Enhancement of mechanical effects of single photons in modulated two-mode optomechanics

Jie-Qiao Liao,1,2,* C. K. Law,3 Le-Man Kuang,4 and Franco Nori2,5
1School of Natural Sciences, University of California, Merced, California 95343, USA
2CEMS, RIKEN, Saitama 351-0198, Japan
3Department of Physics and Institute of Theoretical Physics, The Chinese University of Hong Kong, Shatin, Hong Kong Special Administrative Region, China
4Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education, and Department of Physics, Hunan Normal University, Changsha 410081, China
5Department of Physics, The University of Michigan, Ann Arbor, Michigan 48109-1040, USA

(Received 3 May 2015; published 13 July 2015)

We propose an approach to enhance the mechanical effects of single photons in a two-mode optomechanical system. This is achieved by introducing a resonance-frequency modulation to the cavity fields. When the modulation frequency and amplitude satisfy certain conditions, the mechanical displacement induced by single photons could be larger than the quantum zero-point fluctuation of the oscillating resonator. This method can be used to create distinct mechanical superposition states.

DOI: 10.1103/PhysRevA.92.013822

1. INTRODUCTION

One of the most interesting regimes in cavity quantum optomechanics [1,2] occurs when the radiation pressure of a single photon is strong enough to push a mechanical oscillator with a displacement greater than the oscillator’s zero-point fluctuations before escaping the cavity. In such a regime, one can generate distinct superposition states of a macroscopic object (the oscillator) [3,4], and the quantum nonlinearity in the ultrastrong optomechanical coupling regime can lead to various quantum effects, including photon blockade [5–8], non-Gaussian states of the mechanical oscillator [9–13], modified spectra of photon scattering [6,14], and dressed-state-representation dissipations [15]. Specifically, this regime is defined by the condition

$$g_0 > \omega_M \gg \kappa_c,$$

where $g_0$ is the single-photon optomechanical coupling strength, $\omega_M$ is the frequency of the mechanical oscillator, and $\kappa_c$ is the cavity field damping rate.

The requirement of $g_0 > \omega_M$ in (1) is understood from a generic single-mode time-independent optomechanical system with the Hamiltonian $\mathcal{H} = \omega_c a d a + \omega_M b \hat{b} - g_0 a^\dagger a (b + b^\dagger)$, where $a (b)$ and $a^\dagger (b^\dagger)$ are the annihilation and creation operators of the cavity field (mechanical oscillator). By solving the Schrödinger equation, one finds that the oscillator, when driven by a single photon, can evolve to a coherent state $|2g_0/\omega_M\rangle$ from the ground state after a time $t = \pi/\omega_M$. Therefore, $g_0 > \omega_M$ provides a significant displacement away from the ground state.

However, the realization of condition (1) has been challenging for current experiments because $g_0$ is small in most optomechanical systems, and $\omega_M$ has to be sufficiently high to overcome thermal noise effects. In particular, the quantum limit of phonon number $n_b$ obtained by resolved sideband cooling is $n_b = \kappa_c^2/(16\omega^2_M)$ [16–20]. It should be noted that condition (1) is based on the single-mode time-independent Hamiltonian $\mathcal{H}$, and thus an interesting question is whether it is possible to achieve ultrastrong coupling effects in multimode and time-dependent systems. Indeed, a similar question has been explored in electromechanical systems recently [21,22], and it is found that a suitable modulation can be used to effectively enhance photonic nonlinearities in electromechanical systems. In addition, an interferometric scheme has been proposed to detect optomechanical coherent interaction at the single-photon level [23].

The methods of enhancing radiation pressure effects of a single photon in optomechanical systems have been recently discussed by several authors [24–28]. In this paper, we approach this problem by modulating the photon-tunneling process in the “membrane-in-the-middle” (MIM) configuration. The MIM optomechanical system has been studied experimentally [29–37] and theoretically [38–44]. Here we show that by changing the frequency of the cavities in a time-dependent manner, the photon tunneling oscillation can be modulated. We discover a set of conditions such that a single photon can significantly displace the membrane, even though $g_0/\omega_M$ is much less than 1. Below we derive an effective Hamiltonian of the system and present numerical evidence of the ultrastrong coupling effect.

