Supplementary Information for:

Experimental demonstration of coherence flow in $\mathcal{PT}$- and anti-$\mathcal{PT}$-symmetric systems

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Supplementary Note 1: Phenomenon of stable value and the period of coherent evolution in $\mathcal{PT}$-symmetric systems

Let us first consider the $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian shown in Eq. (1) of the main text. The evolution of quantum states in $\mathcal{PT}$-symmetric systems is described by the time-evolution operator $U_{\mathcal{PT}} = \exp(-i\hat{H}_{\mathcal{PT}}t)$:

$$\begin{aligned} \boldsymbol{U}_{\mathcal{PT}}(t) &= \exp(-i\hat{H}_{\mathcal{PT}}t) \\ &= \exp\left[-is(\sigma_x + ia\sigma_z)t\right] \\ &= \exp\left[s\begin{pmatrix} a & -i \\ -i & -a \end{pmatrix} t\right] \\ &= \begin{pmatrix} A-B & -iC \\ -iC & A+B \end{pmatrix}. \end{aligned} \tag{S1}$$

Here A, B and C are given by:
(i) for $0 < a < 1$,

$$A = \cos(\omega_1 st), \quad B = -\frac{a}{\omega_1}\sin(\omega_1 st), \quad C = \frac{1}{\omega_1}\sin(\omega_1 st), \tag{S2}$$

where $\omega_1 = \sqrt{1-a^2} > 0$.
(ii) for $a > 1$,

$$A = \cosh(\omega_2 st), \quad B = -\frac{a}{\omega_2}\sinh(\omega_2 st), \quad C = \frac{1}{\omega_2}\sinh(\omega_2 st), \tag{S3}$$

where $\omega_2 = \sqrt{a^2-1} > 0$.

In general, the initial state is $|\phi\rangle = \alpha|H\rangle + \beta e^{i\varphi}|V\rangle$, where $\alpha, \beta \in [0,1]$, $\alpha^2 + \beta^2 = 1$ and $\varphi \in [0, 2\pi]$. The time-evolved state is expressed as:

$$\begin{aligned} |\phi(t)\rangle &= \frac{\boldsymbol{U}_{\mathcal{PT}}|\phi\rangle}{\|\boldsymbol{U}_{\mathcal{PT}}|\phi\rangle\|} \\ &= \frac{1}{\sqrt{M}}\begin{pmatrix} \alpha(A-B) - iC\beta e^{i\varphi} \\ (A+B)\beta e^{i\varphi} - iC\alpha \end{pmatrix}, \end{aligned} \tag{S4}$$

where $\||\cdot\rangle\| = \sqrt{\mathrm{Tr}\left(|\cdot\rangle\langle\cdot|\right)}$ denotes the normalization coefficient, and

$$M = \||\cdot\rangle\|^2 = A^2 + B^2 + C^2 + 2AB\left(\beta^2 - \alpha^2\right) - 4\alpha\beta BC\sin\varphi. \tag{S5}$$

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Thus, the coherence of the state $|\phi(t)\rangle$ is given by:

$$C_{l_1}(|\phi(t)\rangle) = \frac{2\left\{\left[\alpha^2(A-B)^2 + C^2\beta^2 + 2\alpha\beta(AC-BC)\sin\varphi\right]\left[C^2\alpha^2 + (A+B)^2\beta^2 - 2\alpha\beta(AC+BC)\sin\varphi\right]\right\}^{1/2}}{A^2+B^2+C^2+2AB\left(\beta^2-\alpha^2\right) - 4\alpha\beta BC\sin\varphi}. \tag{S6}$$

Let us first consider the case when $0 < a < 1$ (i.e., the $\mathcal{PT}$-symmetric-unbroken regime). In this case, A, B, and C are given by Eq. (S2). After inserting Eq. (S2) into Eq. (S6), a simple calculation gives:

$$C_{l_1}(|\phi(t)\rangle) = \frac{2\sqrt{m_1}}{1+m_1}, \tag{S7}$$

where $m_1 = x_1/y_1$, with x_1 and y_1 given below:

$$\begin{aligned} x_1 &= \frac{1}{\omega_1^2}\left[\alpha^2\left(\omega_1^2\cos 2\theta_1 + a\omega_1\sin 2\theta_1\right) + \frac{1-\cos 2\theta_1}{2} + \alpha\beta\sin\varphi\left(\omega_1\sin 2\theta_1 + a(1-\cos 2\theta_1)\right)\right], \\ y_1 &= \frac{1}{\omega_1^2}\left[\beta^2\left(\omega_1^2\cos 2\theta_1 - a\omega_1\sin 2\theta_1\right) + \frac{1-\cos 2\theta_1}{2} - \alpha\beta\sin\varphi\left(\omega_1\sin 2\theta_1 - a(1-\cos 2\theta_1)\right)\right]. \end{aligned} \tag{S8}$$

