Effective Hamiltonian approach to the Kerr nonlinearity in an optomechanical system

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Using the Born-Oppenheimer approximation, we derive an effective Hamiltonian for an optomechanical system that leads to a nonlinear Kerr effect in the system's vacuum. The oscillating mirror at one edge of the optomechanical system induces a squeezing effect in the intensity spectrum of the cavity field. A near-resonant laser field is applied at the other edge to drive the cavity field in order to enhance the Kerr effect. We also propose a quantum-nondemolition-measurement setup to monitor a system with two cavities separated by a common oscillating mirror based on our effective Hamiltonian approach.

I. INTRODUCTION

Recently, experimental and theoretical investigations have been carried out to demonstrate the coherent optomechanical coupling between a quantized cavity field and the mechanical motion of a mirror located at one end of an optical cavity (e.g., see Refs. [1–4]). Many efforts have been made to propose new devices [5] and explore the quantum-classical transition (e.g., see Refs. [6–9]) based on such optomechanical systems. It was also discovered that such systems possess nonlinear optical properties. Our study here focuses on another aspect of optomechanical coupling that can realize the Kerr nonlinear optical effect in a vacuum rather than in a conventional dispersive medium [10,11].

The Kerr effect usually appears in nonlinear dispersive media [12] due to the third-order matter-light interaction. Instead of resulting from higher-order light-matter interactions, we now realize *the nonlinear Kerr effect* from the radiation pressure on an oscillating mirror. This oscillating mirror, located at one end of a Fabry-Pérot (FP) cavity, is driven by an input laser field at the other end. The oscillating mirror is modeled as a quantum-mechanical resonator, whose tiny oscillations are controlled by the radiation pressure of the cavity field. Indeed, it has been pointed out [13,14] that the vacuum cavity with a movable mirror might mimic a Kerr medium when the cavity field is driven by a coherent light field.

In this Brief Report, we use the Born-Oppenheimer (BO) approximation to derive a Kerr-medium-like Hamiltonian, which shows the underlying nonlinear mechanism more clearly than other (equivalent) approaches. We shall first point out that in the conventional BO approximation for a molecule the (slower) nuclear variables are adiabatically separated from the (faster) electronic variables. The stability of the molecular configuration requires the effective potential to have a minimum value. However, the generalized BO theory [15] for spin-orbit systems or cavity QED systems [16] does *not* have this requirement and the effective force could be either attractive or repulsive.

II. INDUCED KERR NONLINEARITY AND BORN-OPPENHEIMER APPROXIMATION

As shown in Fig. 1(a), we study a FP cavity with an oscillating mirror at one end acting as a quantum-mechanical harmonic oscillator. The cavity is driven by a laser field with frequency ω_D. We first study an ideal case without considering the losses of both the cavity field and the oscillating mirror. The total Hamiltonian

$$H = H_P + H_M + H_I \tag{1}$$

contains three parts as

$$H_P = \hbar\omega_0 a^\dagger a - \hbar g a^\dagger a x, \tag{2a}$$

$$H_M = \frac{p^2}{2m} + \frac{1}{2}m\Omega^2 x^2, \tag{2b}$$

$$H_I = i\hbar\lambda(a^\dagger e^{-i\omega_D t} - a e^{i\omega_D t}). \tag{2c}$$

The cavity field with frequency ω_0 is described by bosonic operators a and $a^\dagger$. The symbols m, Ω, x, and p denote, respectively, the mass, frequency, displacement, and momentum of the oscillating mirror (hereafter, we just call it "the
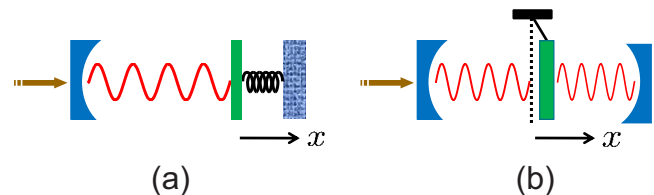

FIG. 1. (Color online) Schematic diagram of (a) a Fabry-Perot cavity with an oscillating mirror (green) in one end and (b) two Fabry-Perot cavities sharing a common oscillating mirror. The mirrors in (a) and (b) are modeled as quantum-mechanical harmonic oscillators. The red sinusoidal curves inside the cavities and the spring schematically represent the cavity field and the harmonic oscillator, respectively. The cavity is driven by an input laser field shown as an (brown) arrow on the left. The two oscillating mirrors have very small displacements, x, around their equilibrium positions.

mirror"). The coupling strength λ between the cavity field and the driving laser field is related to the input laser power P and the decay rate κ of the cavity field via the relation $|\lambda| = \sqrt{2P\kappa/\hbar\omega_D}$. The interaction constant $g = \omega_0/L$ between the cavity field and the mirror stems from a very small change x of the FP cavity length L.

