Entanglement dynamics of two qubits in a common bath

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Abstract We derive a set of hierarchical equations for qubits interacting with a Lorentz-broadened cavity mode at zero temperature, without using the rotating-wave, Born, and Markovian approximations. We use this exact method to reexamine the entanglement dynamics of two qubits interacting with a common bath, which was previously solved only under the rotating-wave and single-excitation approximations. With the exact hierarchy equation method used here, double excitations due to counter-rotating-wave terms are found to have remarkable effects on the dynamics and the steady-state entanglement.

I. INTRODUCTION

Decoherence is one of the most important problems in quantum information processing [1]. The description of this difficult problem usually involves various approximations. During the dynamic evolution, the system and the bath are mixed, and a perturbative treatment is required such that we can trace out the degrees of freedom of the bath. This perturbation is known as the Born approximation. Moreover, if the time scale of the bath is much shorter than that of the system, the Markovian approximation is often applied.

An effective method that avoids the above two approximations was developed by Tanimura et al. [2–4], who established a set of hierarchical equations [4] that includes all orders of system-bath interactions. The derivation of the hierarchy equations requires that the time-correlation function of the bath can be decomposed into a set of exponential functions [4]. At finite temperature, this requirement is fulfilled if the system-bath coupling can be described by a Drude spectrum. The hierarchy equation method is successfully used in describing quantum dynamics of chemical and biophysical systems [3–6], such as the light-harvesting complexes [6], of which the temperature of the environment is high enough, and the coupling between the system and the environment is too strong to enable a Born approximation. However, the powerful hierarchy equation method was seldom used in studying decoherence effects in quantum information [7]. First, the operating temperature of qubit devices is very low. If we use the Drude spectrum, a numerical difficulty arises since the time-correlation function of the bath should be decomposed into a very large set of exponential functions [3]. Actually, the temperature of qubit devices is low enough that we can use a zero-temperature environment to model the decoherence. Second, the Drude spectrum is not quite general in qubit devices, especially when the qubit is placed in a cavity, and its environment is usually modeled by a Lorentz-broadened cavity mode.

In this paper we find that the hierarchy equation can also be derived at zero-temperature if we employ a Lorentz-type system-bath coupling spectrum. The set of hierarchy equations derived here provides an exact treatment of decoherence and employs neither the rotating-wave, Born, nor Markovian approximations. System-bath correlations are here fully accounted during the entire time evolution, as compared to traditional master equation treatments, the correlations are truncated to second order. High-order correlations are shown [8] to be very important, even producing a totally different physics. Moreover, the hierarchy equation we derive here is found to be effective in the single-mode case and is a promising method for studying strong- and ultrastrong-coupling physics [7,9].

II. HIERARCHY EQUATION METHOD

Here we first consider qubits interacting with a common bosonic bath, which is widely considered in studying decoherence-free subspace [10] and bipartite entanglement dynamics [11]. This model was solved exactly [1,12] under the rotating-wave approximation (RWA). It is not surprising that entanglement can be generated for a separable initial state, since the bath induces an effective qubit-qubit interaction. Another observation based on the RWA lies in the steady-state entanglement, which is determined only by the overlap between the initial state and the decoherence-free state, independent of the system-bath coupling [12]. This is because the dynamics of the qubit is restricted to a single-excitation subspace. However, when the counter-rotating terms are accounted, double excitation occurs and the steady-state entanglement vanishes for certain system-bath couplings. We will demonstrate this observation below.
The exact dynamics of the system in the interaction picture can be derived as [4]

$$\rho_S^{(I)}(t) = T \exp \left\{ - \int_0^t dt_2 \int_0^{t_2} dt_1 V(t_2) \{ [C_R(t_2 - t_1) V(t_1)]^{\dagger} C^R(t_2 - t_1) V(t_1) \} \right\} \rho_S(0),$$

(3)

if the qubit and bath are initially in a separable state, i.e., $\rho(0) = \rho_S(0) \otimes \rho_B$, where $\rho_B = \exp(\beta H_B)/Z_B$ is the initial state of the bath, with $\beta = 1/T$ (with $k_B = 1$) and $Z_B$ is the partition function. In Eq. (3), $T$ is the chronological time-ordering operator, which orders the operators inside the integral such that the time arguments increase from right to left. Two superoperators are introduced: $A^s B \equiv \{ A, B \} = AB - BA$ and $A^b B \equiv \{ A, B \} = AB + BA$. Also, $C_R(t_2 - t_1)$ and $C^R(t_2 - t_1)$ are the real and imaginary parts of the bath time-correlation function

$$C(t_2 - t_1) = \exp \{ -(\gamma + i \omega_0) [t_2 - t_1] \},$$

(7)

which is an exponential form that we need to use for the hierarchy equations. In the single-mode limit, $\gamma = 0$ and $C(t_2 - t_1) = \exp(-i \omega_0 |t_2 - t_1|)$, and we see that $\lambda$ is related to the square of the Rabi oscillation frequency.

