Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physica C 388–389 (2003) 661–662
PHYSICA C
www.elsevier.com/locate/physc

Controlling the collective motion of interacting particles: analytical study via the nonlinear Fokker–Planck equation

S. Savel'ev [a,*], F. Marchesoni [a,b], Franco Nori [a,c]

[a] Frontier Research System, The Institute of Physical and Chemical Research (RIKEN), Wako-shi, Saitama 351-0198, Japan
[b] INFM and Physics Department, Universita' di Camerino, I-62032 Camerino, Italy
[c] Center for Theoretical Physics, Department of Physics, Michigan, Ann Arbor, MI 48109-1120, USA

Abstract

We propose a nonlinear Fokker–Planck equation for the description of stochastic transport in systems of short-range interacting particles. We develop a perturbation scheme, valid for high frequencies, for particles on an asymmetric potential driven by a time-oscillating temperature. For a particular type of asymmetric potential, the net DC current shows two current inversions when increasing either the particle density or the interaction strength.

Keywords: Collective transport of short-range interacting particles; Particles driven by a time-dependent temperature

Stochastic transport on spatially asymmetric periodic (''ratchet'') potentials has been intensively studied in systems far from equilibrium, and mostly in the context of molecular motors (see e.g., the reviews [1]). In such Brownian motors, a net motion of particles may occur even in the absence of any DC driving force, due to the rectification of nonequilibrium thermal fluctuations.

Analytical studies of ratchets are usually performed using the *linear* Fokker–Planck equation, which is valid for an assembly of noninteracting particles. Thus, it is important to study the physically relevant case of how particle–particle interactions influence the stochastic transport in Brownian motors. In this paper, we derive and analyze the nonlinear equation describing stochastically moving particles with short-range interaction. Here we consider the so-called temperature ratchet [2], where the time oscillations of the temperature drive the motion of particles.

Our starting point is the overdamped equation of motion for pairwise-interacting particles in an asymmetric periodical potential $\mathscr{U}$,

$$\dot{x}_i = -\frac{\partial \mathscr{U}(x_i)}{\partial x_i} - \sum_{j\neq i} \frac{\partial}{\partial x_i}\mathscr{W}(x_i - x_j) + \sqrt{2k_{\mathrm{B}}T}\xi^{(i)}(t), \quad (1)$$

with temperature T, Boltzmann constant k_{B}, and pair potential $\mathscr{W}$. The Gaussian white noise $\xi^{(i)}$ satisfies the relation $\langle \xi_\alpha^{(i)}(t)\xi_\beta^{(j)}(t+\tau)\rangle = \delta(\tau)\delta_{\alpha\beta}\delta_{i,j}$. Applying the Bogolyubov method, an infinite set of many-particle distribution functions can be constructed. Such a hierarchy can be truncated in the ''mean-field'' approximation by replacing the binary distribution function with the product of two one-particle distribution functions $F_1(t,x)$; hence we obtain the nonlinear integro-differential equation:

$$\frac{\partial F_1(t,x)}{\partial t} = \frac{\partial}{\partial x}\left[F_1(t,x)\frac{\partial \mathscr{U}^{\mathrm{eff}}}{\partial x} + k_{\mathrm{B}}T\frac{\partial F_1(t,x)}{\partial x}\right] \quad (2)$$

with the mean-field potential

* Corresponding author. Address: Digital Materials Laboratory, FRS, RIKEN, 2-1 Hirosawa, Wako-shi, Saitama 351-0198, Japan. Tel.: +81-48-462-1111; fax: +81-48-467-9650.
E-mail address: ssavelev@postman.riken.go.jp (S. Savel'ev).

0921-4534/03/$ - see front matter © 2003 Elsevier Science B.V. All rights reserved.
doi:10.1016/S0921-4534(02)02434-6

$$\mathcal{U}^{\text{eff}} = \mathcal{U}(x) + \int \mathrm{d}x' \mathcal{W}(x - x') F_1(t, x'), \quad (3)$$

which is periodic and has the same spatial period l as the substrate potential $\mathcal{U}$. The distribution function $F_1(t,x)$ can be normalized, for instance, by the total number of particles in the system. In this case the spatial average of the function $F_1(t,x)$ coincides with the density of particles $\bar{n}$. If the interaction range of the particles is the smallest distance in the problem, the interaction potential can be taken in the local limit $\mathcal{W}(x) = g\delta(x)$. This simplifies considerably the nonlinear Fokker–Planck equation under study:

$$\frac{\partial F_1}{\partial t} = \frac{\partial}{\partial x}\left[F_1 \frac{\partial \mathcal{U}}{\partial x} + k_B T(t) \frac{\partial F_1}{\partial x} + g F_1 \frac{\partial F_1}{\partial x}\right]. \quad (4)$$