Usually, the characteristic frequency of the cavity field is about 10^{14} Hz, which is much higher than the nanomechanical resonator frequency 10^9 Hz achieved by current experiments. Nonlinear effects appear when a classical driving field is applied to the cavity. Let us consider the case when the driving field frequency ω_D is close to the cavity field frequency ω_0. We also use the "rotating frame of reference" defined by a unitary transformation $W(t) = \exp(-i\omega_D a^\dagger a t)$, which is very similar to the NMR experiments used to demonstrate the Berry phase [17]. In the rotating frame of reference, the effective form $H^R = W^\dagger(t)HW(t) - iW^\dagger(t)[\partial W(t)/\partial t]$ of the Hamiltonian H in Eq. (1) reads

$$H^R = H_C + H^R_{MR}, \tag{3}$$

with the effective Hamiltonians

$$H_C = \hbar\Delta a^\dagger a + i\hbar\lambda(a^\dagger - a) \tag{4}$$

and

$$H^R_{MR} = \frac{p^2}{2m} + \frac{1}{2}m\Omega^2 x^2 - \hbar g a^\dagger a x. \tag{5}$$

Here, the detuning $\Delta = \omega_0 - \omega_D$ is the effective frequency of the cavity field in the new frame. Clearly, Δ can be controlled by tuning the frequency ω_D of the driving field. Therefore, the effective frequency Δ of the cavity field can be tuned to be much smaller than that of the mechanical resonator. Under such condition, the mechanical resonator can be treated as the fast variable and the BO approximation can be employed [18].

We first study Hamiltonian (5) of the fast variables x and p of the mirror (in the rotating frame) by taking the "slow variables" a and $a^\dagger$ of the cavity field as constants (in the rotating frame). Then Hamiltonian (5) can be rewritten as

$$H^R_{MR} = \hbar\Omega\left(A^\dagger A + \frac{1}{2}\right) - \frac{\hbar^2 g^2}{2m\Omega^2}(a^\dagger a)^2, \tag{6}$$

where the creation operator $A^\dagger$ of the cavity field is defined by

$$A^\dagger = \sqrt{\frac{m\Omega}{2\hbar}}\left(x - \frac{ip}{m\Omega}\right) + \alpha, \tag{7}$$

with $\alpha = -\sqrt{\hbar g^2 N^2/2m\Omega^3}$. Here, $N = a^\dagger a$ is considered as a c number in the first step of the Born-Oppenheimer approximation. Equation (6) shows that the mirror variables are shifted by the amount α due to its interaction with the cavity field. It is clear that the ground state of the effective Hamiltonian in Eq. (6) can be obtained via the ground state of the Hamiltonian H_M in Eq. (2b) for a harmonic oscillator with displacement operator

$$D(\alpha) = \exp(i\alpha A^\dagger - i\alpha^* A). \tag{8}$$

The eigenvectors and the eigenvalues of the mirror, corresponding to Hamiltonian (6), are, respectively,

$$|n\rangle = \frac{1}{\sqrt{n!}}(A^\dagger)^n D(\alpha)|0\rangle \equiv |n(a^\dagger a)\rangle \tag{9}$$

and

$$V_n(a^\dagger a) = \hbar\Omega\left(n + \frac{1}{2}\right) - \frac{\hbar^2 g^2}{2m\Omega^2}(a^\dagger a)^2. \tag{10}$$

Equations (9) and (10) show that $|n\rangle$ and $V_n(a^\dagger a)$ are functions of the slow variables of the cavity field. Equation (9) also shows that the ground state of the fast variables (x and p) of the mirror is a coherent state $D(\alpha)|0\rangle$ resulting from radiation pressure.

According to the lowest-order generalized BO approximation, the total eigenfunction $|\Phi\rangle$ of the coupled system of the cavity field and the mirror can be factorized as $|\Phi\rangle = |\phi_n(\alpha)\rangle|n(a^\dagger a)\rangle$, where $|\phi_n(\alpha)\rangle$ satisfies the Schrödinger equation with the effective Hamiltonian

$$H^R_{ph} = \hbar\Delta a^\dagger a + i\hbar\lambda(a^\dagger - a) + V_n(N)$$
$$= \hbar\Delta a^\dagger a - \hbar\chi(a^\dagger a)^2 + i\hbar\lambda(a^\dagger - a) + \text{const.} \tag{11}$$

The BO adiabatic separation provides an effective potential $V_n(a^\dagger a)$ for the "slow" motion of the cavity field. This potential contains a typical

$$\text{Kerr nonlinear term} = \hbar\chi(a^\dagger a)^2, \tag{12}$$

where the parameter

$$\chi = \hbar\frac{g^2}{2m\Omega^2} \tag{13}$$

plays the role of *the phenomenological third-order susceptibility* as in usual Kerr media.

