y}|$ (x and y labels lattice site along the x and y directions) of the TZMs for different I are shown in (c1-c3). (d) $|\psi_{x,y}|$ for $\gamma_1 = 0.01$, where the TZMs exhibit fully extended spatial distributions at $I = 50^2$. Other parameters used are $J = 1.5$, $\tilde{t}_{i,j} = \tilde{\lambda}_{i,j} = 1.5$, $\alpha = \beta = 0.05$, $\tau = t_d = 2.5$, $N = 6$, $\delta = 1.2$, $u_0 = 0.2$, $v_0 = 0.4$, $L_x = 21$ and $L_y = 21$.

by

$$t_{i,j} = \tilde{t}_{i,j} + \alpha(|a_{i,j}|^2 + |b_{i-1,j}|^2), \tag{R7}$$

$$\lambda_{i,j} = \tilde{\lambda}_{i,j} + \beta(|a_{i,j}|^2 + |b_{i-1,j}|^2). \tag{R8}$$

In the absence of both nonlinearity and non-Hermiticity, the system reduces to an interface model based on the Benalcazar-Bernevig-Hughes (BBH) lattice [Benalcazar2017]. The BBH model exhibits a higher-order topological phase characterized by the emergence of topological zero modes (TZMs) localized at the corners. In the linear Hermitian case, the corresponding 2D interface model supports TZMs localized at the ends of the interface.

When the nonlinearity along the x direction is introduced with $\gamma_1 = 0$, the 2D non-Hermitian nonlinear interface model supports TZMs, as indicated by the red dots in Fig. R2(b). Owing to the strong NHSE, these TZMs are localized from the interface toward the right-bottom corner for weak nonlinearity at $I = 5^2$ [see Fig. R2(c1)]. As the nonlinear intensity increases to $I = 20^2$, the TZMs become more extended and predominantly occupy the bottom edge [see Fig. R2(c2)]. When the nonlinearity is further strengthened to $I = 50^2$, the modes do not spread across the entire lattice but instead become localized again [see Fig. R2(c3)]. In contrast, when nonlinearity is also introduced along the y direction with $\gamma_1 = 0.01$, the TZMs extend over the entire 2D lattice at $I = 50^2$ [see Fig. R2(d)]. Notably, this extended phase emerges without the need to satisfy the constraint $\delta = \delta_c = \tilde{\lambda}_{i,j} - J$. These results demonstrate how the interplay between nonlinearity and non-Hermiticity enables flexible control of topological mode localization in 2D lattices.

The above model and discussion have replaced the original model in the revised manuscript.

[Benalcazar2017] W. A. Benalcazar, B. A. Bernevig and T. L. Hughes, Quantized electric multipole insulators, Science 357, 61 (2017).

Comment 7:

In conclusion, this is a promising work that significantly contributes to the study of nonlinear and non-Hermitian topological systems. I recommend acceptance after revision, provided the authors give satisfactory responses to the points above.

Our reply :

We thank the Reviewer for the positive evaluation and for recommending our paper for publication after revision.

Response to the Reviewer #2

We sincerely thank Reviewer #2 for the detailed and constructive review, as well as for the positive evaluation of our work. We appreciate the Reviewer's comments and their belief that a suitably revised version of the manuscript could be publishable in Communications Physics. We have carefully considered all the feedback and have addressed each point thoroughly. Our detailed responses are provided below.

Comment 1:

In their manuscript "Versatile Control of Nonlinear Topological States in Non-Hermitian Systems," Cai et al present a theoretical and numerical analysis of the interplay between non-Hermitian behaviors that yield the skin effect, localized Hermitian non-linearities, and material topology in a 1D system. They show that it's possible to develop robust systems and input intensities with completely delocalized nonlinear modes, as well as modes whose spatial profiles can be tailored to specific applications. Moreover, as these effects are the result of nonlinearities, they can be dynamically tailored after a device is fabricated. A key strength of this study is its use of the spectral localizer framework, which allows the authors to rigorously demonstrate that the modes they find are, in fact, topological, and can be connected to a provably integer-valued invariant. Overall, I believe that a suitably revised version of this manuscript could be publishable in Comm Phys.

