Quantifying Non-Markovianity with Temporal Steering

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Einstein-Podolsky-Rosen (EPR) steering is a type of quantum correlation which allows one to remotely prepare, or steer, the state of a distant quantum system. While EPR steering can be thought of as a purely spatial correlation, there does exist a temporal analogue, in the form of single-system temporal steering. However, a precise quantification of such temporal steering has been lacking. Here, we show that it can be measured, via semidefinite programing, with a *temporal steerable weight*, in direct analogy to the recently proposed EPR steerable weight. We find a useful property of the temporal steerable weight in that it is a nonincreasing function under completely positive trace-preserving maps and can be used to define a sufficient and practical measure of strong non-Markovianity.

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Quantum entanglement, Einstein-Podolsky-Rosen (EPR) steering, and Bell nonlocality are three of the most intriguing phenomena in quantum physics and, in varying degrees, are thought to act as resources, fuel that powers a range of quantum technologies. Entanglement [1–3] comes hand in hand with the complexity of quantum systems and may be behind the potential speed-up of quantum computation. Bell nonlocality and EPR steering are thought to be the driving power of quantum cryptography and have both been recast in that language. For example, in a quantum key distribution scenario, two parties wish to generate a secret key using shared quantum states as a resource. If one party (Bob) trusts his own experimental apparatus but not that of the other party (Alice), a violation of a steering inequality [2–4] can be used to certify that true quantum correlations exist between their shared states. In stricter terms, such a test proves to Bob that the correlations he observes between his measurement results and Alice's cannot be described by a local hidden-state model; his state is truly being influenced by Alice's measurements in a nonlocal manner. As with entanglement, one quantifies the amount of steering that is possible with a given shared state via a range of possible measures [5–8]. Very recently, a powerful example of such a measure, the steerable weight, was proposed by Skrzypczyk and co-workers [9,10].

In EPR steering, the notion of nonlocality, via spacelike separations between parties, plays an important role. If we relax this constraint and consider timelike separation of measurement events, can the concept of steering still be used in a meaningful way? We can find inspiration in the fact that there do already exist other types of nontrivial temporal quantum correlations complementary to both Bell nonlocality and entanglement. For the former, one of the most well-known examples is the Leggett-Garg inequality [11], which can be used to test the assumption of "macroscopic realism," in contrast to the nonlocal realism tested by Bell's inequality, and for which experimental violations have been observed in a large range of systems [12–14]. For the latter, motivated by the Choi-Jamiolkowski isomorphism [15], which equates the correlations in a bipartite quantum system with two-time correlations of a single quantum system, the notion of temporal entanglement has been proposed in various forms [16–22]. Returning to steering, the concept of temporal steering and a temporal steering inequality were recently introduced by Chen *et al.* [23]. Also inspired by the Choi-Jamiolkowski isomorphism, they showed that, even without the assumption of nonlocality, the concept of one party not trusting the earlier measurements made by another party delineates between certain classical and quantum correlations. Not only does this have direct practical applications in verifying a quantum channel for quantum key distribution, it was recently shown that temporal steering, like EPR steering [24,25], is intimately linked to the concepts of realism and joint measurability [26–29].

Still lacking, however, is a measure to quantify these "temporal steering" quantum correlations. Here, in analogy to the EPR steerable weight [9,10], we define the temporal steerable weight (temporal SW) as a measure of temporal steering. We prove that the temporal SW is nonincreasing under a completely positive trace-preserving (CPT) map and can be used to define a sufficient but not necessary measure of non-Markovianity. In the same way that the spatial steerable weight can be considered a measure of strong entanglement, since not every entangled state is steerable, we define the *temporal* SW as a measure of *strong* non-Markovianity because it vanishes for weak non-Markovian process. (This is also in analogy to, e.g., the phenomenon of *strong nonclassicality*, which can be detected and quantified by a weaker criterion of nonclassicality [30,31]). We show this by comparing the non-Markovianity measured by the temporal SW to an existing entanglement-based measure [32] and find that it is, as expected, less sensitive. However, the temporal SW is, in principle, easier to implement experimentally, as it does not require the use of an ancilla or full process tomography. These results, together with a few illustrative examples discussed mainly in the Supplemental Material [33], suggest that temporal steering can serve as a unique and useful quantum resource.