III. VALIDITY OF THE BO APPROXIMATION

We now verify the validity of the generalized BO approximation applied to the optomechanical system through the squeezing effect, which is induced by the Kerr interaction. This squeezing effect can be demonstrated by the output intensity spectrum $S_I(\omega, \Delta)$. Following the definition of the $S_I(\omega, \Delta)$ in Ref. [14] and the linearization technique of the Langevin equations governed by the Hamiltonian in Eq. (11), we obtain the intensity spectrum under the BO approximation

$$S_I(\omega, \Delta) = \left| 1 - \frac{2\kappa}{D(\omega, \Delta)}[A_-(\omega, \Delta) + iB(\omega, \Delta)e^{i2\theta(\Delta)}] \right|^2, \tag{14}$$

where

$$A_\pm(\omega, \Delta) = -i\omega + i\Delta' \mp 4i|\alpha_s(\Delta)|^2\chi + \kappa, \tag{15a}$$

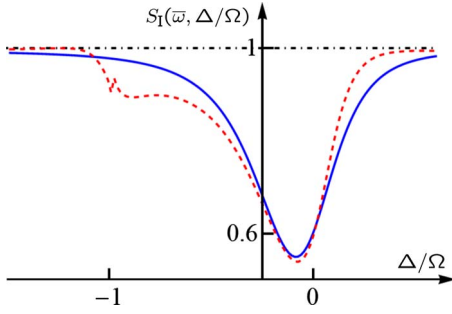$$B(\omega, \Delta) = 2\alpha_s(\Delta)^2\chi, \tag{15b}$$

FIG. 2. (Color online) Intensity spectrum $S_I(\bar{\omega}, \Delta/\Omega)$ versus normalized detuning Δ/Ω for the value $\omega = \bar{\omega}$. Analytical results for the full Hamiltonian are shown with a red dashed curve, while the BO approximation results are shown with a blue solid curve. Note that squeezing occurs when $S_I(\bar{\omega}, \Delta/\Omega) < 1$, and the baseline $S_I(\bar{\omega}, \Delta/\Omega) = 1$ is shown with a horizontal black dot-dashed line.

$$D(\omega, \Delta) = A_+(\omega) A_-(\omega) - |B(\omega, \Delta)|^2, \quad (15c)$$

$$\tan \frac{\theta(\Delta)}{2} = \frac{1}{\kappa} [\Delta' - 2\chi |\alpha_s(\Delta)|^2], \quad (15d)$$

with renormalized detuning $\Delta' = \Delta - \chi$ and the steady-state value $\alpha_s(\Delta)$ of the amplitude of the cavity field. Here, the cavity loss κ has been taken into account. The BO approximation requires that the photons in the cavity can survive a sufficiently long time, which is equivalent to the condition $\kappa \ll \omega_0$.

If there is no Kerr interaction induced by the radiation pressure ($\chi = 0$), $S_I(\omega, \Delta)$ can be simplified to $S_I(\omega, \Delta) = 1$. When the Kerr interaction is induced ($\chi \neq 0$), the intensity spectrum can be less than 1, which displays a squeezing effect [20,21] of the cavity field. Here, $B(\omega, \Delta)$ plays an important role in the reduction of the output intensity fluctuation. Hence, the squeezing effect could be observed experimentally by measuring the intensity spectrum of the output laser.

Figure 2 shows the squeezing of the cavity field in the intensity spectrum with a blue solid curve, where the maximum squeezing occurs in the vicinity of $|\Delta/\Omega| \sim 0$. We consider that the oscillating mirror has mass $m = 100$ ng, frequency $\Omega/\kappa = 40\pi$, and a normalized damping rate $\gamma/\kappa = 0.06$. Figure 2 is plotted for the driving field frequency $\omega_D/2\pi = 282$ THz, the optical cavity length $L = 10^{-2}$ m, finesse $\mathcal{F} = 1.9 \times 10^5$, and decay rate $\kappa \simeq 5 \times 10^5$ s^{-1}; a driving laser wavelength $\lambda = 1064$ nm, normalized frequency $\omega_D/\kappa = 3.54 \times 10^7$, and power $P = 500$ μW. The temperature T of the cavity field and the mechanical resonator is assumed to be zero as in Ref. [19]. Then at a particular frequency $\bar{\omega}$ [chosen at the position where $S_I(\omega, \Delta)$ is minimum], the intensity spectrum has the squeezing effect shown in Fig. 2.