Our reply :

We sincerely thank the Reviewer for their thoughtful and encouraging feedback, as well as for recommending publication after suitable revision.

In response to the Reviewer's comments, we have carefully revised the manuscript to improve the clarity of the underlying physical concepts and enhance the overall presentation. The key changes are clearly highlighted in red in the updated version.

Comment 2:

First, there are a pair of papers by Fulga and co-authors that I believe are highly relevant to the authors' study, as these prior works consider, both theoretically and experimentally the application of the spectral localizer framework to systems exhibiting the non-Hermitian skin effect.

In particular, see

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– N. Chadha, A. G. Moghaddam, J. van den Brink, and C. Fulga, Phys. Rev. B 109, 035425 (2024).

Our reply :

We thank the Reviewer for bringing the works by Fulga and co-authors to our attention. These papers are indeed highly relevant, as they explore both theoretical and experimental applications of the spectral localizer framework to systems exhibiting the non-Hermitian skin effect. We have carefully reviewed these references and now cite them appropriately in the revised manuscript.

Comment 3:

On line 232, the authors state that nonlinear effects are inherently local. I think this is overly broad. This is generally true in photonics, but I'm not sure this is true in all coupled spin systems, for example. (And, there are even some photonic systems that can emulate such non-local interactions, such as those systems mimicking Ising models.)

Our reply :

We thank the Reviewer for pointing this out. We apologize for the unclear wording in our original statement. We did not mean to suggest that all nonlinear effects are local. Rather, we intended to convey that the nonlinear effects in our model are local because the nonlinear hopping is restricted to nearest neighbors. This locality breaks translational symmetry, which in turn invalidates the general definition of topological invariants as used in linear systems. To clarify this point, we have revised the statement to: "nonlinear effects in our model are inherently local."

Comment 4:

On line 275, the authors discuss how the topological zero modes they find are robust against disorder. I would urge the authors to be a bit more specific about exactly what is robust. In Fig. 4g, the index change at $x = 60$ is robust because the local gap μ is non-zero and large on both sides of the index shift. In Fig. 4i, the index change near $x = 1$ is similarly protected, due to the bump in mu for slightly larger x. For Fig. 4h, the situation is, I think, more subtle – you know that the index has to shift somewhere in the range of x $\in$ [3,57], but where it shifts isn't protected. But, the entire range of $\mu \sim 0$ corresponds to a single delocalized state that is still protected from moving into the $x > 60$ region of the system. All of this need to be clarified – a single, simple remark about topological protection I don't believe does justice to these varied phenomena.

Our reply :

We sincerely thank the Reviewer for this insightful comment. We apologize that our original description of the robustness of the TZMs, in the main text, was overly simplified, and the statement is unclear. Let us provide more detailed explanations below.

(1). About the disorder robustness

The statement that "the TZM is robust against disorder" means that *a stable TZM, protected by the band gap, exists as long as the disorder strength applied to the model remains smaller than the band gap, even when strong nonlinearity is present.* The nonlinearity does not break down the topological protection of TZMs.

(a) The local band gap is given by the smallest value $\mu_\zeta = \left|\sigma_{\min}(\tilde{L}_\zeta)\right|$ of the spectral localizer at $\zeta = (x, \bar{\omega})$. If μ_ζ sufficient close to zero, an approximately localized state exists. Conversely, a large value indicates that the system dose not support such a state. This framework allows us to rigorously assess the topological protection of the TZMs. Specifically, the robustness of a TZM can be guaranteed as long as any perturbation to the system remains below the local band gap. This condition is expressed by:

$$\left|\left|\Delta\hat{\mathcal{H}}_{\mathrm{S}}(W)\right|\right| \leq \mu_\zeta^{\max}, \tag{R9}$$

where $\left|\left|\Delta\hat{\mathcal{H}}_{\mathrm{S}}(W)\right|\right|$ is the largest singular value of $\Delta\hat{\mathcal{H}}_{\mathrm{S}}(W) = \hat{\mathcal{H}}_{\mathrm{S}}(W) - \hat{\mathcal{H}}_{\mathrm{S}}$, with $\hat{\mathcal{H}}_{\mathrm{S}}(W) = \hat{\mathcal{H}}_{\mathrm{S}} + W\delta\hat{\mathcal{H}}_{\mathrm{S}}$ representing the perturbed nonlinear Hamiltonian with perturbation strength W, and $\mu_\zeta^{\max} = \max[\mu_\zeta]$ is the maximum value of μ_ζ within the topological region. When this condition is satisfied, a stable TZM, protected by the band gap, exists as long as the disorder strength applied to the model remains smaller than the band gap, even when strong nonlinearity is present.