Temporal steerable weight.—Now, we introduce the concept of a temporal steerable weight in analogy to the spatial steerable weight introduced recently by Skrzypczyk and co-workers [9,10]. In the standard (spatial) EPR steering scenario, Alice performs a positive-operator valued measure measurement $F_{a|x} = M^{\dagger}_{a|x} M_{a|x}$, $\sum_a F_{a|x} = \mathbb{1}$, on a state ρ_{AB} shared with Bob, and creates the assemblage $\{\sigma_{a|x}\}$, where a is the measurement result and x is the basis of the measurement. In defining a steering inequality, or a steerable weight, one assumes that Bob does not trust Alice, nor her experimental apparatus, and wishes to distinguish between true manipulation of his local state via quantum correlations and correlations that cannot be distinguished from some classical theory, typically a local hidden-state model. In temporal steering, we also let Alice perform a positive-operator valued measure measurement $F_{a|x} = M^{\dagger}_{a|x} M_{a|x}$ but on a single system in an initial state ρ_0 at time $t = 0$. After the measurement, the initial state is mapped to $\rho_{a|x}$ (see Fig. 1):

$$\rho_0 \mapsto \rho_{a|x} = \frac{M_{a|x} \rho_0 M^{\dagger}_{a|x}}{p(a|x)}, \qquad (1)$$

with the probability $p(a|x) = \mathrm{tr}(M_{a|x} \rho_0 M^{\dagger}_{a|x})$. After this initial measurement, the state $\rho_{a|x}$ is sent into a quantum channel Λ for a time t. At time t, Bob receives the system and performs quantum state tomography to obtain the state $\sigma_{a|x}$, i.e., $\Lambda(\rho_{a|x}) = \sigma_{a|x}$. To mimic the un-normalized assemblage [9,10] in standard EPR steering, we define the un-normalized states in temporal steering

$$\sigma^{T}_{a|x} \equiv p(a|x) \sigma_{a|x}, \qquad (2)$$

where the superscript T reminds one that the assemblage $\{\sigma^{T}_{a|x}\}$ is for temporal steering.

However, the quantum channel may be noisy, obliterating the influence of Alice's measurement choice, or Alice's measurement results could have been fabricated via classical strategies. In these cases, $\sigma^{T}_{a|x}$ may include, or be entirely described by, an unsteerable assemblage which we define as

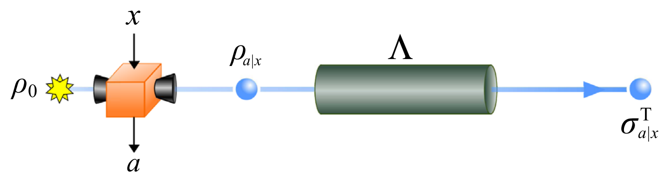

FIG. 1. Schematic diagram of temporal steering. In the beginning, Alice performs the measurement $F_{a|x} = M^{\dagger}_{a|x} M_{a|x}$ on an initial state ρ_0. Then, ρ_0 is mapped to $\rho_{a|x}$ and sent into a quantum channel Λ. Finally, Bob receives the assemblage $\{\sigma^{T}_{a|x}\}$ at time t.

