Majorana approach for driven quantum systems

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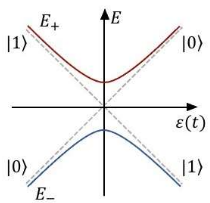
Abstract

The approach by Ettore Majorana for non-adiabatic transitions between two quasi-crossing levels is revisited. This gives the transition probability, known as the Landau-Zener-Stückelberg-Majorana formula, introducing Majorana's approach to modern readers. On top of this, we obtain the full wave function, including its phase, which is important nowadays for quantum control.

Formulation of the problem — I



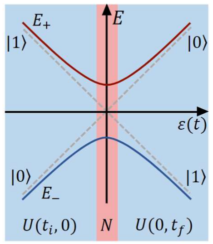
Consider a two-level system or qubit. Its dynamics are important in physics and quantum information. We consider the linear perturbation known as Landau-Zener-Stuckelberg-Majorana transition.

A driven two-level system is described by

$$H(t) = -\frac{\Delta}{2}\sigma_x - \frac{\varepsilon(t)}{2}\sigma_z$$

$$\varepsilon(t) = vt$$

Adiabatic-impulse model — II



We consider the adiabatic-impulse model, where the evolution of the system consists of three parts: the first and the last stages are adiabatic, and the second one is a transition. The transfer matrix N can be written as:

$$N = \begin{pmatrix} \sqrt{\mathcal{P}} & \sqrt{1-\mathcal{P}}e^{-i\phi} \\ -\sqrt{1-\mathcal{P}}e^{i\phi} & \sqrt{\mathcal{P}} \end{pmatrix}$$

The adiabatic evolution can be described by the matrix U_{ad}:

$$U_{ad}(-\tau_a, 0) = \begin{pmatrix} exp\{-i(\Phi(\tau_a) - \Phi_\delta)\} & 0 \\ 0 & exp\{i(\Phi(\tau_a) - \Phi_\delta)\} \end{pmatrix};$$

$$U_{ad}(0, \tau_a) = \begin{pmatrix} exp\{i(\Phi(\tau_a) - \Phi_\delta)\} & 0 \\ 0 & exp\{-i(\Phi(\tau_a) - \Phi_\delta)\} \end{pmatrix}.$$

where $\tau = t\sqrt{\frac{v}{2\hbar}}$

$$\Phi(\tau_0) = \frac{\tau_0^2}{2} + \delta\ln(\sqrt{2}\tau_0). \qquad \Phi_\delta = \frac{1}{2}\delta(\ln\delta - 1).$$

Zener's approach — III

Clarence Zener obtained the full analytical solution is presented in terms of parabolic-cylinder functions $D_{i\delta}(z)$. The wave function is

$$\begin{cases} \alpha & = A_+ D_{-i\delta-1}(z) + A_- D_{-i\delta-1}(-z) \\ \beta & = -A_+ \frac{exp\{-\frac{i\pi}{4}\}}{\sqrt{\delta}} D_{-i\delta}(z) + A_- \frac{exp\{-\frac{i\pi}{4}\}}{\sqrt{\delta}} D_{-i\delta}(-z). \end{cases}$$

where $z = \tau\sqrt{2}e^{\frac{i\pi}{4}}$.

The constants $A_\pm$ are defined from initial conditions

$$A_+ = \frac{\alpha(z_0)D_{-i\delta}(-z_0) - \sqrt{\delta}e^{\frac{i\pi}{4}}\beta(z_0)D_{-i\delta-1}(-z_0)}{D_{-i\delta-1}(-z_0)D_{-i\delta}(z_0) + D_{-i\delta-1}(z_0)D_{-i\delta}(-z_0)};$$

$$A_- = \frac{\alpha(z_0)D_{-i\delta}(z_0) + \sqrt{\delta}e^{\frac{i\pi}{4}}\beta(z_0)D_{-i\delta-1}(z_0)}{D_{-i\delta-1}(-z_0)D_{-i\delta}(z_0) + D_{-i\delta-1}(z_0)D_{-i\delta}(-z_0)}.$$

From the adiabatic-impulse model we obtain the result for the ground state known as Landau-Zener-Stückelberg-Majorana probability $\mathcal{P}_{\mathcal{LZSM}} = e^{-2\pi\delta}$.

And the phase of the wave function equals the Stokes phase $\varphi_S = \frac{\pi}{4} + Arg(\Gamma(1 - i\delta)) + \delta(\ln\delta - 1)$. ◆

In our work, we use this result for comparing the asymptotic method with the analytical solution.