(b) Based on the above discussion, a stable TZM exists as long as the disorder strength applied to the model remains smaller than the band gap. In Sec. V(B) of SI, we provide details on statement *"the TZM is robust against disorder"*. The following is the discussion on robust against disorder of arbitrarily-designed wavefunction profiles in Sec. V(B) of the SI, as shown below

To demonstrate the robustness of arbitrarily-designed wavefunction profiles for the TZMs across the entire lattice under the influence of disorder, we introduce random perturbations to the hopping energies. The hopping energies are modified as $\tilde{t}_j \to \tilde{t}_j(1+W_j)$, and $\tilde{\lambda}_j \to \tilde{\lambda}_j(1+V_j)$, where W_j and V_j are independent random variables uniformly distributed over the range $[-W/2,\ W/2]$. Here,

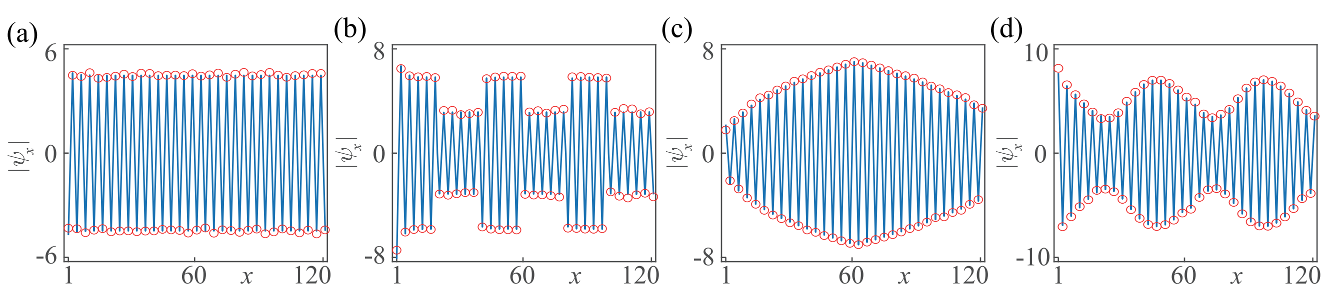

FIG. R3. Spatial distributions $|\psi_x|$ of the TZMs for wavefunction profiles with (a) flat, (b) square, (c) isosceles triangle, and (d) cosine shapes, subject to random disorder applied to the hopping energies $\tilde{t}_j \to \tilde{t}_j(1+W_j)$ and $\tilde{\lambda}_j \to \tilde{\lambda}_j(1+V_j)$. The blue lines indicate the state distributions without perturbation, while the red circles depict the distributions in the presence of disorder. Other parameters used are $J = 1.5$, $\delta = 1.0$, $W = 0.2$, $\alpha = \beta = 0.05$, $\tau = t_d = 2.5$, $N = 31$ and $L = 121$.

W quantifies the disorder strength, providing a controlled parameter to evaluate the stability and resilience of the wavefunction profiles against spatially distributed random disorder.

Figure R3 illustrates the spatial distributions $|\psi_x|$ of the TZMs for wavefunction profiles with shapes (a) flat, (b) square, (c) isosceles triangle, and (d) cosine, both in the absence of disorder (blue lines) and in the presence of disorder (red circles). The disordered state distributions show negligible deviations from the unperturbed ones, highlighting the remarkable robustness of the designed TZMs against disorder due to their topological nature.

In the revised manuscript, we have provided additional details, including Eq. (5), on the disorder robustness, and we now explicitly direct the reader to the more detailed discussion in the Supplementary Information.