II. THE MODEL

We consider an MIM optomechanical system (Fig. 1) in which a vibrational mode of the membrane is coupled to the left and right cavity field modes via the radiation pressure difference between the two cavities, and tunneling of photons through the membrane is allowed. The Hamiltonian of the system [38–44] is modeled by ($\hbar = 1$)

$$H = \omega_L a_L^\dagger a_L + \omega_R a_R^\dagger a_R + J(a_L^\dagger a_R + a_R^\dagger a_L) + \omega_M b \hat{b} + g_0 (a_L^\dagger a_L - a_R^\dagger a_R)(b + b^\dagger),$$

where $a_L (R)$ and $b$ are the annihilation operators of the left (right) cavity and the mechanical membrane, with the respective resonance frequencies $\omega_L (R)$ and $\omega_M$. The parameters $J$ and $g_0$ are the coupling strengths of the photon hopping and the optomechanical interaction, respectively.
The evolution of $|\psi(t)\rangle$ is governed by the transformed Hamiltonian $\tilde{H}(t) = T^\dagger(t)HT(t) - iT^\dagger(t)dT(t)/dt$, which has the form

$$\tilde{H}(t) = \frac{\Delta_0}{4}(a_L^\dagger a_L - a_R^\dagger a_R)(e^{i4Jt} + e^{-i4Jt}) + \frac{\gamma_0}{2}(a_L^\dagger a_L - a_R^\dagger a_R)(be^{-i(\omega_M - 2J)t} + b^\dagger e^{i(\omega_M + 2J)t}) + \frac{\gamma_0}{2}(a_L^\dagger a_L - a_R^\dagger a_R)(be^{-i(\omega_M + 2J)t} + b^\dagger e^{i(\omega_M - 2J)t}) + \frac{\gamma_0}{2}(a_L^\dagger a_R b(e^{i(\omega_M + 2J)t} - e^{-i(\omega_M - 2J)t}) + H.c.).$$

The motivation for performing the above transformation is that the parameters $\omega_M$ and $J$, which are typically much greater than $g_0$ and $\Delta_0$, all appear in oscillating phase factors. The form of $\tilde{H}(t)$ allows us to identify fast oscillating terms and perform the rotating-wave approximation. Specifically, under the conditions $|\omega_M - 2J| \lesssim g_0$, $J \gg \frac{5}{16}\Delta_0$, $\Delta_0 \gg g_0$, we may retain only the slowly oscillating terms $\frac{\gamma_0}{2}(a_L^\dagger a_L - a_R^\dagger a_R)(be^{-i(\omega_M - 2J)t} + b^\dagger e^{i(\omega_M - 2J)t}$ in Eq. (5), and discard all other terms. This is justified within the rotating-wave approximation, because in order to discard those terms, their oscillation frequencies must be much larger than their corresponding coupling coefficients under conditions (6). Hence Eq. (5) is approximated by

$$\tilde{H}(t) \approx \frac{g_0}{2}(a_L^\dagger a_L - a_R^\dagger a_R)(be^{-i(\omega_M - 2J)t} + H.c.).$$

Such a Hamiltonian can be cast into a familiar form in a rotating frame defined by $|\tilde{\psi}(t)\rangle = S(t)|\psi(t)\rangle$, with the unitary operator $S(t) = \exp[i(\omega_M a_L^\dagger a_L + a_R^\dagger a_R) + (\omega_M - 2J)b^\dagger b]$. The transformed Hamiltonian $\tilde{H}' = S(t)\tilde{H}(t)S^{-1}(t)(S(t)\dot{S}(t))/dt$ then reads

$$\tilde{H}' \approx \omega_M(a_L^\dagger a_L + a_R^\dagger a_R) + (\omega_M - 2J)b^\dagger b + \frac{g_0}{2}(a_L^\dagger a_L - a_R^\dagger a_R)(b + b^\dagger).$$

The right side is precisely the same form of the Hamiltonian of the MIM optomechanical system without the tunneling term, but with an effective mechanical frequency $\omega_M - 2J$, which is significantly reduced. Consequently, the mechanical displacement induced by a single photon, which is now proportional to $g_0/[2(\omega_M - 2J)]$, can be significant.

As a remark, our model can be analyzed by working in the Schwinger representation. Define the angular momentum operators $S_x = (a_R^\dagger a_L + a_L^\dagger a_R)/2$, $S_y = i(a_R^\dagger a_L - a_L^\dagger a_R)/2$, and $S_z = (a_L^\dagger a_L - a_R^\dagger a_R)/2$, then the Hamiltonian (2), up to a constant-of-motion term $\omega_M(a_L^\dagger a_L + a_R^\dagger a_R)$, can be expressed
as

$$H = 2\Delta_0 \cos(2Jt)S_z + 2JS_x + \omega_M b^\dagger b + 2g_0 S_x(b + b^\dagger),$$

(9)

which is the quantum Rabi model for $S = 1/2$ (single-photon case) and $\Delta_0 = 0$. The $\Delta_0$ term corresponds to a driving term. It is worth noting that the quantum Rabi model in the ultrastrong-coupling regime, which is difficult to be realized in typical systems, can be simulated by reducing the effective qubit frequency [47].