Here $\theta_1 = \omega_1 st$. From Eq. (S7) and Eq. (S8), one can see that $C_{l_1}(|\phi(t)\rangle)$ is a function of $\sin 2\theta_1$ and $\cos 2\theta_1$. Thus, the period of $C_{l_1}(|\phi\rangle)$ is the same as that of $\sin 2\theta_1$ or $\cos 2\theta_1$. Note that $\sin 2\theta_1$ ($\cos 2\theta_1$) can be written as $\sin 2\omega_1 st$ ($\cos 2\omega_1 st$) with $\omega_1 = \sqrt{1-a^2}$. Therefore, the period of coherent evolution in $\mathcal{PT}$-symmetric systems is:

$$T_{\mathcal{PT}} = \frac{2\pi}{2\omega_1 s} = \frac{\pi}{s\sqrt{1-a^2}}. \tag{S9}$$

Now, let us consider the case $a > 1$ (i.e., the $\mathcal{PT}$-symmetric-broken regime). In this situation, A, B, and C are given by Eq. (S3). Substitution of Eq. (S3) into Eq. (S6) gives:

$$C_{l_1}(|\phi(t)\rangle) = \frac{2\sqrt{m_2}}{1+m_2}, \tag{S10}$$

where $m_2 = x_2/y_2$, with x_2 and y_2 given by

$$\begin{aligned} x_2 &= \alpha^2\left(\cosh\theta_2 + \frac{a}{\omega_2}\sinh\theta_2\right)^2 + \frac{1}{\omega_2^2}\beta^2\sinh^2\theta_2 + 2\alpha\beta\left(\frac{1}{\omega_2}\cosh\theta_2\sinh\theta_2 + \frac{a}{\omega_2}\sinh^2\theta_2\right)\sin\varphi, \\ y_2 &= \beta^2\left(\cosh\theta_2 - \frac{a}{\omega_2}\sinh\theta_2\right)^2 + \frac{1}{\omega_2^2}\alpha^2\sinh^2\theta_2 - 2\alpha\beta\left(\frac{1}{\omega_2}\cosh\theta_2\sinh\theta_2 - \frac{a}{\omega_2^2}\sinh^2\theta_2\right)\sin\varphi. \end{aligned} \tag{S11}$$

Here $\theta_2 = \omega_2 st$. When $t \to \infty$, $\cosh\theta_2 \sim \sinh\theta_2 \to \infty$. Thus, it is straightforward to find from Eqs. (S10) and (S11) that:

$$\begin{aligned} \lim_{t\to\infty} C_{l_1}(|\phi(t)\rangle) &= \frac{2\left\{a^2 + \omega_2^2(\alpha^2-\beta^2)^2 + 2a\omega_2(\alpha^2-\beta^2) + 4\alpha\beta\sin\varphi\left[a + \omega_2(\alpha^2-\beta^2)\right] + 4\alpha^2\beta^2\sin^2\varphi\right\}^{1/2}}{2a^2 + 2a\omega_2\left(\alpha^2-\beta^2\right) + 4a\alpha\beta\sin\varphi} \\ &= \frac{2\left[a + \omega_2(\alpha^2-\beta^2) + 2\alpha\beta\sin\varphi\right]}{2a^2 + 2a\omega_2\left(\alpha^2-\beta^2\right) + 4a\alpha\beta\sin\varphi} \\ &= \frac{1}{a}. \end{aligned} \tag{S12}$$

Equation (S12) shows that the phenomenon of stable value (PSV) of coherence occurs after a long time evolution; that is, the coherence tends to a stable value $1/a$, which is independent of the initial states.

Supplementary Note 2: Phenomenon of stable value and the period of coherent evolution in anti-$\mathcal{PT}$-symmetric systems

Let us now consider the anti-$\mathcal{PT}$($\mathcal{APT}$)-symmetric non-Hermitian Hamiltonian in Eq. (2) of the main text. The evolution of the quantum states in $\mathcal{APT}$-symmetric systems is governed by the operator $U_{\mathcal{APT}} = \exp(-i\hat{H}_{\mathcal{APT}}t)$:

$$\begin{aligned} \boldsymbol{U}_{\mathcal{APT}}(t) &= \exp(-i\hat{H}_{\mathcal{APT}}t) \\ &= \exp\left[-is(i\sigma_x + a\sigma_z)t\right] \\ &= \exp\left[s\begin{pmatrix} -ia & 1 \\ 1 & ia \end{pmatrix}t\right] \\ &= \begin{pmatrix} A+iB & C \\ C & A-iB \end{pmatrix}. \end{aligned} \tag{S13}$$

Here A, B and C are given by:

(i) for $a > 1$,

$$A = \cos(\omega_3 st), \quad B = -\frac{a}{\omega_3}\sin(\omega_3 st)\, t, \quad C = \frac{1}{\omega_3}\sin(\omega_3 st), \tag{S14}$$

where $\omega_3 = \sqrt{a^2 - 1} > 0$.