$$\sigma^{T,\mathrm{US}}_{a|x} = \sum_{\lambda} P(\lambda) P(a_{|x}|\lambda) \sigma_{\lambda}, \qquad (3)$$

where $\sum_{\lambda} P(\lambda) = 1$. We have written the result a, conditional on the basis x, with a subscript notation $a_{|x} \equiv a|x$. In the EPR setting, λ represents a local hidden variable which determines the possible correlations between Alice's and Bob's measurement results from a source which obeys classical realism. As in that case, when Alice reveals her measurement results, Bob can update his knowledge of his state, as indicated by two equal forms (by applying the chain rule) $\sum_{\lambda} P(\lambda) P(a_{|x}|\lambda) \sigma_{\lambda} = \sum_{\lambda} P(a|x) P(\lambda|a_{|x}) \sigma_{\lambda}$. Then, the unsteerable states are those states which obey the classical (realism) chain rule for Alice's joint measurement results, as shown in a recent work on steering witnesses [26]. No matter what happens during the transmission, Bob's task is to check whether the assemblage he receives can be written in the hidden-state form [Eq. (3)] or not. If he can, this means the state Bob receives is independent of the basis x Alice chooses to measure in. As mentioned above, this may be because the quantum channel is too noisy, such that the influence of Alice's measurements is no longer discernable, or Alice's measurement results could have been fabricated via classical strategies. On the other hand, if the assemblage Bob receives cannot be written in the form of Eq. (3), he is convinced that Alice has influenced his state by her choice of measurement. In this case, we call the assemblage Bob receives "temporally steerable" and is symbolized as $\{\sigma^{T,S}_{a|x}\}$.

To determine the steerable weight, one considers the overlap between the state Bob receives and the unsteerable assemblage, such that his state can be written as a mixture

$$\sigma^{T}_{a|x} = \mu \sigma^{T,\mathrm{US}}_{a|x} + (1-\mu) \sigma^{T,S}_{a|x}. \qquad (4)$$

To quantify the "steerability in time" for a given assemblage $\{\sigma^{T}_{a|x}\}$, one has to maximize μ, i.e., maximize the proportion of $\sigma^{T,\mathrm{US}}_{a|x}$. Then, the "temporal steerable weight" can be defined as $\mathrm{TSW} = 1 - \mu^{*}$, in which μ^{*} is the maximum of μ and can be obtained from semidefinite programing [9,10,36]:

$$\begin{aligned} \mu^* = \max \quad & \mathrm{tr}\sum_\lambda \tilde{\sigma}_\lambda \\ \text{subject to} \quad & \sigma^T_{a|x} - \sum_\lambda D_\lambda(a|x)\tilde{\sigma}_\lambda \geq 0 \quad \forall\ a, x \\ & \tilde{\sigma}_\lambda \geq 0 \qquad \forall\ \lambda, \end{aligned} \tag{5}$$

where $\tilde{\sigma}_\lambda = \mu\sigma_\lambda$ and $D_\lambda(a|x)$ are the extremal deterministic values [9] of the conditional probability distributions $P(a_{|x}|\lambda)$. Equation (5), which is formulated as a semidefinite program, can be numerically implemented in various convex optimization packages, e.g., Refs. [37,38].

So far, the formalism is parallel to the standard EPR steerable weight [9]. The primary difference is that $\{\sigma_{a|x}\}$ in Ref. [9] is created through the entanglement between Alice and Bob. Here, $\{\sigma^T_{a|x}\}$ is created through Alice's measurement and the influence of the quantum channel Λ. In the Supplemental Material [33], we give an explicit pedagogical example of how to evaluate the temporal steerable weight.

Measure of non-Markovianity.—Now, we apply the introduced temporal steering weight as a measure of non-Markovianity. Non-Markovianity is a term used to describe the situation when an environment surrounding a quantum system has memory of its past evolution. It is an important concept both because many natural and man-made quantum systems exist in a regime where the assumption of a Markovian (memoryless) environment fails, but also because it can lead to counterintuitive results regarding the decay of quantum effects, particularly when the quantum system is strongly coupled to the surrounding environment. There has been a range of efforts at constructing measures of non-Markovianity, typically based on a scenario where the time evolution of a quantum system is analyzed for non-Markovian properties. Arguably, the most popular measures of non-Markovianity were introduced in Refs. [32,39]. Recently, an attempt to classify these non-Markovianity measures in a unified framework was described in Ref. [40]. Useful for us here is the approach taken in Ref. [39], which is based on observing the behavior of the trace distance between two quantum states. They derived a measure of non-Markovianity by noting that all CPT maps Φ are contractions of the trace distance metric, and a given dynamic process is defined as Markovian if the map is divisible, i.e., $\Phi(\tau+t,0) = \Phi(\tau+t,t)\Phi(t,0)$, for all positive t and τ. These two properties lead to the monotonicity of the trace distance, and violations of this monotonicity indicate the occurrence of non-Markovian dynamics. In a similar way, below we prove that the temporal SW of a system undergoing a CPT map is also a nonincreasing function, i.e.,