III. SINGLE-PHOTON MECHANICAL DISPLACEMENT

Let us consider an initial state of the system in which the membrane is in the ground state and a photon is in the left cavity mode,

$$|\psi(0)\rangle = |\tilde{\psi}(0)\rangle = |1\rangle_L |0\rangle_R |0\rangle_M.$$  (10)

Then, by using the approximate Hamiltonian (8), we have

$$|\psi(t)\rangle \approx T(t)S(t)e^{-iHt}|\tilde{\psi}(0)\rangle \equiv |\psi_{\text{approx}}(t)\rangle$$  (11)

after transforming back to the original frame. Explicitly, the approximate state $|\psi_{\text{approx}}(t)\rangle$ is given by

$$|\psi_{\text{approx}}(t)\rangle = e^{-i\theta(t)}e^{-i\omega_M t}|\psi(Jt)\rangle_{[1]}|0\rangle_R - i \sin(Jt)|0\rangle_L |1\rangle_R \beta(t)|M\rangle.$$  (12)

where $|\beta(t)\rangle_M$ is a coherent state of the membrane, and $\theta(t)$ and $\beta(t)$ are given by

$$\theta(t) = \left(\frac{\Delta_0}{2} - \frac{g_0^2}{2(\omega_M - 2J)}\right) t + \frac{g_0^2}{4(\omega_M - 2J)^2} \left[\sin(\omega_M t) - \sin(2J t)\right].$$  (13a)

$$\beta(t) = \frac{g_0}{2(\omega_M - 2J)} \left( e^{-i\omega_M t} - e^{-2iJ t} \right).$$  (13b)

Hence the membrane can reach a maximum displacement $g_0/(\omega_M - 2J)$ at the time $t_m = \pi/(\omega_M - 2J)$, assuming $\omega_M > 2J$. A large displacement amplitude of the membrane can be obtained if $2J$ is close to $\omega_M$. In particular, at $2J = \omega_M$, the displacement amplitude in Eq. (13b) takes the form

$$\beta(t) = -i g_0 t \exp(-i\omega_M t/2),$$

which grows linearly in time until it becomes too large that Hamiltonian (2) breaks down. This is in contrast to unmodulated systems (i.e., $\Delta_0 = 0$) in which the membrane can gain less than one phonon for the same initial condition [48].

To check the validity of our approximation, we performed a numerical calculation of the state $|\psi(t)\rangle$ with the Hamiltonian $H$ in Eq. (2) subjected to the frequency modulation defined by Eq. (3) and the same initial condition as above. The numerically exact $|\psi(t)\rangle$ is then compared with the $|\psi_{\text{approx}}(t)\rangle$ of Eq. (12). Specifically, we examine the fidelity $F(t) = ||\psi(t)\rangle |\psi_{\text{approx}}(t)\rangle ||^2$.

FIG. 2. (Color online) (a) The fidelity $F(t)$ versus the scaled time $g_0 t$ when the parameter $\Delta_0/g_0$ takes various values. (b) Comparison of the mechanical displacements $|\beta(t)|$ in the numerical case (for $\Delta_0/g_0 = 20$, 40, and 60) with that in the analytical case. Other parameters are given by $\omega_M/g_0 = 201$ and $J/g_0 = 100.25$.

Then, by the Schrödinger equation, the probability amplitudes $A_m$ and $B_m$ are governed by the following set of coupled differential equations:

$$\dot{A}_m(t) = -i[\omega_c + m \omega_M + \Delta_0 \cos(2Jt)]A_m(t) - iJ B_m(t)$$  (15a)

$$- i g_0 [\sqrt{m + 1} A_{m+1}(t) + \sqrt{m} A_{m-1}(t)],$$

$$\dot{B}_m(t) = -i[\omega_c + m \omega_M - \Delta_0 \cos(2Jt)]B_m(t) - iJ A_m(t)$$  (15b)

$$+ i g_0 [\sqrt{m + 1} B_{m+1}(t) + \sqrt{m} B_{m-1}(t)],$$

which are solved numerically. The fidelity between the two states $|\psi(t)\rangle$ and $|\psi_{\text{approx}}(t)\rangle$ can be obtained as

$$F(t) = \left| \langle \psi(t) | \psi_{\text{approx}}(t) \rangle \right|^2$$

$$= \sum_{m=0}^{\infty} \frac{\beta_m(t)}{\sqrt{m!}} \left[ \cos(Jt) A_m^*(t) - i \sin(Jt) B_m^*(t) \right].$$  (16)

