

Scalable quantum computation via local control of only two qubits

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We apply quantum control techniques to a long spin chain by acting only on two qubits at one of its ends, thereby implementing universal quantum computation by a combination of quantum gates on these qubits and indirect SWAP operations across the chain. It is shown that the control sequences can be computed and implemented efficiently. We discuss the application of these ideas to physical systems such as superconducting qubits in which full control of long chains is challenging.

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The control of quantum systems at will has been an aspiration for physicists for a long time. The achievement of quantum control not only clears the path toward a thorough understanding of quantum mechanics but also allows the exploration of devices whose functions are based on exotic quantum-mechanical effects. Among other applications, the future success of quantum information processing depends largely on our ability to tame many-body quantum systems that are highly fragile. Although the progress of technology allows us to manipulate small quantum systems quite well, the control of larger ones still represents a considerable challenge. Here, we show that it is possible to efficiently control a large system *indirectly*. Therefore, the current control of a few qubits could already suffice to achieve scalable quantum computation.

One encouraging result is that almost any coupled quantum system is controllable in principle, even by the steering of only a single particle [1]. The question of controllability in this type of situation can be described by the theory of quantum control [2], which uses Lie algebraic arguments. This is interesting from the theoretical point of view; however, can it be practically useful from the quantum computing perspective? Problems we need to contemplate before attempting to build a large quantum computer using quantum control are as follows. First, the control criterion is generally not computable for large systems. Second, even if the question of controllability can be answered positively for specific systems [3], the precise sequence of actual controls (or “control pulses”) is generally not computable. And third, even if it can be computed, the theory of control tells us nothing about the overall *duration* of the control pulses needed to achieve a given task, and it might take far too long to be practically relevant.

The usual approach to circumvent these problems focuses on systems that are sufficiently small so that we do not already require a quantum computer to check their controllability and to design control pulses. In such a case, the theory of time-optimal control [4] can be used to achieve impressive improvements in terms of total time or type of pulses required in comparison with the standard gate model. More complicated desired operations on larger systems are then decomposed (“compiled”) into sequences of smaller ones. Yet

the feasibility of this approach is ultimately limited by the power of our classical computers and is therefore constrained to low-dimensional many-body systems only.

The first step to achieve control over large-scale systems by controlling only a few particles was made in the context of quantum state transfer in spin chains. In these simple models, the issues mentioned here were avoided by restriction of the analysis to the subspace with a single excitation. Most of the proposed schemes for quantum state transfer [5,6] are actually not based on the framework of control theory, but on smart tricks from various fields of physics, often using classical intuition about the dynamics. Also, the theory of optimal control was recently applied to state transfer [7,8].

A few of these proposals were then applied to more general tasks than state transfer, that is, the use of spin chains for entanglement purification [9], for cloning transformations [10], or even for fully fledged quantum computation with little control [6]. Thus, such schemes are no longer restricted to a small subspace of the full Hilbert space. These schemes for quantum computation use clever methods to design the control pulses analytically, that is, without relying on control theory, but at the price of limitations on their applicability. In particular, they assume specific coupling parameters and design of the system. Nonetheless, such methods are intriguing from a theoretical perspective (e.g., in relation to complexity questions) and give hope for the feasibility of quantum control of larger systems.

The goal of this paper is to efficiently compute control pulses for a large system, using the full Hilbert space, and to show that the duration of the pulses scales efficiently (i.e., polynomially) with the system size. There are various trade-offs to consider. For instance, if we allow more means for external control and/or higher ability in designing the system Hamiltonian, the problem will become more tractable theoretically (on paper), but more demanding and less relevant from an experimental perspective.