(ii) for $0 < a < 1$,

$$A = \cosh(\omega_4 st), \quad B = -\frac{a}{\omega_4}\sinh(\omega_4 st), \quad C = \frac{1}{\omega_4}\sinh(\omega_4 st), \tag{S15}$$

where $\omega_4 = \sqrt{1 - a^2} > 0$.

In general, the initial state is $|\phi\rangle = \alpha|H\rangle + \beta e^{i\varphi}|V\rangle$. The time-evolved state is given by:

$$\begin{aligned} |\phi(t)\rangle &= \frac{\boldsymbol{U}_{\mathcal{APT}}|\phi\rangle}{\|\boldsymbol{U}_{\mathcal{APT}}|\phi\rangle\rangle\|} \\ &= \frac{1}{\sqrt{M}}\begin{pmatrix} \alpha(A + Bi) + C\beta e^{i\varphi} \\ (A - Bi)\beta e^{i\varphi} + C\alpha \end{pmatrix}, \end{aligned} \tag{S16}$$

where $M = A^2 + B^2 + C^2 + 4C(A\cos\varphi + B\sin\varphi)\alpha\beta$. The coherence of $|\phi(t)\rangle$ is:

$$C_{l_1}(|\phi(t)\rangle) = \frac{2\left\{\left[(A\alpha + C\beta\cos\varphi)^2 + (B\alpha + C\beta\sin\varphi)^2\right]\left[((A\cos\varphi + B\sin\varphi)\beta + C\alpha)^2 + (A\sin\varphi - B\cos\varphi)^2\beta^2\right]\right\}^{1/2}}{A^2 + B^2 + C^2 + 4C(A\cos\varphi + B\sin\varphi)\alpha\beta}. \tag{S17}$$

Let us first consider the case $a > 1$ (i.e., the $\mathcal{APT}$-symmetric-unbroken regime). In this case, A, B and C are given by Eq. (S14). After inserting Eq. (S14) into Eq. (S17), we obtain:

$$C_{l_1}(|\phi(t)\rangle) = \frac{2\sqrt{m_3}}{1 + m_3}, \tag{S18}$$

where $m_3 = x_3/y_3$, with x_3 and y_3 given below:

$$\begin{aligned} x_3 &= \frac{1}{\omega_3^2}\left[\omega_3^2\alpha^2 + \alpha\beta\omega_3\cos\varphi\sin 2\theta_3 + \frac{1 - \cos 2\theta_3}{2}(1 - a\alpha\beta\sin\varphi)\right], \\ y_3 &= \frac{1}{\omega_3^2}\left[\omega_3^2\beta^2 + \alpha\beta\omega_3\cos\varphi\sin 2\theta_3 + \frac{1 - \cos 2\theta_3}{2}(1 - a\alpha\beta\sin\varphi)\right]. \end{aligned} \tag{S19}$$

Here $\theta_3 = \omega_3 st$. Based on Eq. (S18) and Eq. (S19), one sees that $C_{l_1}(|\phi(t)\rangle)$ is a function of $\sin 2\theta_3$ and $\cos 2\theta_3$; that is, $\sin 2\omega_3 st$ and $\cos 2\omega_3 st$. Hence, the period of coherent evolution in $\mathcal{APT}$-symmetric systems is:

$$T_{\mathcal{APT}} = \frac{2\pi}{2\omega_3 s} = \frac{\pi}{s\sqrt{a^2 - 1}}. \tag{S20}$$

Let us now consider the case of $0 < a < 1$ (i.e., the $\mathcal{APT}$-symmetric-broken regime). In this situation, A, B and C are given by Eq. (S15). Substitution of Eq. (S15) into Eq. (S17) leads to:

$$C_{l_1}(|\phi(t)\rangle) = \frac{2\sqrt{m_4}}{1 + m_4}, \tag{S21}$$

where $m_4 = x_4/y_4$, with x_4 and y_4 given below:

$$\begin{aligned} x_4 &= \frac{1}{\omega_4^2}\left[\left(\omega_4^2\cosh^2\theta_4 + a^2\sinh^2\theta_4\right)\alpha^2 + \beta^2\sinh^2\theta_4 + 2\alpha\beta\sinh\theta_4\left(\omega_4\cos\varphi\cosh\theta_4 - a\sinh\theta_4\sin\varphi\right)\right], \\ y_4 &= \frac{1}{\omega_4^2}\left[\left(\omega_4^2\cosh^2\theta_4 + a^2\sinh^2\theta_4\right)\beta^2 + \alpha^2\sinh^2\theta_4 + 2\alpha\beta\sinh\theta_4\left(\omega_4\cos\varphi\cosh\theta_4 - a\sinh\theta_4\sin\varphi\right)\right]. \end{aligned} \tag{S22}$$

Here $\theta_4 = \omega_4 st$. When $t \to \infty$, $\cosh\theta_4 \sim \sinh\theta_4 \to \infty$. Thus, it follows from Eq. (S22) that:

$$x_4 \sim y_4 \sim \frac{1 + 2\alpha\beta(\sqrt{1 - a^2}\cos\varphi - a\sin\varphi)}{1 - a^2}\sinh^2\theta_4. \tag{S23}$$

Accordingly, it follows from Eq. (S21) that:

$$\lim_{t\to\infty} C_{l_1}\left(|\phi(t)\rangle\right) = \lim_{t\to\infty} \frac{2\sqrt{x_4/y_4}}{1+x_4/y_4} = 1. \tag{S24}$$

Equation (S24) shows that the phenomenon of stable value (PSV) of coherence occurs after a long time evolution; that is, the coherence tends to 1, which is independent of the initial states.

