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Quantum tomography for solid-state qubits

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Abstract. – We propose a method for the tomographic reconstruction of qubit states for a general class of solid-state systems in which the Hamiltonians are represented by spin operators, *e.g.*, with Heisenberg-, XXZ-, or XY-type exchange interactions. We analyze the implementation of the projective operator measurements, or spin measurements, on qubit states. All the qubit states for the spin Hamiltonians can be reconstructed by using experimental data.

Quantum information processing requires the effective measurement of quantum states. However, a single quantum measurement can only obtain partial information of a quantum state. The reconstruction of a quantum state requires measuring a complete set of observables on an ensemble of identically prepared copies of the system. This method called quantum state tomography [1], is very important because any unknown state can be ascertained by tomographic measurements. Moreover, the full description of qubit states can increase the accuracy of quantum operations. Tomographic measurements have been experimentally implemented for, *e.g.*, the nuclear spin state of an NMR system [2], the electromagnetic field and photon state [3], the vibration state of molecules [4], the motional quantum state of a trapped atom [5], and atomic wave packets [6].

Experimental investigations on solid-state qubits are very promising, especially in superconducting [7,8] and quantum dot structures [9]. These recent achievements make it necessary to experimentally determine quantum states in solid-state systems. Although there are many theoretical studies on tomography (*e.g.*, refs. [10] and references therein), to our knowledge, these are not specific to solid-state systems. Here, we focus on this question for quantum computing models using standard spin representations for solid-state qubits. Our proposal is related to tomographic measurements using NMR. The measurements of the density matrix in NMR experiments are obtained from the NMR spectrum of the linear combination of "product operators", *i.e.* products of the angular-momentum operators [11]. However, experiments in solid-state systems usually involve the local single-qubit projective operator measurement (POM) or spin measurement. So, we study the method of tomographic reconstruction of solid-state qubits by POM or spin measurement for a number of promising solid-state quantum computing models [12–19]. We will investigate how the multi-qubit correlation measurements can be realized by virtue of an appropriate two-qubit operation combined with single-qubit operations.

State and measurements. – Using the density matrix form, an n-qubit state ρ can be expressed as

$$\rho = \frac{1}{2^n} \sum_{l_1,\cdots,l_n=0,x,y,z} r_{l_1\cdots l_n}\ \sigma_{l_1} \otimes \sigma_{l_2} \otimes \cdots \otimes \sigma_{l_n}\,, \tag{1}$$

where $r_{l_1\cdots l_n}$ are 4^n real parameters, $\sigma_{l_m=x,y,z}$ and $\sigma_{l_m=0}$ $(0 \leq m \leq n)$ are the Pauli spin and identity operators of the m-th qubit, respectively. We adopt the convention $|0\rangle = |\uparrow\rangle$ and $|1\rangle = |\downarrow\rangle$ to denote the computational basis states of each qubit. The normalization condition $\mathrm{Tr}\,\rho = 1$ makes $r_{0,\cdots,0} = 1$, which means that ρ can be specified by $(4^n - 1)$ real parameters. These parameters correspond to the expectation values of the measurements given by the operators $\sigma_{j_1} \otimes \cdots \otimes \sigma_{j_n}$; that is, $\mathrm{Tr}\{\rho(\sigma_{j_1} \otimes \cdots \otimes \sigma_{j_n})\} = r_{l_1\cdots l_n}\delta_{j_1 l_1} \cdots \delta_{j_n l_n}$, where $l_1, \cdots, l_n$ are not simultaneously taken as zero. If there are $n - m$ identity operators among $\sigma_{l_1}\otimes\cdots\otimes\sigma_{l_n}$, the measurement is really done by the m-qubits and it can be abbreviated by the tensor product of only the m Pauli operators, which is denoted hereafter by $\sigma_{1\,l_1} \otimes \cdots \otimes \sigma_{m\,l_m}$. The $(4^n - 1)$ measurements required to reconstruct the n-qubit state can be decomposed into a summation from the single-qubit to n-qubit measurements as $\sum_{j=1}^{n} 3^j \binom{n}{j}$, where $3^j \binom{n}{j}$ is the number of j-qubit measurements and $\binom{n}{j}$ is the binomial coefficient.

To reconstruct the n-qubit state ρ, we need to determine all of its expanded coefficients $\{r_{l_1,\cdots,l_n}\}$. In solid-state systems, the correlated multi-qubit measurement is not realizable now, and the experimental readout is often done via single-qubit POM (*e.g.*, refs. [12–15]) or single spin measurement (*e.g.*, refs. [16, 17]). Without loss of generality, we assume that the POM is denoted by $|1\rangle\langle 1|$ and the spin measurement is presented by σ_x or σ_y. Below, we will discuss how to determine the $(4^n - 1)$ coefficients of an n-qubit state by using experimental data of the POM and then generalize it to the spin case. Our goal is to build a correspondence between the above measurements and actual measurements done via $|1\rangle\langle 1|$.