Now the temperature T is chosen to be a periodic time-dependent function [2], say $T(t) = T(1 + a\cos(\omega t))$, $a < 1$. In the high frequency limit, $[\max_x(\mathcal{U}) - \min_x(\mathcal{U})]/(\omega l^2) \ll 1$, $k_B T/(\omega l^2) \ll 1$, and $g\bar{n}/(\omega l^2) \ll 1$, the solution of Eq. (4) can be expanded with respect to the reciprocal of the frequency $1/\omega$:

$$F_1(\tau, x) = \sum_{i=0}^{\infty} \frac{1}{\omega^i} \phi_i(\tau, x), \quad (5)$$

where $\tau = \omega t$ is a dimensionless time. The following periodic and normalizing conditions can be taken: $\phi_i(\tau + 2\pi, x) = \phi_i(\tau, x) = \phi_i(\tau, x + l)$, $\int_0^l \mathrm{d}x\, \phi_{i\neq 0}(x) = 0$, and $\int_0^l \mathrm{d}x\, \phi_0(x)/l = \bar{n}$. Omitting the detailed description of the iterative procedure, here we concentrate on the physical results. Instead of the usual Boltzmann distribution, the equilibrium distribution function $\phi_0(x)$ of the very short-range interacting particles is described by the transcendental equation:

$$\phi_0(x) = C(\bar{n}) \exp\left(-\frac{\mathcal{U}(x) + g\phi_0(x)}{k_B T}\right) \quad (6)$$

with a constant $C(\bar{n})$ defined by the normalization condition. By solving Eq. (4) up to the third approximation with respect to $1/\omega$, we obtain the equation for the DC net current related to the nonequilibrium state induced by the time oscillating temperature (Fig. 1):

$$J(\bar{n}, \omega, T) = \frac{k_B^2 T^2 a^2}{2\omega^2 \int_0^l \frac{\mathrm{d}x}{\phi_0}} \int_0^l \mathrm{d}x \left\{ \frac{(\mathcal{U}'')^2 \mathcal{U}'(4k_B T + 5g\phi_0)}{(k_B T + g\phi_0)^3} - \frac{2k_B T(\mathcal{U}')^3 \mathcal{U}''(4k_B T + 5g\phi_0)}{(k_B T + g\phi_0)^5} - \frac{g\phi_0(\mathcal{U}')^5(6k_B^2 T^2 + 10g k_B T\phi_0 + 3g^2\phi_0^2)}{(k_B T + g\phi_0)^7} \right\}, \quad (7)$$

with $' \equiv \partial/\partial x$. This expression can be simplified in the limit of zero interaction, giving the known result [1]:

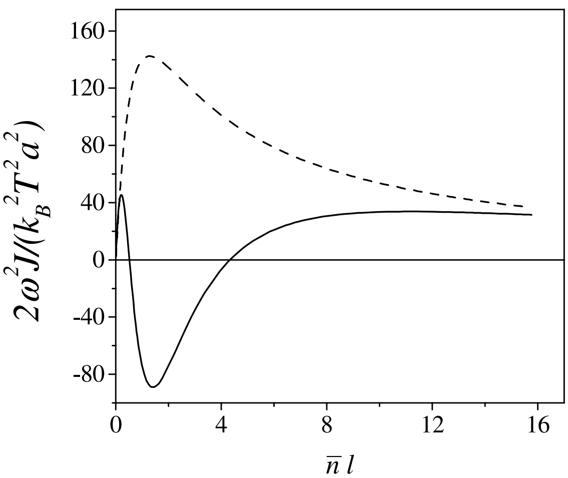

Fig. 1. The DC probability current versus particle density for the potential $\mathcal{U}/k_B T = a_0 \sin(2\pi x/l) + a_1 \sin(4\pi x/l - \beta_1) + a_2 \sin(6\pi x/l - \beta_2)$. Two sets of parameters are shown here: $a_0 = 1, a_1 = 0.2, a_2 = -0.06, \beta_1 = 0.45, \beta_2 = 0.45$ for the solid curve with *two current inversions*; and $a_0 = 1, a_1 = 0.01, a_2 = 0, \beta_1 = 0.45$ for the dashed curve with *no current inversions*. These two examples illustrate two types of allowed $J(n)$'s for interacting particles driven by a time-oscillating temperature $T(t)$ and moving on a spatially asymmetric periodic potential.

$$J = \frac{2a^2 l\bar{n} \int_0^l \mathrm{d}x\, \mathcal{U}'(\mathcal{U}'')^2}{\omega^2 \int_0^l \mathrm{d}x \exp\left(\frac{\mathcal{U}}{k_B T}\right) \int_0^l \mathrm{d}x \exp\left(-\frac{\mathcal{U}}{k_B T}\right)} \quad (8)$$

and can also provide the current in the strong interaction limit, $\max(k_B T; \max_x \mathcal{U} - \min_x \mathcal{U}) \ll g\bar{n} \ll \omega l^2$,

$$J = \frac{5k_B^2 T^2 a^2}{2\omega^2 g^2 l^2 \bar{n}} \int_0^l \mathrm{d}x\, \mathcal{U}'(\mathcal{U}'')^2. \quad (9)$$

It is clear from the last two equations, that the sign of the current is the same in the case of weak and strong interactions for *any* asymmetric potential. This means that 'either there is no current inversion or that the current inverts an even number of times when increasing either the strength g of interaction or the particle density $\bar{n}$'. Numerical calculations of the integral in expression (7) show examples of these two scenarios (Fig. 1).

In conclusion, we have derived the nonlinear Fokker–Planck equation for systems of locally interacting particles and obtained the expression for the net DC current for temperature ratchet in high frequency limit.

References

[1] P. Reimann, Phys. Rep. 361 (2002) 57; R.D. Astumian, Science 276 (1997) 917.
[2] P. Reimann, R. Bartussek, R. Haussler, P. Hanggi, Phys. Lett. A 215 (1996) 26.