(2). Spectral localizer at different nonlinear intensity in Fig. 4

Figure 4(g), 4(h), and 4(i) represent the local topological invariant C_ζ and the local band gap μ_ζ at different nonlinear intensities, from weak to strong, illustrating the transition of TZMs from localized to extended states. The topological nature is determined by whether an integer-valued change of the local topological invariant C_ζ is achieved. As shown in Fig. 4(g), 4(h), and 4(i), the change of C_ζ is indeed 1, indicating that the system remains in the topological regime despite strong nonlinearity. Consequently, these TZMs are topologically protected and robust against disorder, meaning that a stable TZM persists as long as the disorder strength remains smaller than the *maximum value of the local band gap* μ_ζ. In addition, if μ_ζ is sufficiently close to zero, an approximately localized state exists. Therefore, the regions where the local band gap μ_ζ is closest to zero differ for each intensity, signifying the emergence of extended TZMs at strong nonlinearity. It should be emphasized that topological protection is a global property of the system, rather than a purely local one. As the nonlinearity increases, the band gap remains open, and the in-gap zero modes stay topologically protected within our considered parameter regimes.

Comment 5:

On lines 317-319, the authors state that their method enables the creation of arbitrary extended waveforms. But, this method requires significant overhead in terms of needing both nonlinearities and non-Hermiticity. It seems like one could create a similar arbitrary extended waveform by simply adding on-site perturbations to a lattice until a single state had the desired shape. This would require similar lattice tuning. The authors should discuss the advantages of their proposed method over this naïve solution.

Our reply :

We thank the Reviewer for this comment.

While it is true that on-site perturbations can slightly modify the spatial profiles of lattice eigenstates, such modifications alone cannot, in general, produce arbitrary extended waveforms with the precise control and programmability achieved by our method.

The inherent localization of topological states in Hermitian linear systems (without nonlinearity and non-Hermiticity) prohibits this from happening: Topological in-gap states are typically localized at the boundaries or interface and remain robust against weak symmetry-protected perturbations, without the emergence of extended wavefunctions in the absence of nonlinearity or non-Hermiticity. Weak perturbations, protected by the topology and band gap, cannot delocalize these states; they merely shift the energy or slightly modify the localized profile, while the states remain bound to the boundary or interface. Such perturbations cannot transform the states into extended bulk-like waveforms. Strong perturbations, on the other hand, destroy the topological in-gap states altogether. Therefore, it is not possible to achieve extended wavefunction profiles for topological boundary states by simply adding on-site perturbations. We emphasize that our focus is on the versatile control of originally localized topological in-gap states by utilizing nonlinearity and non-Hermiticity. In contrast, controlling an originally extended and continuum bulk state is relatively trivial.

The unique advantage of our approach arises from the synergistic combination of non-Hermiticity and nonlinearity. Non-Hermiticity enables a critical transition where TZMs become extended through the competition between topological edge localization and non-Hermitian skin-mode localization. However, this extension alone does not allow arbitrary waveform control because it is tied to the specific conditions at the critical point. Nonlinearity plays a complementary and essential role by enabling amplitude-dependent, self-consistent shaping of the wavefunction. Together, these ingredients enable robust, dynamically stable, and fully programmable extended states of TZMs across the entire lattice—something that cannot be achieved through linear Hermitian perturbations alone.

Comment 6:

On lines 400-404, I do not find this contrast to be obvious. E.g., Figs. 6e3 and 6e4 look very similar to me. I do believe an analysis of a 2D system is important for the manuscript, but I'm not sure I yet see the value in the current 2D system the authors are studying.

Our reply :

We appreciate the insightful comment and suggestion regarding the 2D model. Indeed, in the original 2D model, the stacking of 1D topological chains primarily results in trivial topology along the y axis. Interestingly, in an SSH-type configuration along the y direction, we find that extended TZMs with nontrivial topology can emerge along both directions. We provide further discussion on this point below.