$$\mathrm{TSW}_\rho \geq \mathrm{TSW}_{\Phi(\tau)\rho} \tag{6}$$

for a CPT map $\Phi(\tau)$. Together with the property of divisibility, one can conclude that the temporal SW decreases monotonically under Markovian dynamics. Therefore, our measure of non-Markovianity is defined by integrating the positive slope of the temporal SW

$$\mathcal{N}_{\mathrm{TSW}} \equiv \int_{\sigma_{\mathrm{TSW}}>0} dt\,\sigma_{\mathrm{TSW}}(t,\rho_0,\Phi), \tag{7}$$

where $\sigma_{\mathrm{TSW}}(t,\rho_0,\Phi) = \frac{d}{dt}\mathrm{TSW}_{\Phi(t)\rho_0}$ is the rate of change of the temporal steerable weight. In the examples discussed in the Supplemental Material [33], we demonstrate explicitly how one can use this as a practical measure of strong non-Markovianity. Here, we discuss only the following example.

Proof of the monotonicity of temporal SW under Markovian dynamics.—First, we prove that the temporal SW of a system undergoing a CPT map is a nonincreasing function, as given by Eq. (6). To obtain the temporal SW of a qubit at time t_1, one needs the quantity $\sigma^T_{a|x}(t_1) - \sum_{\lambda_1} D_{\lambda_1}(a|x)\tilde{\sigma}_{\lambda_1}$, in which the set $\{\tilde{\sigma}_{\lambda_1}\}$ is chosen to maximize $\mathrm{Tr}(\sum_{\lambda_1}\tilde{\sigma}_{\lambda_1})$ at time t_1. Summing all the measurement outcomes a and taking the trace, we have

$$\begin{aligned} &\mathrm{Tr}\left[\sum_a \sigma^T_{a|x}(t_1) - \sum_a\sum_{\lambda_1} D_{\lambda_1}(a|x)\tilde{\sigma}_{\lambda_1}\right] \\ &\quad = \mathrm{Tr}\left[\sum_a \sigma^T_{a|x}(t_1) - \sum_{\lambda_1}\tilde{\sigma}_{\lambda_1}\right] = 1 - \mu_1^*, \end{aligned} \tag{8}$$

where we have used the properties $\sum_a D_\lambda(a|x) = 1$ and $\mathrm{Tr}[\sum_a \sigma^T_{a|x}(t_1)] = 1$. Similarly, to obtain the temporal SW of the qubit at a later time $t_2 = t_1 + \tau$, one also has

$$\mathrm{Tr}\left[\sum_a \sigma^T_{a|x}(t_2) - \sum_a\sum_{\lambda_2} D_{\lambda_2}(a|x)\tilde{\sigma}_{\lambda_2}\right] = 1 - \mu_2^*, \tag{9}$$

where $\{\tilde{\sigma}_{\lambda_2}\}$ is chosen to maximize $\mathrm{Tr}(\sum_{\lambda_2}\tilde{\sigma}_{\lambda_2})$ at time t_2. One can also perform a CPT map $\Phi(\tau)$ to Eq. (8), giving

$$\begin{aligned} &\mathrm{Tr}\left[\sum_a \Phi(\tau)\sigma^T_{a|x}(t_1) - \sum_a\sum_{\lambda_1} \Phi(\tau)D_{\lambda_1}(a|x)\tilde{\sigma}_{\lambda_1}\right] \\ &\quad = \mathrm{Tr}\left[\sum_a \sigma^T_{a|x}(t_2) - \sum_a\sum_{\lambda_1} D_{\lambda_1}(a|x)(\Phi(\tau)\tilde{\sigma}_{\lambda_1})\right]. \end{aligned} \tag{10}$$