In Fig. 2(a), we plot the fidelity as a function of the evolution time for various values of $\Delta_0/g_0$. We see that a good fidelity [i.e., $F(t) \approx 1$] can be obtained for larger values of $\Delta_0/g_0$, which is in accordance with the condition in Eq. (6). The fidelity curves exhibit some fast oscillations, which are caused by the time factors $\exp(\pm i2Jt)$ and $\exp(\pm i\omega_M t)$ ($\omega_M, 2J \gg g_0$). We also plot the time dependence of the
mechanical displacement $|\langle b(t) \rangle|$ in Fig. 2(b). Here the solid curve corresponds to the analytical case $|\langle b(t) \rangle| = |\beta(t)\rangle$, which is independent of $\Delta_0$, as given in Eq. (13b). The other three curves in panel (b) are determined by the numerical solutions with the corresponding values of $\Delta_0/g_0$ given in panel (a). It can be seen that the exact numerical results match well the approximate analytical result, except a slight shift of the period.

Although a high ratio $\Delta_0/g_0$ would increase the fidelity as shown in Fig. 2, there is an upper limit of $\Delta_0$ according to Eq. (6). Specifically, for the approximate Hamiltonian (7) to be valid, we also need $\Delta_0 \ll 16J/5$. In other words, there is a finite range of $\Delta_0$ for our model to operate with. This is illustrated in Fig. 3 where the fidelity $F(t_1)$ at $t_1 = \pi/(\omega_M - 2J)$ is plotted as a function of $\Delta_0/g_0$. The choice of time $t_1$ is useful because it is the time for the first maximum displacement. In plotting all the curves in Fig. 3, we use the parameters with $\omega_M - 2J = g_0/2$, so that the maximum mechanical displacement is 2. From Fig. 3, we see that fidelity is almost 1 in a range of $\Delta_0/g_0$ when the conditions in Eq. (6) are satisfied. The range of $\Delta_0/g_0$ giving $F(t_1) \approx 1$ becomes wider as the ratio $J/g_0$ increases, as illustrated by the blue dash-dotted curve in Fig. 3.

![Graph](image)

**FIG. 3.** (Color online) The fidelity $F(t_1)$ at time $t_1$ versus the ratio $\Delta_0/g_0$ for various values of $J/g_0$, where $t_1 = \pi/(\omega_M - 2J)$ is the time for the first peak value of $|\beta(t)\rangle$. For different values of $J$, the values of $\omega_M$ are taken to keep $\omega_M - 2J = g_0/2$.

### IV. DISCUSSIONS

Our analysis so far has focused on the coherent evolution of the modulated system. To address the effect of cavity-field damping, we let $\kappa_c$ be the cavity-field damping rate for both the left and right cavities, then with $\omega_M - 2J \approx g_0/2$, $t \approx 1/g_0$ is the time scale for the membrane displacement being greater than the width of the mechanical ground state. Therefore the condition $g_0 \gg \kappa_c$ would ensure that the photon has a sufficient time to yield a significant mechanical displacement (with $|\beta| > 1$) before it leaks out of the cavity.

More specifically, based on the approximate Hamiltonian (8) and the initial state (10), it is not difficult to estimate that the phonon number $<b|b'>$ is of the order $g_0^2/(4(\omega_M - 2J)^2 + 4\kappa_c^2)$ at a time long compared with the cavity-field lifetime, but short compared with the lifetime of the mechanical oscillator, i.e., $1/\gamma_M \gg t \gg 1/\kappa_c$, where $\gamma_M$ is the damping rate of the membrane. Therefore, if $g_0 \gg \kappa_c$ and $\omega_M - 2J < g_0$, then the average phonon number of our modulated system can be larger than 1, after the photon leaks out of the cavity.

We point out that our approach can be used to create distinct mechanical superposition states [49–56]. The procedure consists of three steps: (i) Prepare the system in initial states $(1/\sqrt{2})(|1\rangle_L|0\rangle_R \pm |0\rangle_L|1\rangle_R)|0\rangle_M$. (ii) Then let the system evolve for a suitable time, for example, $t = 2\pi/(\omega_M - 2J)$. (iii) Measure the state of the photon [44]. As long as the photon is measured in either the left cavity or the right cavity, then the mechanical resonator will be projected to a mechanical superposition state.

