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ENTANGLEMENT OF TWO COUPLED CHARGE QUBITS

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We have analyzed the entanglement of a system of two coupled charge qubits. We calculate the amount of entanglement using several different approaches. We show that in the ideal case the system remains entangled most of the time and the amount of entanglement reaches almost unity, i.e. the system becomes maximally entangled at certain instances.

Keywords: Quantum computation; quantum bits; entanglement.

1. Introduction

Entanglement is a peculiar, yet natural, nonclassical correlation that is possible between separated quantum systems. Although entanglement has been known for many years as a purely theoretical subject, it turned into a practical issue recently after it was realized that it plays a crucial role in quantum computation and quantum communication.

Entanglement of two quantum systems can be understood by using the following example. Let us consider two single qubits A and B whose states can be presented as a superposition of the basis states: $|\psi_A\rangle = a_1|0\rangle_1 + a_2|1\rangle_1$ and $|\psi_B\rangle = b_1|0\rangle_2 + b_2|1\rangle_2$, where $|0\rangle_{1(2)}$ and $|1\rangle_{1(2)}$ are the basis states of the first (second) qubits and $a_{1,2}$ and $b_{1,2}$ are the corresponding amplitudes. In the unentangled case, the state $|\psi\rangle$ of a composite two-qubit system can be described as a product of two single-qubit states:

$$|\psi\rangle = |\psi_A\rangle|\psi_B\rangle = c_1|00\rangle + c_2|10\rangle + c_3|01\rangle + c_4|11\rangle \tag{1}$$

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where c_i ($i = 1, 2, 3, 4$) are the amplitudes of the two-qubit states that are the product of the corresponding amplitudes of the single-qubit states (e.g. $c_1 = a_1 b_1$, etc.). Here we use the following notations: $|00\rangle = |0\rangle_1|0\rangle_2$, $|10\rangle = |1\rangle_1|0\rangle_2$, $|01\rangle = |0\rangle_1|1\rangle_2$ and $|11\rangle = |1\rangle_1|1\rangle_2$. There exist certain two-qubit states for which $|\psi\rangle \neq |\psi_A\rangle|\psi_B\rangle$. For example, the state $|\psi\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$ cannot be reduced to a product of two single-qubit states described by Eq. (1). Such a state is called an entangled state.

For practical purposes, one needs to quantify entanglement. A few measures of entanglement have been introduced: *negativity*, *concurrence*, *entanglement of formation* and *entropy of entanglement*.

Here we consider a system of two coupled charge qubits and calculate the time evolution of entanglement using these criteria.

2. Measures of Entanglement

Let us note first that there is a simple qualitative test for entanglement provided by Peres[1] and Horodecki.[2] For 2×2 systems like the one we consider here, the necessary and sufficient condition for entanglement is the negativity of the partial transposition of a state of the system. That is, if the partially transposed density matrix[a]

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

has negative eigenvalues then the system is entangled, if the eigenvalues are zero or positive then the system is unentangled.

Based on the qualitative partial transpose approach, a quantitative entanglement measure, called *negativity*, can be introduced:[3]

$$N(\rho) = \max(0, -2\lambda_{\text{min}}), \tag{2}$$

where λ_{min} is the smallest eigenvalue of the partial transpose of the state ρ.

The concept of *concurrence* originates from Ref. 4 and is defined for a pair of qubits as[5]

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{3}$$

where the λ_i's are the square roots of the eigenvalues of $\rho\tilde{\rho}$ in descending order. Here $\tilde{\rho}$ is the result of applying the spin-flip operation to ρ: $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, where ρ^* is a complex conjugate of ρ.

For a pure state $|\psi\rangle$ of a composite system [see Eq. (1)], the concurrence can be expressed as[6]

$$C(\psi) = 2|c_1 c_4 - c_2 c_3|.$$

[a](p_i are the state probabilities satisfying the normalization condition $\sum_i p_i = 1$).

For a pair of qubits, there exists an explicit formula for the *entanglement of formation* E_f based on the concept of concurrence. For a pure state ρ, the entanglement of formation is related to concurrence as[6]

$$E_f = \mathcal{E}(C(\rho)), \tag{4}$$

where the function $\mathcal{E}$ is defined by

$$\mathcal{E}(C) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right); \tag{5}$$

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x). \tag{6}$$

This definition can be further extended to the more general case of mixed states.[7]

The simplest case of an entangled system is a pair of qubits in a pure but nonfactorizable state. In this case the entanglement can be defined via the von Neuman *entropy* of either qubit A or B:[8]

$$E_e = -\mathrm{Tr}\rho_A \log_2 \rho_A = -\mathrm{Tr}\rho_B \log_2 \rho_B, \tag{7}$$

where $\rho_A = |\psi_A\rangle\langle\psi_A|$ and $\rho_B = |\psi_B\rangle\langle\psi_B|$ are the density matrices of subsystem A and B, respectively. For such a sytem, the entanglement of formation defined in Eq. (4) coincides with the entropy of entanglement.