Here, we present a solution to achieve feasible control both theoretically and experimentally. We will use a Hamiltonian that can be efficiently diagonalized for large systems through the Jordan-Wigner transformation. The control pulses are

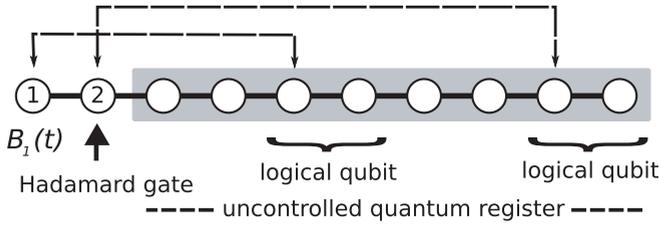


FIG. 1. Our approach for universal quantum computation on a chain of N spins. By modulating the magnetic field $B_1(t)$ on qubit 1, we induce information transfer and SWAP gates on the chain (dashed lines). The states of the qubits from the uncontrolled register can be brought to the controlled part. There, the gates from a quantum algorithm are performed by local operations. Afterward, the (modified) states are swapped back into their original positions.

applied only to the first two spins of a chain (see Fig. 1). The control consists of two parts: one where we will use the Jordan-Wigner transformation to efficiently compute and control the information transfer through the chain (thus using it as a quantum data bus) and a second part where we will use some local gates acting on the chain end to implement two-qubit operations. To be efficiently computable, these local gates need to be fast with respect to the natural dynamics of the chain. Combining the two actions allows us to implement any unitary operation described in the gate model.

System and information transfer problem. We consider a chain of N spin-1/2 particles coupled by the Hamiltonian

$$H = \frac{1}{2} \sum_{n=1}^{N-1} c_n [(1 + \gamma)XX + (1 - \gamma)YY]_{n,n+1} + \sum_{n=1}^N B_n Z_n,$$

where X, Y, Z are the Pauli matrices, the c_n are generic coupling constants, and the B_n represent a magnetic field. Variation of the parameter γ encompasses a wide range of experimentally relevant Hamiltonians [11–13], including the transverse Ising model ($\gamma = 1$; for this case we require the fields $B_n \neq 0$) and the XX model ($\gamma = 0$). We assume that the value of B_1 can be controlled externally. This control will be used to induce information transfer on the chain and realize SWAP gates¹ between arbitrary spins and the two “control” spins 1 and 2 at one chain end. Hence such SWAP gates are steered *indirectly* because we control only the first qubit.

In order to focus on the main idea we now present our method for $\gamma = 0$ and $B_n = 0$ for $n > 1$. The general case follows along the same lines, although it is more technically involved. Our first task is to show that by tuning only $B_1(t)$ we can perform SWAP gates between arbitrary pairs of qubits. First we rewrite the Hamiltonian, using the Jordan-Wigner transformation $a_n = \sigma_n^+ \prod_{m < n} Z_m$, as $H = \sum_{n=1}^{N-1} c_n \{a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n\}$. The operators a_n obey the canonical anticommutation relations $\{a_n, a_m^\dagger\} = \delta_{nm}$ and $\{a_n, a_m\} = 0$. The term we control by modulating $B_1(t)$ is

$h_1 = Z_1 = 1 - 2a_1^\dagger a_1$. From the theory of quantum control [2] we know that the reachable set of unitary time-evolution operators on the chain can be obtained by computing the *dynamical Lie algebra* generated by ih_1 and iH . It contains all possible commutators of these operators, of any order, and their real linear combinations. For example, it contains the anti-Hermitian operators $ih_{12} \equiv [ih_1, [ih_1, iH]]/(4c_1) = i(a_1^\dagger a_2 + a_2^\dagger a_1)$, $ih_{13} \equiv [iH, ih_{12}]/c_2 = a_1^\dagger a_3 - a_3^\dagger a_1$, and $ih_{23} \equiv [ih_{12}, ih_{13}] = i(a_2^\dagger a_3 + a_3^\dagger a_2)$. We observe that taking the commutator with h_{12} exchanges the index 1 of h_{13} with 2. Taking the commutator with iH we find that $ih_{14} \equiv [ih_{13}, iH] + ic_1 h_{23} - ic_2 h_{12} = i(a_1^\dagger a_4 + a_4^\dagger a_1)$ and $ih_{24} \equiv a_2^\dagger a_4 - a_4^\dagger a_2$ are also elements of the dynamical Lie algebra. Hence the effect of taking the commutator with H is to raise the index of the h_{kl} . Generalizing this, we find that the algebra contains the elements ih_{kl} , with $k < l$, $ih_{kl} \equiv a_k^\dagger a_l - a_l^\dagger a_k$ for $(k - l)$ even, $ih_{kl} \equiv i(a_k^\dagger a_l + a_l^\dagger a_k)$ for $(k - l)$ odd, and $h_k = Z_k = 1 - 2a_k^\dagger a_k$. From control theory [2] we thus know that the time-evolution operators $\exp(-\pi ih_{kl}/2)$ (which will turn out to be very similar to SWAP gates) can be achieved through tuning $B_1(t)$. The main point is that, because both h_1 and H are free-fermion Hamiltonians, the corresponding control functions can be computed *efficiently* in a $2N$ -dimensional space (which we will do explicitly later). Ultimately, we need to transform the operators back to the canonical spin representation. Using $a_k^\dagger a_l = \sigma_k^- \sigma_l^+ \prod_{k < j < l} Z_j$, we find $\exp(-\pi ih_{kl}/2) = (|00\rangle_{kl}\langle 00| + |11\rangle_{kl}\langle 11|) \otimes \mathbb{1} + (|01\rangle_{kl}\langle 10| - |10\rangle_{kl}\langle 01|) \otimes L_{kl}$ for $(k - l)$ even. The operator $L_{kl} = \prod_{k < j < l} Z_j$ arises from the nonlocal tail of the Jordan-Wigner transformation and acts only on the state of the spins *between* k and l , *controlled* by the state of the qubits k, j in the odd-parity sector.