Supplementary Note 3: Proof for the characteristics of each backflow in the $\mathcal{PT}$-symmetric-unbroken regime

For an arbitrary initial state $|\phi\rangle = \alpha|H\rangle + \beta e^{i\varphi}|V\rangle$, the coherence of the evolved state $|\phi(t)\rangle$ in the $\mathcal{PT}$-symmetric unbroken regime is given by Eq. (S7). According to Eq. (S7), the derivative of $C_{l_1}\left(|\phi(t)\rangle\right)$ can be decomposed into

$$\frac{dC_{l_1}\left(|\phi(t)\rangle\right)}{dt} = \frac{dC_{l_1}\left(|\phi(t)\rangle\right)}{dm_1} \times \frac{dm_1}{d\theta_1} \times \frac{d\theta_1}{dt}. \tag{S25}$$

Because of $\frac{d\theta_1}{dt} = \omega_1 s > 0$, the condition for $\frac{dC_{l_1}(|\phi(t)\rangle)}{dt} = 0$ turns into:

$$\frac{dC_{l_1}\left(|\phi(t)\rangle\right)}{dm_1} = 0 \tag{S26}$$

or

$$\frac{dm_1}{d\theta_1} = 0. \tag{S27}$$

First, we consider the case of $\frac{dC_{l_1}(|\phi(t)\rangle)}{dm} = 0$. According to Eq. (S7), we have

$$\frac{dC_{l_1}\left(|\phi(t)\rangle\right)}{dm_1} = \frac{1-m_1}{(1+m_1)^2\sqrt{m_1}} = 0. \tag{S28}$$

Because of $m_1 = x_1/y_1$, it follows from Eq. (S8) that:

$$m_1 = \frac{\alpha^2\left(\omega_1^2\cos 2\theta_1 + a\omega_1 \sin 2\theta_1\right) + \frac{1-\cos 2\theta_1}{2} + \alpha\beta\sin\varphi\left[\omega_1\sin 2\theta_1 + a(1-\cos 2\theta_1)\right]}{\beta^2\left(\omega_1^2\cos 2\theta_1 - a\omega_1\sin 2\theta_1\right) + \frac{1-\cos 2\theta_1}{2} - \alpha\beta\sin\varphi\left[\omega_1\sin 2\theta_1 - a(1-\cos 2\theta_1)\right]}. \tag{S29}$$

After inserting Eq. (S29) into Eq. (S28), we obtain

$$\tan 2\theta_1 = -\frac{\left(\alpha^2-\beta^2\right)\omega_1}{a+2\alpha\beta\sin\varphi}. \tag{S30}$$

Note that the period of $\tan 2\theta_1$ is $\frac{\pi}{2}$ with respect to θ_1, while the period of $C_{l_1}\left(|\phi(t)\rangle\right)$ is $T_{\mathcal{PT}}{=}\pi/(\omega_1 s)$ with respect to t. Because of $\theta_1{=}\omega_1 st$, the period $T_{\mathcal{PT}}{=}\pi/(\omega_1 s)$ can be expressed as $T_{\theta_1}{=}\pi$ with respect to θ_1. Thus, one period of $C_{l_1}\left(|\phi(t)\rangle\right)$ includes two periods of $\tan 2\theta_1$; that is, there are two different values of θ_1 (or t) satisfying Eq. (S30) or Eq. (S26) within one period of $C_{l_1}\left(|\phi(t)\rangle\right)$.

Now, we consider the case of $\frac{dm_1}{d\theta_1} = 0$. Based on $m_1 = x_1/y_1$, one has

$$\frac{dm_1}{d\theta_1} = \frac{x_1' y_1 - x_1 y_1'}{y_1^2}, \tag{S31}$$

where $x_1' = \frac{dx_1}{d\theta_1}$ and $y_1' = \frac{dy_1}{d\theta_1}$. It follows from Eq. (S8) that:

$$\begin{aligned} x_1' &= \frac{1}{\omega_1^2}\left[\alpha^2\left(-2\omega_1^2\sin 2\theta_1 + 2a\omega_1\cos 2\theta_1\right) + \sin 2\theta_1 + \alpha\beta\sin\varphi\left(2\omega_1\cos 2\theta_1 + 2a\sin 2\theta_1\right)\right], \\ y_1' &= \frac{1}{\omega_1^2}\left[\beta^2\left(-2\omega_1^2\sin 2\theta_1 - 2a\omega_1\cos 2\theta_1\right) + \sin 2\theta_1 - \alpha\beta\sin\varphi\left(2\omega_1\cos 2\theta_1 - 2a\sin 2\theta_1\right)\right]. \end{aligned} \tag{S32}$$