Majorana's approach — IV

This method is based on applying Laplace transform for solving differential equations. We apply the following substitution

$$\alpha = f\exp\left(\frac{i}{2}\tau^2\right); \beta = g\exp\left(-\frac{i}{2}\tau^2\right). \quad (1)$$

Then the Schrodinger equation becomes

$$\dot{f} = i\sqrt{2\delta}g\exp\left(-i\tau^2\right) \qquad (2)$$
$$\dot{g} = i\sqrt{2\delta}f\exp\left(i\tau^2\right).$$

Applying Laplace transform — V

The system could be transformed into two second-order differential equations:

$$\ddot{f} + 2i\tau\dot{f} + 2\delta f = 0 \qquad (3)$$
$$\ddot{g} - 2i\tau\dot{g} + 2\delta g = 0.$$

We applied the Laplace transform for the first equation and then the solution is obtained:

$$F(s) = C\cdot\exp\left(-i\frac{s^2}{4}\right)s^{-(i\delta+1)}. \qquad (4)$$

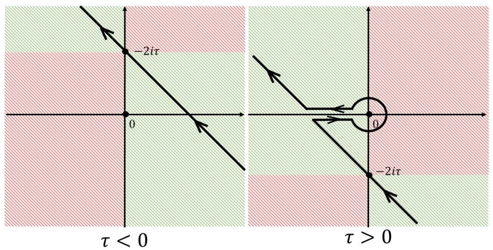
Steepest descent method — VI

For obtaining the evolution of the wave function the inverse Laplace transform should be applied

$$f(\tau) = C\int_C e^{s\tau}\exp\left(-i\frac{s^2}{4}\right)s^{-(i\delta+1)}ds. \qquad (5)$$

The result from this integral could be obtained using the steepest descent method. This method is asymptotic, which means that we get the result far from zero times (far from the transition). $C = constant$.

This method could be applied if some conditions are fulfilled. According to it, the integration is on two special contours, before and after the transitions:



In the second case, the result comes not only from the steepest descend method but also from integration near zero.

The full asymptotic dependence — VII

The part in the near-zero region could be transformed. In this limit, the integral is a Hankel-type integral. The second part of the wave function is obtained from equation (2):

$\tau < 0$ ▲

$$\alpha(\tau) = C(-2i\tau)^{-i\delta-1}exp\left(-\frac{i\tau^2}{2}\right)\sqrt{-4\pi i};$$

$$\beta(\tau) = -C\sqrt{\frac{2\pi}{\delta}}(-2i)^{-i\delta-1}\tau^{-i\delta}exp\left(-i\frac{3\pi}{4} - \frac{i\tau^2}{2}\right)\left[(i\delta+1)\tau^{-2} + 2i\right].$$

$\tau > 0$ ▲

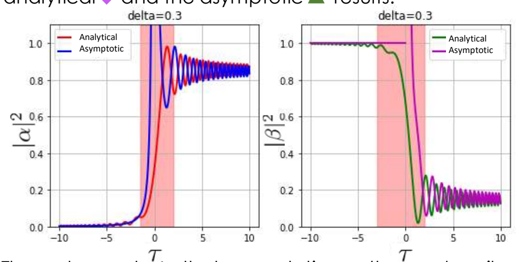
$$\alpha(\tau) = C\sqrt{-4\pi i}(-2i\tau)^{-i\delta-1}exp\left(-\frac{i\tau^2}{2}\right) + $$
$$+C\frac{2\pi i}{\Gamma(i\delta+1)}\tau^{i\delta}exp\left(\frac{i\tau^2}{2}\right);$$

$$\beta(\tau) = -C\sqrt{\frac{2\pi i}{\delta}}(\tau)^{-i\delta}(-2i)^{-i\delta-1}[(i\delta+1)\tau^{-2} + 2i]exp\left(-\frac{i\tau^2}{2}\right) +$$
$$+C\sqrt{\frac{\delta}{2}}\frac{2\pi i}{\Gamma(i\delta+1)}\tau^{i\delta-1}\exp\left(-\frac{i\tau^2}{2}\right).$$

It is the general result obtained by the asymptotic approach. The C is obtained using the initial state. If initially, the state is ground, then the result ▲ comes to ◆

Comparing the asymptotic and analytical results — VIII

The equations above allow the comparison of the analytical ◆ and the asymptotic ▲ results.



These demonstrate that asymptotic results can describe the quantum system in a long time interval.

- Landau-Zener-Stuckelberg-Majorana physics is important for both fundamental science and quantum technologies.
- We developed (asymptotic) Majorana's approach up to the adiabatic-impulse model and compared this with Zener's (analytic) approach.