Now, we also study the intensity spectrum without the BO approximation. Starting from the total Hamiltonian in Eq. (3), a full derivation shows that the intensity spectrum has a similar form as in Eq. (15) except that the renormalized detuning Δ' and the Kerr interaction strength χ are replaced, respectively, by $\Delta - g x_s$ and $\chi \Omega^2 \zeta(\omega)$. Here, $x_s = \chi |a_s|^2$ is the

steady-state value of the oscillation amplitude of the mirror and

$$\zeta(\omega) = \frac{1}{\Omega^2 - \omega^2 - i \gamma \omega} \quad (16)$$

is the mechanical susceptibility of the oscillating mirror. The maximum squeezing effect in the intensity spectrum, plotted by the red dashed curve in Fig. 2, can also be observed in the vicinity of $|\Delta/\Omega| \sim 0$. We use the same parameters for calculating the intensity spectrum under the BO approximation. Therefore, the BO approximation is valid when the frequency of the oscillating mirror Ω is much larger than the frequency of the light field Δ.

The effect of the BO approximation can also be understood via $\zeta(\omega)$. If the mirror frequency Ω is much larger than the detuning Δ of the cavity field in the rotating frame of reference, $\zeta(\omega)$ plays an important role in the vicinity of $\omega \approx \Delta$. When the BO approximation is valid under the condition $\Omega \gg \Delta$, and when the macroscopic displacement x_s is extremely small, the mechanical susceptibility is approximately equal to

$$\zeta(\omega) \approx 1/\Omega^2, \quad (17)$$

which leads the intensity spectrum to have the same form as in Eq. (14).

IV. QUANTUM NONDEMOLITION MEASUREMENT WITH TWO-MODE INDUCED KERR EFFECT

Within the BO approximation, the cavity field inside the flexible Fabry-Pérot cavity, driven by an input laser, exhibits a Kerr-like nonlinear property. It is not difficult to generate a two-mode induced Kerr interaction, which is useful for quantum nondemolition (QND) measurements [22].

As shown in Fig. 1(b), we consider two FP cavities, referred to below as the left and the right cavities with subindexes L and R, sharing one common oscillating mirror. This mirror is assumed to oscillate with a very small displacement x around its equilibrium position. Thus, two cavity fields, with frequencies ω_L and ω_R in the laboratory reference frame, indirectly interact with each other via this oscillating mirror.

In the rotating frame of reference, using the BO approximation discussed above, we can derive the induced effective interaction between the two cavity fields

$$H_{eff}(n_L, n_R) = -\hbar(\chi_L n_L^2 + \chi_R n_R^2 + 2\sqrt{\chi_L \chi_R} n_L n_R), \quad (18)$$

where the

$$\chi_i = \hbar g_i^2/(2m\Omega^2) \quad (19)$$

and n_i with $i = L, R$ denote the photon number operator of the left and the right cavity fields. To perform a QND measurement, the self-modulation term $\chi_i n_i^2$ of the probe field can be ignored [22]. The parameters χ_i result from the nonresonant mechanical states transition. If the oscillating mirror has an actual level $\hbar(\Delta_L + \Delta_R)$, only the two-mode Kerr term dominates the resonant process and both two self-modulation terms are off-resonant. Here, $\Delta_L = \omega_L - \omega_D$ and $\Delta_R = \omega_R - \omega_D$

are the detunings of the left and right cavity fields relevant to the frequency ω_D of the driving field. Under this condition, the system Hamiltonian for the QND measurement can be written as

$$H_{QND} = \hbar\Delta_L a_L^\dagger a_L + \hbar\Delta_R a_R^\dagger a_R + 2\hbar\chi a_L^\dagger a_L a_R^\dagger a_R. \quad (20)$$

The Kerr interaction in Eq. (20) satisfies the QND measurement conditions [22]. Since this two-mode Kerr interaction commutes with the free Hamiltonians of both cavities, we can nondestructively measure the photon number by observing the other cavity's conjugate observable.

V. CONCLUSION

We have studied the Kerr nonlinearity in the vacuum induced by the radiation pressure in typical optomechanical systems. Such nonlinear interaction can be explicitly obtained via a BO approximation for an ideal case. Through a squeezing effect, which should be experimentally observable via the intensity spectrum, we verify the validity of the generalized BO approximation by taking into account the dissipation and fluctuation of the cavity field. Furthermore, we propose a two-mode Kerr interaction between two cavity fields for quantum nondemolition measurements, which can be realized by using two cavities sharing one oscillating mirror. It is possible to nondestructively measure a cavity photon number by observing another cavity's conjugate observable.

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