Substituting Eq. (S8) and Eq. (S32) into Eq. (S31), one can easily find that the condition for $dm_1/d\theta_1 = 0$ is:

$$\left[2a\omega_1^2\alpha^2\beta^2 - a(4\alpha^2\beta^2\sin^2\varphi + 1) - \alpha\beta\sin\varphi(3a^2+1)\right]\tan^2\theta_1 - \omega_1(1-2\beta^2)(1-2a\alpha\beta\sin\varphi)\tan\theta_1$$
$$+2a\omega_1\alpha^2\beta^2 + \alpha\beta\sin\varphi = 0. \tag{S33}$$

Thus, the discriminant of Eq. (S33) is given by:

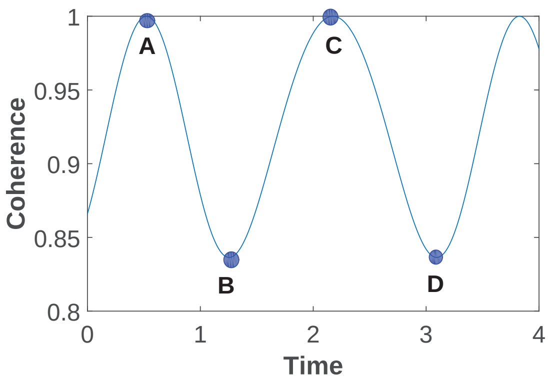
$$\Delta = g + 4(p+q)r, \tag{S34}$$

with

$$\begin{aligned} g &= \omega_1^2(\alpha^2-\beta^2)^2(1-2a\alpha\beta\sin\varphi)^2, \\ p &= 4a\alpha^2\beta^2\sin^2\varphi + \alpha\beta\sin\varphi(3a^2+1), \\ q &= a\left[(\alpha^2-\beta^2)^2 + 2(1+a^2)\alpha^2\beta^2\right], \\ r &= 2a\omega_1^2\alpha^2\beta^2 + \alpha\beta\sin\varphi. \end{aligned} \tag{S35}$$

Here, $g \geq 0$ and $q > 0$. Without loss of generality, we consider $\sin\varphi \geq 0$ and $\alpha, \beta \in (0,\ 1)$. In this case, $p \geq 0$ and $r > 0$. Hence, we have $\Delta > 0$, which implies that $\tan\theta_1$ has two different values to satisfy either Eq. (S33) or Eq. (S27). As mentioned above, the period $T_{\mathcal{PT}}$=$\frac{\pi}{s\sqrt{1-a^2}}$ of $C_{l_1}(|\phi(t)\rangle)$ can be expressed as T_{θ_1}=π with respect to θ_1. Note that the period of $\tan\theta_1$ and the period of $C_{l_1}(|\phi(t)\rangle)$ are π with respect to θ_1, and $\tan\theta_1$ has two different values to satisfy Eq. (S27). Thus, there exist two different values θ_1 (or t) to satisfy Eq. (S27) within one period of $C_{l_1}(|\phi(t)\rangle)$.

From the above discussion, one can conclude that for a wide rangle of initial states $\alpha|H\rangle + \beta e^{i\varphi}|V\rangle$, with $\alpha, \beta \in (0,\ 1)$ and $\sin\varphi \geq 0$, the $\frac{dC_{l_1}(|\phi(t)\rangle)}{dt}$ has four zero points in one period (i.e., T=$\frac{\pi}{s\sqrt{1-a^2}}$) of coherent evolution. Therefore, in the $\mathcal{PT}$-symmetric-unbroken regime, there indeed exists the phenomenon of two backflows of coherence in a period of coherent evolution (e.g., see Supplementary Figure 1).



Supplementary Figure 1: The points A, B, C and D are four extreme points within one period. Note that in the $\mathcal{PT}$-symmetric-unbroken regime, there are two backflows of coherence inside a simple period of coherent evolution.