In order to use the chain as a quantum data bus, our goal is to implement SWAP gates $S_{kl} = |00\rangle_{kl}\langle 00| + |11\rangle_{kl}\langle 11| + |10\rangle_{kl}\langle 01| + |01\rangle_{kl}\langle 10|$, so the fact that we have achieved some modified operators with different phases on k, l instead, and also the controlled nonlocal phases L_{kl} , could potentially be worrisome. We will use a method suggested in [14] that allows us to tackle these complications. That is, rather than using the physical qubits, we encode in *logical* qubits, consisting of two neighboring physical qubits each. They are encoded in the odd-parity subspace $|01\rangle, |10\rangle$. Although this encoding sacrifices half of the qubits, the Hilbert space remains large enough for quantum computation, and the encoding has the further advantage of avoiding macroscopic superpositions of magnetization, which would be very unstable. The swapping of a logical qubit n to the control end of the chain then consists of two physical SWAPs $\exp(-\pi ih_{1,2n-1}/2)$ and $\exp(-\pi ih_{2,2n}/2)$. Since both physical SWAPs give the same phases, the resulting operation is indeed a full logical SWAP. Any single-qubit operation on the logical qubits can be implemented by bringing the target qubit to the control end, performing the gate there, and bringing it back again. We could equally decide to perform single logical qubit gates directly, without bringing them to the control end. This is possible because $\exp(-ih_{2n-1,2n}t)$ in the physical picture translates to $\exp(-iX_{L,n}t)$ in the logical picture, and because Z_{2n-1} is in the algebra generated by Z_1 , which allows us to perform the operation $\exp(-iZ_{2n-1}t) = \exp(-iZ_{L,n}t)$.

¹We remark that a SWAP gate between two sites k and l is much more specific than achieving quantum state transfer [5] between them. A SWAP performs *bidirectional* quantum state transfer and does not change the state of the remaining sites.

For quantum computation, we need to be able to perform at least one entangling two-qubit operation. We choose a controlled-Z operation, which can be performed by operating on only one physical qubit from each of the two logical qubits involved; to perform a controlled-Z operation between logical qubits n and m , we bring the physical qubits $(2n - 1)$ and $(2m - 1)$ to the control end, perform a controlled-Z operation between them, and bring them back. It is easy to check that again all unwanted phases cancel out. The controlled-Z operation could not be efficiently computed in the interplay with the many-body Hamiltonian H , because it cannot be generated by a quadratic Hamiltonian in the Jordan-Wigner picture. Therefore, this gate must be implemented on a time scale t_g much faster than the natural evolution of the chain, that is, $t_g \ll \min_j\{1/c_j\}$. We can soften this requirement by using control theory to generate $\exp(-iZ_1 X_2 t)$ by modulating $\beta_1(t)Y_1$ (which is a linear term in the Jordan-Wigner picture), and then using a fast Hadamard gate on the second site to obtain $\exp(-iZ_1 Z_2 t)$, which, together with $\exp(-iZ_1 t)$ and $\exp(-iZ_2 t)$, gives the controlled-Z gate. This leads to a remarkable conclusion: except for a fast Hadamard gate on the second qubit, all other controls required for quantum computation can be computed efficiently in the Jordan-Wigner picture.