To derive the hierarchy equation in a convenient form, we further write the real and imaginary parts of the time-correlation function (7) as

$$C^R(t) = \sum_{k=1}^{2} e^{-\nu_k t}, \quad C^I(t) = \sum_{k=1}^{2} (-1)^k \frac{\lambda}{2t} e^{-\nu_k t},$$

(8)

where $\nu_k = \gamma + (-1)^k i \omega_0$. Then, following procedures shown in Appendix and Ref. [2,4], the hierarchy equations of the qubits are obtained:

$$\frac{\partial}{\partial t} \varrho_{\vec{n}}(t) = -(i H^x_S + \vec{n} \cdot \vec{V}) \varrho_{\vec{n}}(t) - i \sum_{k=1}^{2} V^x \varrho_{\vec{n}+\vec{e}_k}(t)$$

$$- i \frac{\lambda}{2} \sum_{k=1}^{2} n_k [V^x + (-1)^k V^z] \varrho_{\vec{n}+\vec{e}_k}(t),$$

(9)

where the subscript $\vec{n} = (n_1, n_2)$ is a two-dimensional index, with $n_{1,2} \geq 0$, and $\varrho_{\vec{n}}(t) \equiv \varrho_{(\vec{n},0)}(t)$. The vectors $\vec{e}_1 = (1, 0)$, $\vec{e}_2 = (0, 1)$, and $\vec{V} = (v_1, v_2) = (\gamma - i \omega_0, \gamma + i \omega_0)$. We emphasize that $\varrho_{\vec{n}}(t)$ with $\vec{n} \neq 0$ are auxiliary operators introduced only for the sake of computing, they are not density matrices, and are all set to be zero at $t = 0$. The hierarchy equations are a set of linear differential equations and can be solved by using the Runge-Kutta method. The contributions of the bath to the dynamics of the system, including both dissipation and Lamb shift, are fully contained in the hierarchy equation (9). The Lamb shift term [13], which is related to the imaginary part of the bath correlation function, can be written explicitly in the common non-Markovian equations. Since the real and imaginary parts of the bath correlation function are taken into considered here, the effects of the Lamb shift exist in the hierarchy equations, although not in an explicit form.

For numerical computations, the hierarchy equation (9) must be truncated for large enough $\vec{n}$. We can increase the hierarchy order $\vec{n}$ until the results of $\varrho_{\vec{n}}(t)$ converge. The terminator of the hierarchy equation is

$$\frac{\partial}{\partial t} \varrho_{\vec{n}}(t) = -(i H^x_S + \vec{N} \cdot \vec{V}) \varrho_{\vec{n}}(t)$$

$$- i \frac{\lambda}{2} \sum_{k=1}^{2} n_k [V^x + (-1)^k V^z] \varrho_{\vec{n}+\vec{e}_k}(t),$$

(10)

where we dropped the deeper auxiliary operators $\varrho_{\vec{n}+\vec{e}_k}$. The numerical results in this paper were all tested and converged, and the density matrix $\rho_S(t)$ is positive.
III. ENTANGLEMENT OF TWO QUBITS IN A COMMON BATH

Below we apply the hierarchy equation (9) to a widely studied model: two qubits interacting with a common bath. The model is used to study decoherence-free space [10], bath-induced entanglement [12], and other related topics [11]. In previous works [12,14,15], the RWA was used, and the exact dynamics could be found only in a single-excitation subspace. Without using the RWA, the model was also studied [16–18] in a perturbative way. However, if the system-bath coupling becomes strong enough, which is explored in recent experiments [9], both the RWA and perturbation methods fail. Therefore the hierarchy method is very suitable in such conditions.