3. Circuit and Model

We consider a system of two coupled charge qubits.[9] Each qubit is a Cooper-pair box whose charge states are quantized when the charging energy of the box $E_{c1,2} = (2e)^2/2C_{1,2}$ exceeds the Josephson energy $E_{J1,2}$ of its coupling to a reservoir. Here $C_{1,2}$ is the total capacitance of the corresponding Cooper-pair box and $2e$ is the Cooper-pair charge. The qubits are coupled by an on-chip capacitor giving a mutual coupling energy E_m. At low temeperature and in a proper voltage range, each qubit is reduced to a two-level system and, under condition $E_{J1,2} \sim E_m$, the whole system can be described by the two-qubit Hamiltonian

$$\begin{aligned} H = &\sum_{n_1,n_2=0}^{1} E_{n_1n_2}|n_1n_2\rangle\langle n_1n_2| - \frac{E_{J1}}{2}\sum_{n_2=0}^{1}(|0\rangle\langle 1| + |1\rangle\langle 0|)\otimes|n_2\rangle\langle n_2| \\ &- \frac{E_{J2}}{2}\sum_{n_1=0}^{1}|n_1\rangle\langle n_1|\otimes(|0\rangle\langle 1| + |1\rangle\langle 0|) \end{aligned} \tag{8}$$

where

$$E_{n_1n_2} = E_{c1}(n_{g_1} - n_1) + E_{c2}(n_{g_2} - n_2) + E_m(n_{g_1} - n_1)(n_{g_2} - n_2)$$

are the electrostatic energies corresponding to different charge configurations, n_{g_1} and n_{g2}, are the normalized gate-induced charges on the corresponding qubit. The system has double degeneracy of electrostatic energies at $n_{g1} = n_{g2} = 0.5$:

$E_{00} = E_{11}$ and $E_{10} = E_{01}$. If the system is driven nonadiabatically from the $|00\rangle$ state to this point, it evolves coherently as was demonstrated in Ref. 9. The values of electrostatic energies at the double degeneracy point as well as Josephson energies were determined from the independent measurements: $E_{00} = E_{11} \approx 70.9$ GHz, $E_{10} = E_{01} \approx 63.6$ GHz and $E_{J1} = E_{J2} \approx 9.1$ GHz. Josephson energies were made equal by applying an external magnetic field to suppress E_{J1} from its maximum value of 13.4 GHz at zero magnetic field.

Then we calculate the evolution of entanglement using the criteria described above.

4. Results and Discussion

We consider an ideal case of a pure bipartite system. The evolution of the system starts from the $|00\rangle$ state, $\rho_0 = |00\rangle\langle 00|$. Neglecting decoherence, we calculate the time dependence of the density matrix using the Hamiltonian given in Eq. (8):

$$\rho(t) = U^{TC} \rho_0 U\,, \tag{9}$$

where $U = \exp(-iHt/\hbar)$ and U^{TC} is a transpose conjugate of U. Then we use $\rho(t)$ to check the Peres–Horodecki criteria and to calculate the amount of entanglement using Eqs. (2)–(4) and (7).

The result of the Peres–Horodecki test is shown in Fig. 1. It is clear that most of the time the product of eigenvalues is negative,[10] therefore our system passes the test. We then proceed with the calculation of the amount of entanglement.

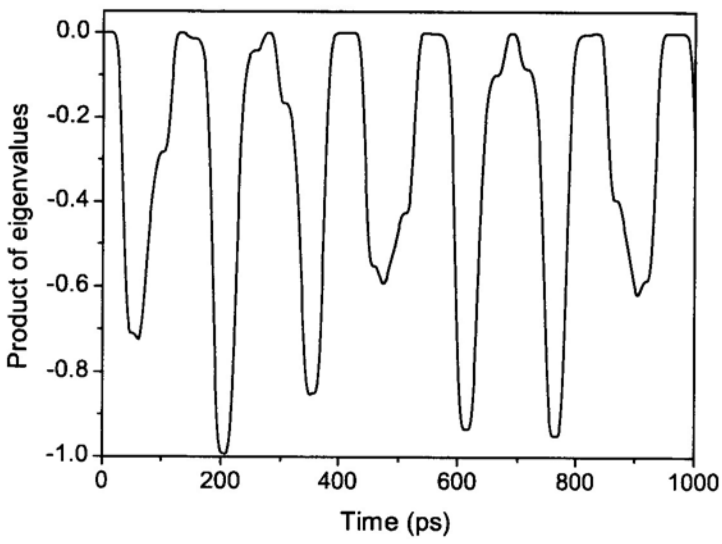

Fig. 1. Peres–Horodecki test for a coupled qubit system. Here the product has been normalized to the interval [0, 1].