Efficiency. The crucial question left open then is the length of time needed actually to implement the gates. Unfortunately, the theory of quantum control does not provide a general answer, though some interesting progress was recently reported [15]. In order to evaluate the efficiency, we have numerically simulated a range of chain lengths and studied the scaling of the logical SWAP operation time T with the (physical) chain length N . We set the coupling strength constant, namely, $c_n = J \forall n$. To provide evidence of a polynomial scaling, we set the simulation time $T_N = N^2$ (with all times in units of $1/J$ and $\hbar = 1$) and verify for each N that we can find a specific $B_1^*(t)$ that performs the logical SWAP operation.²

We quantify our success by calculating the gate fidelity $F = (|\text{tr} U^\dagger U_g|/N)^2$ between the time evolution U and the goal unitary U_g . This standard choice of fidelity is used for evaluating generic unitaries, and it is well suited for our case, confirming that the SWAP gate $S_{kl} \otimes \mathbb{1}_{\text{rest}}$ acts as the identity almost everywhere. However, the normalization factor $1/N^2$ could in principle wash out errors in the part of the gate that acts on qubits k and l only, resulting in the wrong scaling. Therefore, we checked the reduced gate fidelity (tracing out the rest of the system) on those qubits alone, finding that its fidelity remains above $1 - 10^{-4}$ for all N considered.

The function $B_1(t)$ is obtained using techniques from optimal control theory [2,4,16]. Briefly, the procedure is as follows: (1) an initial guess is made for the function $B_1(t)$; (2) we run the optimal control algorithm to generate a new $B_1(t)$ which decreases the error of our operation; (3) steps 1 and 2 are iterated until the final error reaches a preselected threshold fidelity. In practice, it suffices to choose a threshold which is of the same order of magnitude as the error introduced by the Hadamard gate.

²Note that the time T_N is the time it takes to perform a physical SWAP operation; the logical SWAP will take twice this time.

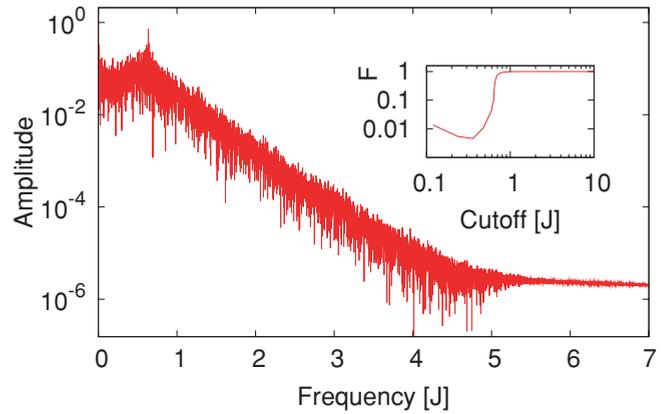


FIG. 2. (Color online) Fourier transform of the optimized function $B_1^*(t)$ that produces the physical SWAP between $n = 1$ and $n = 29$ for the chain $N = 30$ with a fidelity of $F = 1 - 10^{-4}$ in a total time $T_N = 870$. The inset shows the effect of a high-frequency cutoff on the fidelity; frequency components above $2J$ can be neglected.