Supplementary Note 4: Proof for the characteristics of backflow in the $\mathcal{APT}$-symmetric-unbroken regime

For an arbitrary initial state $|\phi\rangle = \alpha|H\rangle + \beta e^{i\varphi}|V\rangle$, the coherence of the evolved state in the $\mathcal{APT}$-symmetric systems is given by Eq. (S17). In the $\mathcal{APT}$-symmetric-unbroken regime (i.e., $a > 1$), A, B and C are given by Eq. (S14). In view of Eq. (S18), the derivative of $C_{l_1}(|\phi(t)\rangle)$ can be expressed as:

$$\frac{dC_{l_1}(|\phi(t)\rangle)}{dt} = \frac{dC_{l_1}(|\phi(t)\rangle)}{dm_3} \times \frac{dm_3}{d\theta_3} \times \frac{d\theta_3}{dt}. \tag{S36}$$

Note that $\frac{d\theta_3}{dt} = \omega_3 s > 0$. Thus, to meet $\frac{dC_{l_1}(|\phi(t)\rangle)}{dt} = 0$, it follows from Eq. (S36):

$$\frac{dC_{l_1}(|\phi(t)\rangle)}{dm_3} = 0, \tag{S37}$$

or

$$\frac{dm_3}{d\theta_3} = 0. \tag{S38}$$

First, we consider the case when $\frac{dC_{l_1}(|\phi(t)\rangle)}{dm_3} = 0$. According to Eq. (S18), we have

$$\frac{dC_{l_1}(|\phi(t)\rangle)}{dm_3} = \frac{1-m_3}{(1+m_3)^2\sqrt{m_3}} = 0. \tag{S39}$$

Because of $m_3 = x_3/y_3$ and according to Eq. (S19), we have

$$m_3 = \frac{\left[\omega_3^2\alpha^2 + \alpha\beta\omega_3\cos\varphi\sin 2\theta_3 + \frac{1-\cos 2\theta_3}{2}(1-a\alpha\beta\sin\varphi)\right]}{\left[\omega_3^2\beta^2 + \alpha\beta\omega_3\cos\varphi\sin 2\theta_3 + \frac{1-\cos 2\theta_3}{2}(1-a\alpha\beta\sin\varphi)\right]}. \tag{S40}$$

Substituting Eq. (S40) into Eq. (S39) leads to

$$\alpha^2 - \beta^2 = 0. \tag{S41}$$

In general, Eq. (S41) is not satisfied for an arbitrary initial state $\alpha|H\rangle + \beta e^{i\varphi}|V\rangle$.

Now, we consider the other case of $dm_3/d\theta_3 = 0$. Because of $m_3 = x_3/y_3$ and according to Eq. (S19), one has

$$\frac{dm_3}{d\theta_3} = \frac{x_3' y_3 - x_3 y_3'}{y_3^2}, \tag{S42}$$

where

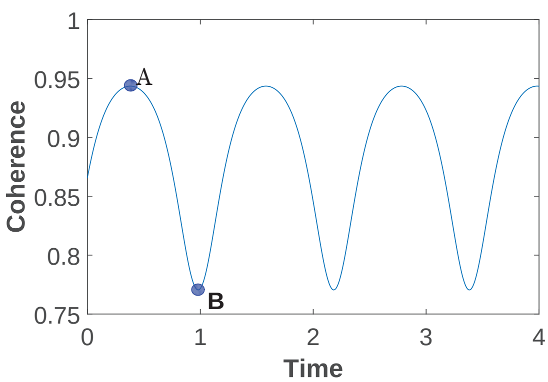
$$\begin{aligned} x_3' &= \frac{1}{\omega_3^2}\left[2\alpha\beta\omega_3\cos\varphi\cos 2\theta_3 + \sin 2\theta_3(1-a\alpha\beta\sin\varphi)\right], \\ y_3' &= \frac{1}{\omega_3^2}\left[2\alpha\beta\omega_3\cos\varphi\cos 2\theta_3 + \sin 2\theta_3(1-a\alpha\beta\sin\varphi)\right]. \end{aligned} \tag{S43}$$

According to Eqs. (S19, S42, S43), one can easily find that the condition for $dm_1/d\theta_1 = 0$ is:

$$\tan 2\theta_3 = -\frac{2\alpha\beta\omega_3\cos\varphi}{1-a\alpha\beta\sin\varphi}. \tag{S44}$$

Because the period of $\tan 2\theta_3$ is $\frac{\pi}{2}$ and the period of $C_{l_1}(|\phi(t)\rangle)$ is T_{θ_3}=π (i.e., $T_{\mathcal{APT}}$=$\frac{\pi}{s\sqrt{a^2-1}}$), one period of $C_{l_1}(|\phi(t)\rangle)$ includes two periods of $\tan 2\theta_3$. Thus, there exist two different values of θ_3 (or t) satisfying Eq. (S44) or Eq. (S38) within one period of $C_{l_1}(|\phi(t)\rangle)$.