We can extend the 1D non-Hermitian nonlinear model to two dimensions (2D) by stacking 1D topological interface chains along the y direction, as shown in Fig. R2(a). The blue dashed box marks a unit cell containing four sublattices, with odd and even indices j distinguishing sites along y. Hopping amplitudes along the y direction are both nonlinear and site-dependent, where dashed purple lines indicate hopping terms with negative signs. The nonlinear intracell coupling is given

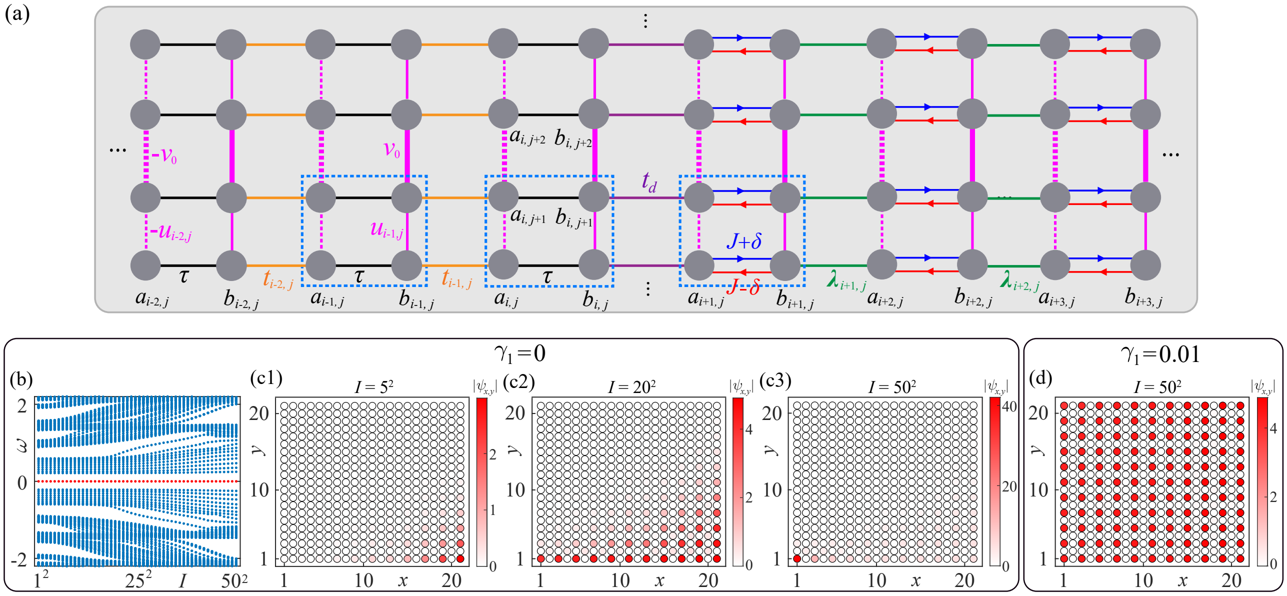

FIG. R4. **Delocalization of TZMs in a 2D non-Hermitian nonlinear lattice**. (a) Schematic of a 2D non-Hermitian nonlinear interface model, formed by stacking 1D topological interface chains with staggered nonlinear hopping along the y direction. Each unit cell, indicated by a blue dashed box, contains four sublatties. The nonlinear intracell hopping, along the y direction, is given by $u_{i,j} = u_0 + \gamma_1(|\eta_{i,j}|^2 + |\eta_{i,j+1}|^2)$ ($j \in$ odd), where $\eta_{i,j} \in \{a_{i,j}, b_{i,j}\}$ are the state amplitudes of the two sites linked by the corresponding nonlinear coupling. Dashed purple lines indicate hopping terms with negative signs along the y direction. (b) ω versus I for $\gamma_1 = 0$, where red dots mark TZMs. The corresponding spatial distributions $|\psi_{x,y}|$ (x and y labels lattice site along the x and y directions) of the TZMs for different I are shown in (c1-c3). (d) $|\psi_{x,y}|$ for $\gamma_1 = 0.01$, where the TZMs exhibit fully extended spatial distributions at $I = 50^2$. Other parameters used are $J = 1.5$, $\tilde{t}_{i,j} = \tilde{\lambda}_{i,j} = 1.5$, $\alpha = \beta = 0.05$, $\tau = t_d = 2.5$, $N = 6$, $\delta = 1.2$, $u_0 = 0.2$, $v_0 = 0.4$, $L_x = 21$ and $L_y = 21$.

by

$$u_{i,j} = u_0 + \gamma_1(|\eta_{i,j}|^2 + |\eta_{i,j+1}|^2), \ j \in \text{odd}, \tag{R10}$$

where $\eta_{i,j} \in \{a_{i,j}, b_{i,j}\}$ represents the state amplitudes of the two sites linked by the corresponding nonlinear coupling, and the intercell coupling, characterized by strength v_0, is linear.