If the algorithm converges for each N and the corresponding T_N , giving the optimal pulse sequence $B_1^*(t)$, then we can assert that the scaling of the operation time is at least as good as $T_N = N^2$, up to a given precision. We initialize the algorithm with a guess for the function $B_1(t)$, for example, $B_1(t) = 1$. This is, of course, a poor choice for performing the SWAP gate. However, the optimization algorithm exponentially improves the choice of $B_1(t)$ and reaches the desired precision. It is worth pointing out that the frequencies of the pulse are of the same order as the natural dynamics of the chain, so the control function does not need to be fast. This is expected, because the local gates make use of the evolution of the chain in order to become global. The Fourier transform of a typical optimized function $B_1^*(t)$ is given in Fig. 2. Low-pass filtering confirmed that only frequencies of up to $\sim J$ are required. Furthermore, such pulses are robust against small fluctuations [4]. Simulating chain lengths up to $N = 40$, we find that $T_N = N^2$ can be achieved (Fig. 3). We stress here that the chosen scaling law T_N may not necessarily describe the shortest time on which the physical SWAP gate can be performed. However, the dynamical Lie algebra of quasifree fermions has a dimension of the order N^2 , indicating that such scaling might be optimal.

A final remark on the robustness against imperfections of the results presented is needed, as perfect homogeneity or even fine tuning of individual couplings might be very hard

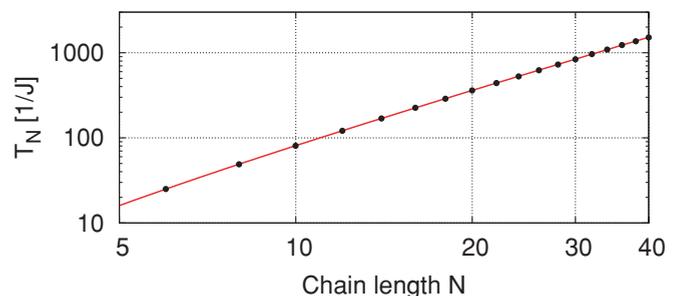


FIG. 3. (Color online) The SWAP operation time T_N versus the chain length N : the red continuous line is $(N - 1)^2$ while the black dots are the lengths for which we numerically verified this scaling with a fidelity $F > 1 - 10^{-4}$.

to achieve in most systems and might prevent information transfer schemes from working [17]. In the previous analysis, for simplicity, we considered a chain with uniform couplings $c_n = J$ for $n > 1$, but the results for arbitrary couplings are similar as long as localization effects can be neglected, that is, as long as N is smaller than the localization length. We can assume that the disordered Hamiltonian parameters are known, because they can be estimated efficiently by controlling only the chain end [18,19], but we need to confirm that the time scale of the control does not change. Therefore, we performed the optimization with an off-site disorder of 10%, uniformly distributed, and found that it leads to the same results, for most realizations, until at least $N = 40$.

Conclusion. We have shown how to *efficiently* compute control pulses for large spin chains described by a vast class of Hamiltonians. The pulses are computed for a $2N$ -dimensional system but can be applied to the full 2^N -dimensional system. Full quantum computation is possible by controlling only two spins at one end of the chain. The only price for this indirect control is that the quantum computation takes quadratically longer than for direct control. Given the large benefit of requiring so little control for a quantum computer, we believe that this scheme would be very useful for future implementations. As a further application, we remark that our proposal can also be applied to use the spin chain as a quantum memory, storing qubits by moving states from the controlled part to the rest of the chain and by applying control pulses on qubits 1 and 2 to achieve the identity on the register (effectively switching off the chain Hamiltonian). For future studies, we would like to probe the ultimate limit at which one can perform SWAP operations using optimal control [2,4,8,16], and try to obtain simple (possibly analytic [15]) pulses.

As a possible application of our proposal, we note that in Josephson qubit implementations much progress has been reported on the control and readout of two qubits [11] but that the control and readout of *many* qubits is currently impossible. In fact, it is expected to be difficult to construct fully addressable long arrays of Josephson junctions, while it is conceivable to produce long chains of qubits with always-on interaction where only one or two qubits are fully controllable and readable [20]. In those systems the decoherence time is (optimistically) ~ 1000 times larger than the time scale of the interqubit coupling [21,22], which would make our scheme applicable for up to ~ 30 qubits to achieve a single SWAP gate (though, of course, fewer qubits for full computation). This means that simple tasks such as a quantum state transfer [5] can already be implemented in such systems with current technology. Hence, the optimal control ideas presented here have the potential to address a serious limitation in such implementations and thus open an additional avenue toward quantum information processing in solid state devices.

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