Consider two qubits interacting with a common bosonic bath. The system Hamiltonian in Eq. (1) is now given by

\[ H_S = \frac{\omega_1}{2} \sigma_1^z + \frac{\omega_2}{2} \sigma_2^z, \tag{11} \]

and below we consider \( \omega_1 = \omega_2 = \omega_0 \), i.e., the resonant case. The system operator in Eq. (2) is set by \( V = \alpha_1 \sigma_1^x + \alpha_2 \sigma_2^x \), and for simplicity, we consider \( \alpha_1 = \alpha_2 = 1 \). This model is exactly solvable [12] when the RWA is applied and the initial state is of the form

\[ |\psi(0)\rangle = |c_1(0)\rangle |0_1\rangle |0_2\rangle + |c_2(0)\rangle |0_1\rangle |1_2\rangle \bigotimes_k |0_k\rangle. \tag{12} \]

The time evolution is then given by

\[ |\psi(t)\rangle = |c_1(t)\rangle |1_1\rangle |0_2\rangle + |c_2(t)\rangle |0_1\rangle |1_2\rangle \bigotimes_k |0_k\rangle \]

\[ + \sum_k c_k(t) |0_1\rangle |0_2\rangle |1_k\rangle, \tag{13} \]

where \( |1_k\rangle \) denotes that only the \( k \)-th mode of the bath is excited. The explicit forms of \( c_1(t) \) and \( c_2(t) \) are given in Ref. [12]. The time evolution of the density matrix is

\[ \rho(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |c_1(t)|^2 & c_1(t)\sigma_1^z(t) & 0 \\ 0 & c_2(t)\sigma_1^z(t) & |c_2(t)|^2 & 0 \\ 0 & 0 & 0 & 1 - |c_1(t)|^2 - |c_2(t)|^2 \end{pmatrix}. \tag{14} \]

which is obviously restricted to a single-excitation space, and thus the concurrence of the above density matrix is

\[ C(t) = 2|c_1(t)| |c_2(t)|. \tag{15} \]

![FIG. 1. (Color online) Concurrence versus time for the initial state \( |\psi(0)\rangle = |0_1\rangle |0_2\rangle \) with different values of \( \gamma \). Here \( \lambda = 0.1 \omega_0 \) is in the strong-coupling regime. In the single-mode limit, \( \gamma = 0 \), the result of the hierarchy equation (solid) coincides with direct numerical calculations (circles). The entanglement suddenly vanishes, and revivals are observed. When increasing \( \gamma \), the oscillations and the maximum entanglement are suppressed. Under the RWA, the initial state does not evolve, and the entanglement stays at zero. exact numerical results obtained by solving the single-mode Hamiltonian directly. Therefore, by using a unified method, we can study the dynamics of the system interacting with a bath from the single-mode to multimode regime.

Another interesting result here is about the steady-state entanglement. Under the RWA, the dynamics is in the single-excitation subspace, only two states are independent, \( |\psi_{\pm}\rangle = (|0_1\rangle |1_2\rangle \pm |1_1\rangle |0_2\rangle) / \sqrt{2} \). The state \( |\psi_-\rangle \) is decoherence-free; this means that if the initial state has a nonvanishing overlap with \( |\psi_-\rangle \), the steady state is entangled, and the concurrence becomes

\[ C(t \to \infty) = C(|\langle \psi_- | \psi(0) \rangle|^2) = |\langle \psi_- | \psi(0) \rangle|^2, \tag{16} \]

which is independent of the system-bath coupling strength \( \lambda \) and the bath-decay rate \( \gamma \). However, if \( \lambda \) is not very small, although \( |\psi_-\rangle \) is also decoherence-free, Eq. (16) should be reexamined by using a more rigorous treatment, since double excitations need to be accounted. Actually, the reliability of the RWA was discussed in many papers [16–24]. As shown in Refs. [19,24], counter-rotating-wave terms can induced a significant shift in the population of the steady state even in the bad-cavity case.

In Fig. 2 we show the results given by the hierarchy method. The initial state there is \( |\psi(0)\rangle = (2|0_1\rangle |1_2\rangle + |1_1\rangle |0_2\rangle) / \sqrt{3} \). According to Eq. (16), the concurrence of the steady state is 0.1. We can see in Fig. 2(a)–2(c) that increasing \( \gamma \) the concurrence of the steady state decreases. In Fig. 2(d), we show that for a given \( \lambda = 0.01 \omega_0 \), the steady-state entanglement vanishes when \( \gamma \) is larger than a critical value. This reflects the importance of the counter-rotating-wave terms, which break the single-excitation condition and give a totally different steady-state entanglement. Similar results are obtained in Ref. [24], where the increase of the cavity decay rate is found to decrease the maximum of induced entanglement, and the steady state that computed without RWA has no entanglement but finite discord. This simple example indicates that some
In summary, we derive a set of hierarchy equations at zero temperature with a Lorentz spectrum. This set of equations is very suitable for qubit-cavity systems, especially when the interaction is so strong that the RWA and perturbative methods break down. It even works well when the bath has only one single mode. Moreover, this equation is very flexible. For example, if the qubits interact with several cavity modes, each broadened into a Lorentz form, then the bath correlation functions can also be expanded as several exponential functions. Thus the form of the hierarchy equations remains. The hierarchy equations are applied to reexamine the dynamics of two qubits interacting with a common bath. Previous works usually employed the RWA, and the results were restricted to the single-excitation space. This is not the case in this paper, since we do not use the RWA, and the counter-rotating-wave terms will cause double excitations. We found that the steady-state entanglement depends on the system-bath coupling spectrum. For a given coupling strength $\lambda$, there will be no steady-state entanglement when $\gamma$ is larger than a critical value. The exact dynamics exhibits a totally different physics, compared to the RWA model, which motivates the reexamination of many previous approximate studies.