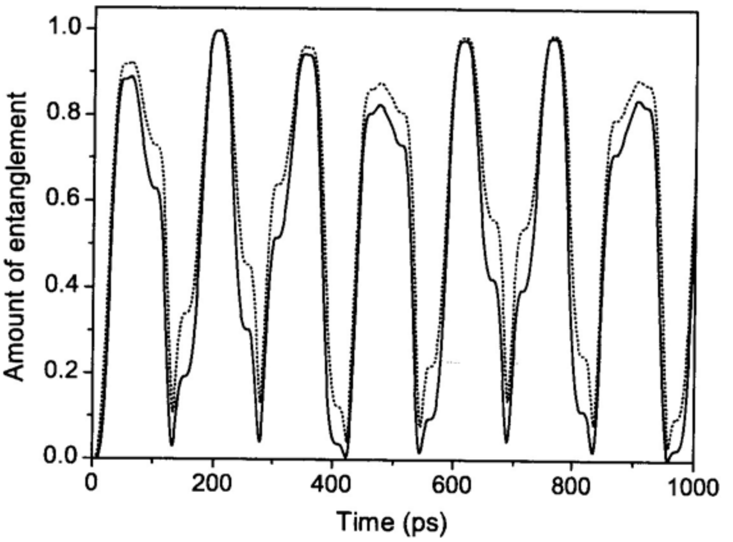

Fig. 2. Amount of entanglement in a system of two coupled charge qubits calculated by negativity, concurrence, entanglement of formation and entropy of entanglement. Dotted line — negativity and concurrence; solid line — entanglement of formation and entropy of entanglement.

Dependence of the amount of entanglement on time is presented in Fig. 2. All four criteria have been used, however, only two curves are seen in Fig. 2. This is due to the fact that for the considered case of pure states different approaches give similar results. For example, negativity coincides with concurrence and entanglement of formation coincides with the entropy of entanglement. Note that the results in Fig. 2 are consistent: despite small difference in absolute value the amout of entanglement basically coinsides and reaches maxima and minima at the same time instances. The results in Fig. 2 are also consistent with the results in Fig. 1. Amount of entanglement in Fig. 2 reaches maximum values when the product of eigenvalues in Fig. 1 reaches minimum values and vice versa.

Comparing the time evolution of the amount of entanglement with the time evolution of the state probabilities ρ_{00}, ρ_{10}, etc. we can conclude that the amount of entanglement reaches maximum values when the probabilities ρ_{00} and ρ_{11} are close to $1/2$ while the probabilities of the states $|10\rangle$ and $|01\rangle$ almost vanish. On the other hand, the amount of entanglement is close to zero, when the probability of only one state, $|00\rangle$ or $|11\rangle$, approaches unity while the rest three being almost zero.

We stress finally that entanglement of two qubits is a result of their interaction described by the $E_m(n_{g_1} - n_1)(n_{g_2} - n_2)$ term in the Hamiltonian given in Eq. (8). If we switch off the interaction, i.e. set E_m equal to zero, then entanglement vanishes. With the non-zero interaction between the qubits, entanglement oscillates with the same frequencies as do other quantities of the system like state probabilities.

5. Conclusions

Our results show that the ideal coupled two-qubit system remains entangled most of the time during its coherent evolution. The amount of entanglement oscillates between zero (completely unentangled qubits) and unity (maximally entangled qubits). This is an optimistic scenario for entanglement because we considered only pure states and neglected decoherence. Still these results are basically true at least for the first $100, \ldots, 200$ picoseconds when decoherence is weak. In a more realistic approach, one should consider mixed states and take into account decoherence and real pulse shape.

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References

1. A. Peres, *Phys. Rev. Lett.* **77**, 1413 (1996).
2. M. Horodecki, P. Horodecki and R. Horodecki, *Phys. Lett.* **A223**, 1 (1996).
3. G. Vidal and R. F. Werner, *Phys. Rev.* **A65**, 032314 (2002).
4. S. Hill and W. K. Wootters, *Phys. Rev. Lett.* **78**, 5022 (1997).
5. W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
6. W. K. Wootters, *Quant. Inform. Comput.* **1**, 27 (2001).
7. H. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, *Phys. Rev.* **A54**, 3824 (1996).
8. H. H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher, *Phys. Rev.* **A53**, 2046 (1996).
9. Yu. A. Pashkin, T. Yamamoto, O. Astafiev, Y. Nakamura, D. V. Averin and J. S. Tsai, *Nature* **421**, 823 (2003).
10. T. Tilma, M. Byrd and E. C. G. Sudarshan, *J. Phys. A.: Math. Gen.* **35**, 10445 (2002).