Along the x direction, for each chain, the intercell couplings, $t_{i,j}$ and $\lambda_{i,j}$, are nonlinear and given by

$$t_{i,j} = \tilde{t}_{i,j} + \alpha(|a_{i,j}|^2 + |b_{i-1,j}|^2), \tag{R11}$$

$$\lambda_{i,j} = \tilde{\lambda}_{i,j} + \beta(|a_{i,j}|^2 + |b_{i-1,j}|^2). \tag{R12}$$

In the absence of both nonlinearity and non-Hermiticity, the system reduces to an interface model based on the Benalcazar-Bernevig-Hughes (BBH) lattice [Benalcazar2017]. The BBH model exhibits a higher-order topological phase characterized by the emergence of topological zero modes (TZMs) localized at the corners. In the linear Hermitian case, the corresponding 2D interface model supports TZMs localized at the ends of the interface.

When the nonlinearity along the x direction is introduced with $\gamma_1 = 0$, the 2D non-Hermitian nonlinear interface model supports TZMs, as indicated by the red dots in Fig. R4(b). Owing to the strong NHSE, these TZMs are localized from the interface toward the right-bottom corner for weak nonlinearity at $I = 5^2$ [see Fig. R4(c1)]. As the nonlinear intensity increases to $I = 20^2$, the TZMs become more extended and predominantly occupy the bottom edge [see Fig. R4(c2)]. When the nonlinearity is further strengthened to $I = 50^2$, the modes do not spread across the entire lattice but instead become localized again [see Fig. R4(c3)]. In contrast, when nonlinearity is also introduced along the y direction with $\gamma_1 = 0.01$, the TZMs extend over the entire 2D lattice at $I = 50^2$ [see Fig. R4(d)]. Notably, this extended phase emerges without the need to satisfy the constraint $\delta = \delta_c = \tilde{\lambda}_{i,j} - J$. These results demonstrate how the interplay between nonlinearity and non-Hermiticity enables flexible control of topological mode localization in 2D lattices.

The above model and discussion have replaced the original model in the revised manuscript.

[Benalcazar2017] W. A. Benalcazar, B. A. Bernevig and T. L. Hughes, Quantized electric multipole insulators, Science 357, 61 (2017).

List of main changes

1. We have revised the equation (5) as

$$\left|\left|\Delta\hat{\mathcal{H}}_{\mathrm{S}}(W)\right|\right| < \mu_{\zeta}^{\max}, \tag{R1}$$

Response to the Reviewer #1

Comment 1:

The authors have addressed all my comments point by point, and I am satisfied with their responses. I recommend acceptance.

Our reply :

We sincerely thank the Reviewer for the recommendation.

Response to the Reviewer #2

Comment 1:

I greatly appreciate the authors' detailed responses to my questions and comments. Overall, I believe this manuscript can proceed to publication at Comm Phys.

But, there is one minor correction I'd encourage the authors to make: In Eq. 5, the equality condition corresponds to a "perfect" perturbation driving the smallest singular value of L exactly to 0. Thus, exactly at this point, the system's local topological invariant (eg Eq. 4) is undefined, and so it's not clear if any zero mode is topological under such a perturbation. I would strongly recommend that the authors alter Eq. 5 to be strictly less than.

Our reply :

We sincerely thank the Reviewer for the positive assessment and recommendation for publication.

Regarding the condition in Eq. 5, we agree with the reviewer's observation that the equality corresponds to a "perfect" perturbation, which drives the smallest singular value of the spectral localizer exactly to zero, thereby rendering the local topological invariant undefined. Following this suggestion, we have revised Eq. 5 to use an inequality, which more accurately captures the intended physical interpretation.