IV. CONCLUSION

Below we derive the hierarchy equations. First, inserting the correlation function (7) into Eq. (3), we find

$$
\rhoS(t) = U(t) \left\{ T \exp \left( -\int_0^t dt_1 \int_0^{t_1} dt_2 V(t_2)^\omega \sum_{k=1}^{2} \frac{\lambda}{2} e^{-\gamma_k (t_2-t_1)} [V(t_1)^\omega + (-1)^k V(t_1)^\nu] \right) \rhoS(0) \right\} U^\dagger(t) 
$$

where $U(t) = \exp\{-i (H_S + H_B) t\}$, and the two new superoperators

$$
\Phi(t) = -i V(t)^\omega, \\
\Theta_k(t) = -\frac{i}{2} [V(t)^\omega + (-1)^k V(t)^\nu]
$$

in order to make the following discussion clearer and simpler. Equation (A1) is a time-ordered integral equation, which is not easy to solve directly. The idea of the hierarchy equation method [2,4] is to transform such an integral equation to a group of ordinary differential equations. The derivation of the hierarchy equations is straightforward: taking the time derivative of Eq. (A1) repeatedly.

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APPENDIX: DERIVATION OF THE HIERARCHY EQUATIONS

Below we derive the hierarchy equations. First, inserting the correlation function (7) into Eq. (3), we find

$$
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$$

where $U(t) = \exp\{-i (H_S + H_B) t\}$, and the two new superoperators

$$
\Phi(t) = -i V(t)^\omega, \\
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in order to make the following discussion clearer and simpler. Equation (A1) is a time-ordered integral equation, which is not easy to solve directly. The idea of the hierarchy equation method [2,4] is to transform such an integral equation to a group of ordinary differential equations. The derivation of the hierarchy equations is straightforward: taking the time derivative of Eq. (A1) repeatedly.
We first take the time derivative of Eq. (A1) and obtain

$$\frac{\partial}{\partial t} \rho_S(t) = -i H_S^x \rho_S(t) + \Phi \sum_{k=1}^{2} F_k(t),$$  \hspace{1cm} (A3)

where

$$F_k(t) = U(t) T \left\{ \int_0^t dt e^{-\nu_k(t-\tau)} \Theta_1(\tau) \exp \left[ \int_0^\tau dt_2 \int_0^{t_2} dt_1 \Phi(t_2) \sum_{n=1}^{2} e^{\nu_n(t_2-t_1)} \Theta_n(t_1) \right] \right\} \rho_S(0) U(t)^\dagger.$$  \hspace{1cm} (A4)

Thus the solution of $\rho_S(t)$ is determined by (1) its own free evolution and (2) the dynamics of $F_k(t)$. The initial condition of $F_k(t)$ is

$$F_k(0) = 0,$$  \hspace{1cm} (A5)

which is a direct result of Eq. (A4). To solve for $F_k(t)$, we first introduce the following useful notations [4]:

$$\varrho_{(0,0)}(t) \equiv \rho_S(t), \quad \varrho_{(1,0)}(t) \equiv F_1(t), \quad \varrho_{(0,1)}(t) \equiv F_2(t).$$  \hspace{1cm} (A6)

Then Eq. (A3) can be rewritten as

$$\frac{\partial}{\partial t} \rho_S(t) = -i H_S^x \rho_S(t) + \Phi \sum_{k=1}^{2} \varrho_{(k,0)+\tilde{e}_k}(t),$$  \hspace{1cm} (A7)

where $\tilde{e}_1 = (1, 0)\text{ and } \tilde{e}_2 = (0, 1)$.