Since $\Phi(\tau)$ is a trace-preserving map, the value of Eq. (10) is still $1-\mu_1^*$. However, we know that the set $\{\tilde{\sigma}_{\lambda_2}\}$ is the optimal way to maximize $\mathrm{Tr}(\sum_{\lambda_2}\tilde{\sigma}_{\lambda_2})$ at time t_2 for Eq. (9). Therefore, comparing Eq. (9) with Eq. (10) would give

$$1 - \mu_1^* \geq 1 - \mu_2^*. \tag{11}$$

This proves the theorem given in Eq. (6). Employing the divisibility of Markovian dynamics leads to the monotonicity of the temporal SW:

$$\mathrm{TSW}\{\Phi(\tau+t,0)\sigma_{a|x}\} = \mathrm{TSW}\{\Phi(\tau+t,t)\Phi(t,0)\sigma_{a|x}\} \leq \mathrm{TSW}\{\Phi(t,0)\sigma_{a|x}\}. \qquad (12)$$

An example of non-Markovianity of a spin-boson problem.—Exact solutions to the general spin-boson problem have applications in a huge range of systems, from quantum computing to physical chemistry and photosynthesis [41]. Various techniques and methods exist to numerically acquire such solutions, one of the most powerful of which is the hierarchy equations of motion [42,43]. Here, we use those equations to model a two-level system coupled to a bosonic environment or reservoir. The general Hamiltonian is written as

$$H_{\mathrm{SB}} = \frac{E}{2}\sigma_z + \Delta\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \sum_{\mathbf{k}}\sigma_z \otimes l_{\mathbf{k}}(a_{\mathbf{k}}^{\dagger} + a_{\mathbf{k}}), \qquad (13)$$

where Δ is the two-level system tunneling amplitude and E is the two-level system splitting. The environment modes are described with creation ($a_{\mathbf{k}}^{\dagger}$) and annihilation operators ($a_{\mathbf{k}}$) with energy $\omega_{\mathbf{k}}$, which couple to the system, described by the Pauli operators σ_z and σ_x, with strength $l_{\mathbf{k}}$. By assuming that the environment modes are well described by a Drude-Lorentz spectral density $J(\omega) = 2\alpha\omega_c(\omega/\omega^2 + \omega_c^2)$, where α is the system-reservoir coupling strength and ω_c is the bath cutoff frequency, we can exactly solve the dynamics of the two-level system (details can be found in Refs. [41–43]). We can then compare the non-Markovianity as measured via the temporal SW to that given by the nonmonotonic behavior of the entanglement, as given by the concurrence [1], between the two-level system and an isolated ancilla [32]. One important difference in the two approaches is that in the temporal SW case, there is no ancilla. In the ancilla case, the initial condition between the system and ancilla is that of a maximally entangled state; to mimic that in the temporal SW case, we assume the two-level system is initially in a maximally mixed state. We then evolve the entire system-reservoir equations of motion, using parameters relevant to energy transfer in photosynthesis [41], and plot both measures in Fig. 2.

For both measures, we see similar behavior, particularly as a function of reservoir cutoff frequency and reservoir temperature. However, as a function of system-reservoir coupling, the entanglement measure has a larger window of detection. This may be attributed to the hierarchical relationship between EPR steering and entanglement. For example, Ref. [2] has shown that EPR steerable states are a superset of Bell nonlocal states and a subset of entangled states. This hierarchy links together these three different notions of quantum correlations. Therefore, the fact that the concurrence-based measure of non-Markovianity is more sensitive to the non-Markovianity than the temporal SW measure seems linked, intuitively, to the notion that steering, in its EPR form, is a subset of entangled states. Also note that the sharp features in both measures are typical and arise because of the sudden vanishing and reappearance of both quantities in the temporal domain. Note that here, for consistency with Ref. [32], we plot $\mathcal{N}_{\mathrm{TSW}}$ and $\mathcal{N}_C$ using