We note that mechanical superposed states can be measured by using quantum state tomography [57,58]. By strongly driving another cavity mode (avoiding the cross talk with the single-photon state) at red sideband, the optomechanical coupling can be linearized into a beam-splitter interaction. Assume a large decay of the assistant cavity field such that it can be eliminated adiabatically, then the state of the mechanical mode can be mapped to the output light [59]. As a result the mechanical state can be known by reconstructing the state of the output light beam [60]. In addition, the technique of spectrometric reconstruction can also be used to obtain the state information of the mechanical mode by detecting the single-photon spectrum [57].

Moreover, we would like to mention that the approach in this paper can also be used to enhance the optical nonlinearities [21]. In optomechanics, the magnitude of the optical nonlinearity is $g_0^2/\omega_M$. It can be increased by effectively decreasing the frequency, from $\omega_M$ to $\omega_M - 2J$, under the same coupling strength $g_0$.

Finally, we discuss an example that illustrates the conditions required for the experimental demonstration of the mechanism presented in this paper. The example consists of a vibrational mode of a silicon nitride membrane with an eigenfrequency $\omega_M/2\pi = 1$ MHz [36,37], and the mechanical damping rate is less than a few hertz and so it may be ignored. The membrane is placed in the middle of a Fabry-Perot cavity of length $L$. A pair of nearly degenerate normal modes of the cavity is chosen such that the frequency splitting $2J \approx \omega_M$ at the avoided crossing. To obtain mode splitting in the megahertz range, the transverse modes of the cavity may be exploited [30,33,36]. Consider $L = 1$ mm, then $g_0$ is about 1 kHz, which is much smaller than $\omega_M$. However, by applying a weak modulation with $\Delta_0 = 40g_0 \approx 40$ kHz, the effect of radiation pressure from a single photon can be significantly enhanced according to our theory, provided that the cavity-field damping rate $\kappa_c$ is smaller than $g_0$. Since $\kappa_c$ is of the order of few megahertz in experiments [37], this is still a main challenge to achieve $g_0 > \kappa_c$. Note that $g_0 > \kappa_c$ is also the condition for the MIM optomechanical system to exhibit enhanced photon-photon interactions [42]. Therefore the progress of designs and fabrications of ultrahigh $Q$ cavities will be crucial to access ultrastrong-coupling quantum effects in optomechanics.
V. CONCLUSION

In conclusion, we have proposed a method to enhance the mechanical effects of single photons in a two-mode optomechanical system by introducing a frequency modulation to the cavity fields. The quantum dynamics, which is driven by photon hopping between the two modulated cavities and optomechanical interactions, can be captured approximately by the Hamiltonian (8) under the condition (6). Such a Hamiltonian indicates that a single photon can displace the membrane with $\beta > 1$ even though $g_0/\omega_M \ll 1$. This method can be used to create distinct superposition states of the mechanical resonator and enhance the optical nonlinearities. Finally we remark that these types of systems can be studied via quantum emulations or quantum simulations [61,62], as described in [63,64]. These describe circuit analogs of optomechanical systems, allowing an additional way to study the effects predicted here.

ACKNOWLEDGMENTS

J.Q.L. would like to thank Hunan Normal University for its hospitality where part of this work was carried out. J.Q.L. is supported by the DARPA ORCHID program through AFOSR, the National Science Foundation under Award No. NSF-DMR-0956064, and the NSF-COINS program under Grant No. NSF-EEC-0832819. C.K.L. is partially supported by a grant from the Research Grants Council of Hong Kong Special Administrative Region of China (Project No. CUHK401812). L.M.K. is partially supported by the National 973 Program under Grant No. 2013CB921804 and the NSF under Grants No. 11375060 and No. 11434011. F.N. is partially supported by the RIKEN iTHERS Project, the MURI Center for Dynamic Magneto-Optics via the AFOSR Award No. FA9550-14-1-0040, the Impact program of JST, and a Grant-in-Aid for Scientific Research (A).


[48] When $\Delta_0 = 0$, the Hamiltonian (9) in the single-photon subspace can be identified as a quantum Rabi model. Under the condition $|\omega_m - 2J| < \frac{\eta}{2}$ and $J \gg \xi_0$, the model can further be reduced to the Jaynes-Cummings model by using the rotating-wave approximation. In this picture, the initial state (10) corresponds to a vacuum field state and the two-level atom is in an equal superposition of excited and ground states. Therefore the atom can only deliver at most half a quanta to the field (membrane).