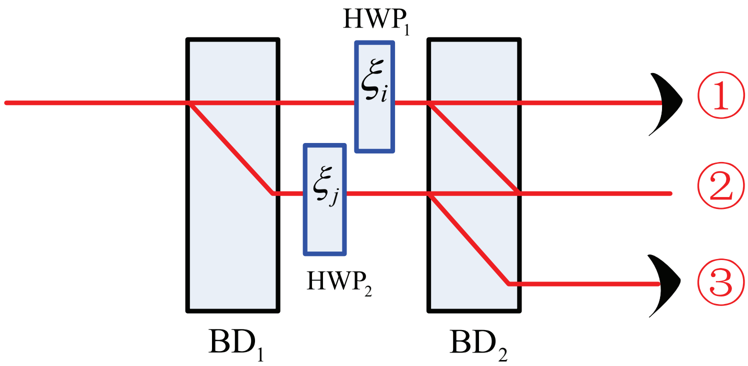
From the above discussion, one can conclude that $\frac{dC_{l_1}(|\phi(t)\rangle)}{dt}$ has two zero points in one period (i.e., $T_{\mathcal{APT}}$=$\frac{\pi}{s\sqrt{a^2-1}}$) of coherent evolution. Therefore, the coherent oscillation of quantum states in the $\mathcal{APT}$-symmetric-unbroken regime has only one backflow within one period (eg., see Supplementary Figure 2).



Supplementary Figure 2: The points A and B are two extreme points in one period. The coherent oscillation of quantum states in the $\mathcal{APT}$-symmetric-unbroken regime has only one backflow within one period.

Supplementary Note 5: Experimental implementation of the loss operator L

As illustrated in Supplementary Figure 3, we experimentally implement the loss operator L by a combination of two beam displacers (BD_1 and BD_2) and two half-wave plates (HWP_1 and HWP_2). Here, the optical axes of the BDs are cut so that the vertically polarized photons are transmitted directly, while the horizontally polarized photons are displaced into the lower path. In addition, the HWP_1 and HWP_2 with setting angles ξ_i and ξ_j are, respectively, inserted into the upper and lower paths between the two BDs. The rotation operations on the photon polarization states, performed by the HWP_1 and HWP_2, are given as follows:



Supplementary Figure 3: Experimental setup to realize a loss operator, where ξ_i and ξ_j are the two tunable setting angles for the half-wave plates HWP_1 and HWP_2, respectively.

$$R_{\mathrm{HWP}}\left(\xi_i\right)=\begin{pmatrix}\cos 2\xi_i & \sin 2\xi_i\\ \sin 2\xi_i & -\cos 2\xi_i\end{pmatrix},\quad R_{\mathrm{HWP}}\left(\xi_j\right)=\begin{pmatrix}\cos 2\xi_j & \sin 2\xi_j\\ \sin 2\xi_j & -\cos 2\xi_j\end{pmatrix}. \tag{S45}$$

In this case, when a horizontally polarized photon passes through the experimental setup, one can find that

$$|H\rangle \xrightarrow{\mathrm{BD}_1} |H\rangle_{\mathrm{lower}} \xrightarrow{R_{\mathrm{HWP}}(\xi_j)} R_{\mathrm{HWP}}(\xi_j)|H\rangle \xrightarrow{\mathrm{BD}_2} \cos 2\xi_j|H\rangle_3+\sin 2\xi_j|V\rangle_2, \tag{S46}$$

where the subscript "lower" represents the lower path between the two BDs, while subscripts "2" and "3" represent the two paths 2 and 3 after the second BD, respectively. Similarly, when a vertically polarized photon pass the experimental setup, one can find that

$$|V\rangle \xrightarrow{\mathrm{BD}_1} |H\rangle_{\mathrm{upper}} \xrightarrow{R_{\mathrm{HWP}}(\xi_i)} R_{\mathrm{HWP}}(\xi_i)|V\rangle \xrightarrow{\mathrm{BD}_2} \sin 2\xi_i|H\rangle_2-\cos 2\xi_i|V\rangle_1, \tag{S47}$$

where the subscript "upper" represents the upper path between the two BDs, while subscripts "1" and "2" represent the two paths 1 and 2 after the second BD, respectively. That is, only horizontally polarized photons in the upper path and vertically polarized photons in the lower path are transmitted through the second BD and then combined onto path 2, while the other photons transmitted onto path 1 or 3 are blocked, i.e., they are discarded and lost from the system.