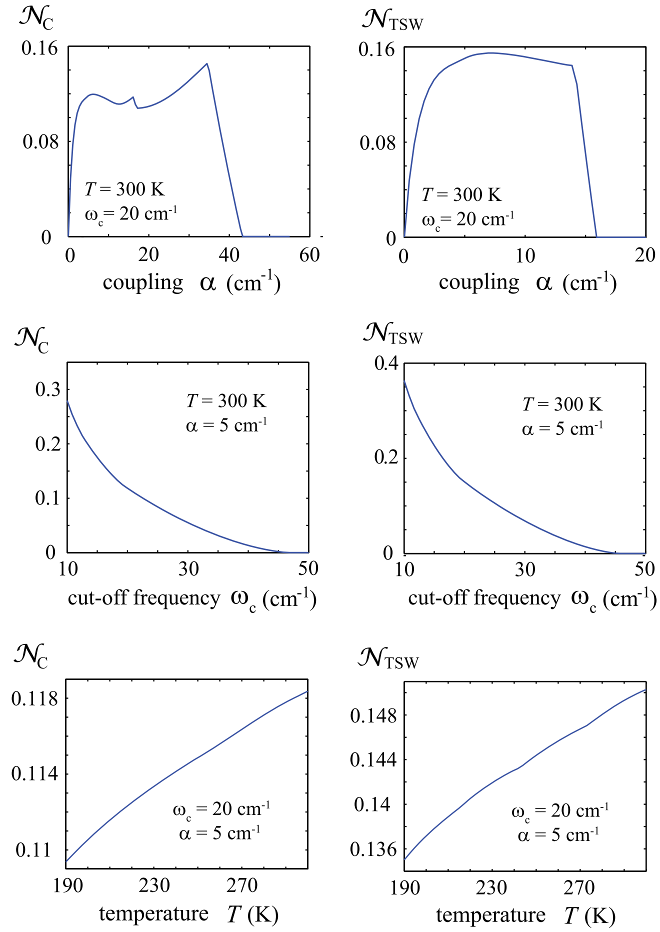

FIG. 2. The non-Markovianity measures $\mathcal{N}_{\mathrm{TSW}}$ (based on the temporal steerable weight) and $\mathcal{N}_C$ (based on the entanglement with an ancilla), as a function of system-reservoir coupling α, reservoir cutoff frequency ω_c, and temperature T, for a two-level system coupled to a bosonic reservoir with Drude-Lorentz spectral density. The system parameters are chosen to maximize the affect of the reservoir memory, with $E = 0$ and $\Delta = 100$ cm^{-1}. The magnitudes of these parameters are typical for excitation energy transfer in photosynthesis [41], where the memory effect and structure of the environment is thought to play an important role.

$$\mathcal{N}_i = \int_{t_0}^{t_f} \left| \frac{df_i[\rho(t)]}{dt} \right| dt + f_i[\rho(t_f)] - f_i[\rho(t_0)], \qquad (14)$$

where for the temporal SW measure $i = \mathrm{TSW}$, the function $f_i[\rho(t)]$ is the temporal SW at time t, while for the concurrence measure $i = C$, the function $f_i[\rho(t)]$ is the concurrence between system and ancilla at time t. This definition for the integral differs from Eq. (7) by a trivial factor of $1/2$.

Conclusions.—To summarize, we have discussed the concepts of "temporal" steering and how this can be quantified in a similar way to that of the original spatial EPR steering. We further proved that the temporal steerable weight is a nonincreasing function under a CPT map and can be used as a measure of non-Markovianity, suggesting that both forms of steering can act as a quantum resource, similar to entanglement. Finally we note that, in parallel, the temporal steerable weight has been recently implemented experimentally [44].

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