The differential equations of $\varrho_{(1,0)}(t)$ and $\varrho_{(0,1)}(t)$ are obtained as

$$\frac{\partial}{\partial t} \varrho_{(1,0)}(t) = -(i H_S^x + \nu_1) \varrho_{(1,0)}(t) + \Phi \sum_{k=1}^{2} \varrho_{(1,0)+\tilde{e}_k}(t) + \Theta_1 \varrho_{(0,0)}(t),$$  \hspace{1cm} (A8)

$$\frac{\partial}{\partial t} \varrho_{(0,1)}(t) = -(i H_S^x + \nu_2) \varrho_{(0,1)}(t) + \Phi \sum_{k=1}^{2} \varrho_{(0,1)+\tilde{e}_k}(t) + \Theta_2 \varrho_{(0,0)}(t),$$  \hspace{1cm} (A9)

where we find three new auxiliary matrices:

$$\varrho_{(2,0)}(t) = U(t) T \left[ \int_0^t dt e^{-\nu_1(t-\tau)} \Theta_1(\tau) \right] \left[ \int_0^t dt e^{-\nu_2(t-\tau)} \Theta_2(\tau) \right] \left\{ \int_0^{t_2} dt_1 \Phi(t_2) \sum_{k=1}^{2} e^{\nu_k(t_2-t_1)} \Theta_k(t_1) \right\} \rho_S(0) U(t)^\dagger, \hspace{1cm} \text{(A10)}$$

$$\varrho_{(1,1)}(t) = U(t) T \left[ \int_0^t dt e^{-\nu_1(t-\tau)} \Theta_1(\tau) \right] \left[ \int_0^t dt e^{-\nu_2(t-\tau)} \Theta_2(\tau) \right] \times \left\{ \int_0^{t_2} dt_2 \int_0^{t_2} dt_1 \Phi(t_2) \sum_{k=1}^{2} e^{\nu_k(t_2-t_1)} \Theta_k(t_1) \right\} \rho_S(0) U(t)^\dagger, \hspace{1cm} \text{(A11)}$$

By repeating the above procedures, we find

$$\frac{\partial}{\partial t} \varrho_{\mathbf{\tilde{n}}}(t) = -(i H_S^x + \tilde{n} \cdot \tilde{\nu}) \varrho_{\mathbf{\tilde{n}}}(t) + \Phi \sum_{k=1}^{2} \varrho_{\mathbf{\tilde{n}}+\tilde{e}_k}(t) + \sum_{k=1}^{2} n_k \Theta_k \varrho_{\mathbf{\tilde{n}}-\tilde{e}_k}(t),$$  \hspace{1cm} (A12)

where $\mathbf{\tilde{n}} = (n_1, n_2)$ is a two-dimensional index, with $n_{1(2)} \geq 0$. The two-dimensional vector $\tilde{\nu} = (v_1, v_2) = (\gamma - i \omega_0, \gamma + i \omega_0)$. The auxiliary matrix is

$$\varrho_{\mathbf{\tilde{n}}}(t) = U(t) T \left[ \int_0^t dt e^{-\nu_1(t-\tau)} \Theta_1(\tau) \right]^{n_1} \left[ \int_0^t dt e^{-\nu_2(t-\tau)} \Theta_2(\tau) \right]^{n_2} \times \left\{ \int_0^{t_2} dt_2 \int_0^{t_2} dt_1 \Phi(t_2) \sum_{k=1}^{2} e^{\nu_k(t_2-t_1)} \Theta_k(t_1) \right\} \rho_S(0) U(t)^\dagger.$$  \hspace{1cm} (A13)
Inserting Eq. (A2) into Eq. (A12), we obtain the explicit form of the hierarchy equation as

$$\frac{\partial}{\partial t} \rho_{\hat{n}}(t) = -i (H_S^x + \mathbf{n} \cdot \mathbf{v}) \rho_{\hat{n}}(t) - \sum_{k=1}^{2} V^x \rho_{\hat{n} + \mathbf{v}_k}(t) - i \sum_{k=1}^{2} \sum_{\ell} n_k (V^x + (-1)^{k} V^y) \rho_{\hat{n} - \mathbf{v}_k}(t).$$  (A14)

The initial conditions are

$$\rho_{\hat{n}}(0) = \begin{cases} \rho_2(0), & \text{for } n_1 = n_2 = 0, \\ 0, & \text{for } n_1 > 0, n_2 > 0. \end{cases}$$

Although the explicit form of $\rho_{\hat{n}}(t)$ is complicated, we need only to focus on its differential equations, which can be solved directly by using the traditional Runge-Kutta method.