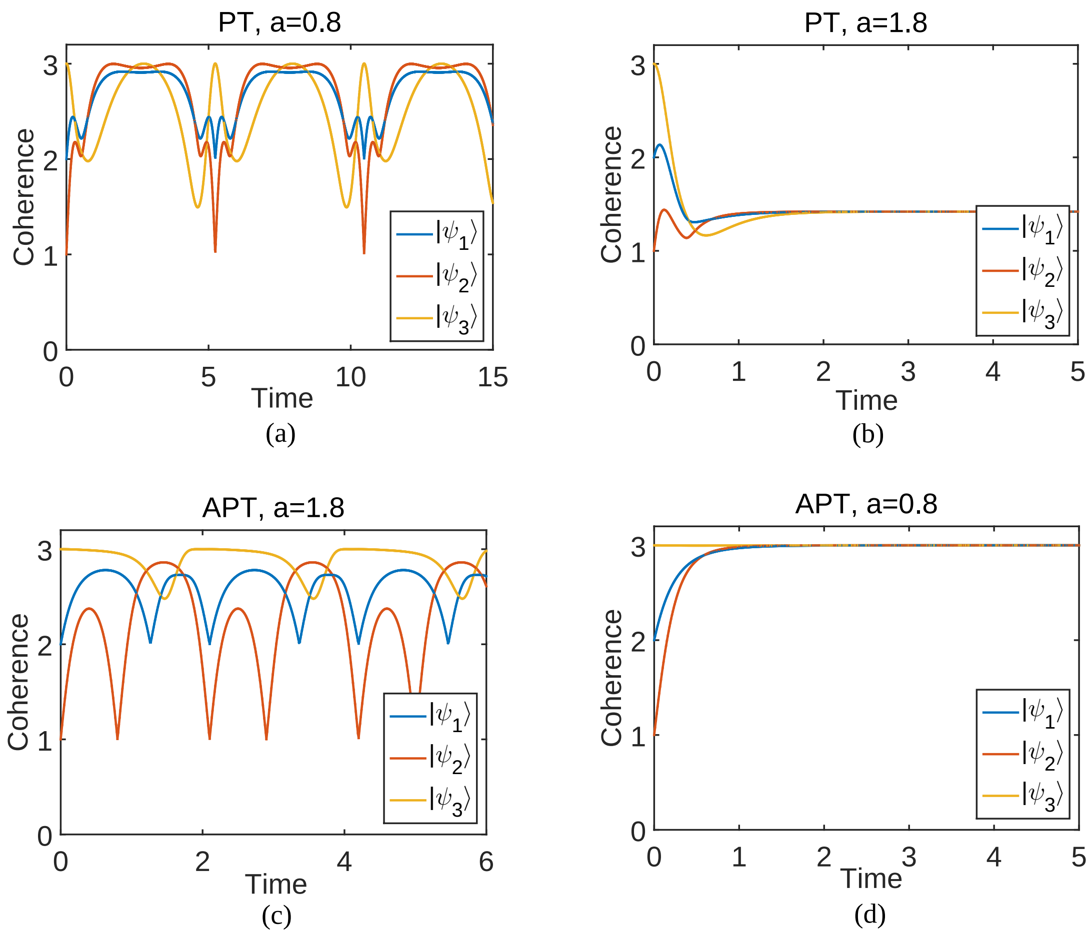
In this sense, according to Eqs. (S46) and (S47), when the input photon is initially in the state $|\phi\rangle_{\mathrm{in}}=\alpha|H\rangle+\beta e^{i\varphi}|V\rangle$, then the output photon appearing in the path 2 would be in the state $|\phi\rangle_{\mathrm{out}}=\alpha\sin 2\xi_j|V\rangle_2+\beta e^{i\varphi}\sin 2\xi_i|H\rangle_2$. It is obvious that this state transformation can be written as $|\phi\rangle_{\mathrm{out}}=L|\phi\rangle_{\mathrm{in}}$, with a polarization-dependent photon loss operator L, given by

$$L\left(\xi_i,\ \xi_j\right)=\begin{pmatrix}0 & \sin 2\xi_i\\ \sin 2\xi_j & 0\end{pmatrix}, \tag{S48}$$

where ξ_i and ξ_j are, respectively, the two tunable setting angles for the half-wave plates HWP_1 and HWP_2 (Supplementary Figure 3).

Supplementary Note 6: Coherence flow for two-qubit $\mathcal{PT}$- and anti-$\mathcal{PT}$- symmetric systems

We have numerically simulated the dynamics of coherence for two-qubit $\mathcal{PT}/\mathcal{APT}$ systems. As shown in Supplementary Figures 4(a, c), there exist different periodic oscillations of coherence (including one coherence backflow, two coherence backflows, and multiple coherence backflows in one period) for $\mathcal{PT}/\mathcal{APT}$-symmetric systems in the unbroken regime. In addition, as illustrated in Supplementary Figures 4(b, d), there exists PSV for both $\mathcal{PT}$ -and $\mathcal{APT}$ -symmetric systems in the broken regime, which are independent of the initial states.



Supplementary Figure 4: The evolution of coherence for three different initial states in a two-qubit $\mathcal{PT}$ /$\mathcal{APT}$ -symmetric system. We consider the two qubits undergoing the same $\mathcal{PT}/\mathcal{APT}$ -symmetric dynamic process, i.e., the parameters a involved in the Hamiltonians (1) and (2) of the main text are set to be the same for both qubits. (a) $a = 0.8$, the $\mathcal{PT}$ symmetry unbroken regime; (b) $a = 1.8$, the $\mathcal{PT}$ symmetry broken regime; (c) $a = 1.8$, the $\mathcal{APT}$ symmetry unbroken regime; (d) $a = 0.8$, the $\mathcal{APT}$ symmetry broken regime. The three initial states are $|\psi_1\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |11\rangle)$ (blue curves), $|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + e^{i\pi/5}|11\rangle\right)$ (red curves), and $|\psi_3\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + e^{i\pi/5}|11\rangle\right)$ (yellow curves).