Single-qubit measurements. – The single-qubit measurements σ_{lz} $(l = 1, 2, \cdots, n)$ can be done by the projectors $(|1\rangle\langle 1|)_l$ due to the equivalence $\sigma_{lz} = \sigma_{l0} - 2(|1\rangle\langle 1|)_l$, with identity operators σ_{l0}. Thus, the POM experiments can directly determine n coefficients via n outcomes of the measurements $(|1\rangle\langle 1|)_l$. The measurements corresponding to the remaining $(4^n - n - 1)$ coefficients cannot be directly performed because of the limitations of the current experiment. In order to obtain these coefficients, a sequence of quantum operations W is needed such that each coefficient can be transformed to a position that is measurable by a POM experiment. The probability p of the l-th single-qubit measurement $(|1\rangle\langle 1|)_l$ on the operated state ρ can be expressed as

$$p = \mathrm{Tr}\left\{W\rho W^{\dagger}(|1\rangle\langle 1|)_l\right\} = \mathrm{Tr}\left\{\rho W^{\dagger}(|1\rangle\langle 1|)_l W\right\}, \tag{2}$$

which means that the experimental POM $(|1\rangle\langle 1|)_l$ on the state $W\rho W^{\dagger}$ can be equivalently described as $W^{\dagger}(|1\rangle\langle 1|)_l W = [\frac{1}{2} - \frac{1}{2}W^{\dagger}\sigma_{lz}W]$ on the original state ρ. Because the POM experiment is equivalent to $W^{\dagger}\sigma_{lz}W$, we can call it an equivalent m-qubit measurement of $(|1\rangle\langle 1|)_l$, if the largest number of Pauli operators in the operator product expansion of $W^{\dagger}\sigma_{lz}W$ is an m-qubit. Thus, the $(4^n - n - 1)$ coefficients can be obtained by the single-qubit projection $(|1\rangle\langle 1|)_l$ on the state ρ with appropriate operations W implemented by the dynamical evolution of the system with experimentally controllable parameters.

For all universal quantum computing proposals, the most general Hamiltonian of the system can be described as

$$H = \sum_{l=1}^{n} \sum_{\alpha=x,y,z} \varepsilon_{l\,\alpha}\, \sigma_{l\,\alpha} + \sum_{1=l<m}^{n} \sum_{\alpha,\beta=x,y,x} J_{l\,m}^{\alpha\beta}\, \sigma_{l\,\alpha} \otimes \sigma_{m\,\beta}\,, \tag{3}$$

where $\{\varepsilon_{l\,\alpha}\}$ and $\{J_{l\,m}^{\alpha\beta}\}$ are controllable and tunable system-specific one-qubit and exchange coupling parameters, which are required by the universality of quantum computing [20], $\sigma_{l\,\alpha=x,y,z}$ denote the Pauli operators of the l-th qubit. Without loss of generality, all parameters are assumed to be positive real numbers.

In order to obtain each coefficient corresponding to single-qubit measurements σ_{lx} or σ_{ly}, all single-qubit operations need to be performed separately by controlling the one-qubit parameters $\varepsilon_{l\,\alpha}$ while turning off all interactions in the Hamiltonian (3), that is, $J_{l\,m}^{\alpha\beta} = 0$. For n single-qubit measurements $\{\sigma_{ly}\}$, each σ_{ly} can be equivalently obtained by $(|1\rangle\langle 1|)_l$, after the l-th qubit is rotated $\pi/2$ about the x-axis, the latter expressed as $X_l = \exp[-i\pi\sigma_{lx}/4]$. This rotation can be realized within the evolution time $t = \hbar\pi/4\varepsilon_{lx}$, after the one-qubit parameters ε_{ly} and ε_{lz} are adjusted to zero. Other n single-qubit measurements $\{\sigma_{lx}\}$ can also be obtained by measuring $(|1\rangle\langle 1|)_l$ on the state, within the evolution time $t = \hbar\pi/4\varepsilon_{ly}$, after ε_{lx} and ε_{lz} are set to zero. This quantum operation is equivalent to a $\pi/2$ rotation of the l-th qubit about the y-axis, which is denoted by $Y_l = \exp[-i\pi\sigma_{ly}/4]$. However, not all of the three one-qubit parameters ε_{lx}, ε_{ly} and ε_{lz} appear in the Hamiltonian of most physical systems. For example, the parameter ε_{ly} is always zero in the charge-qubit system [12]. For this case, to obtain the rotation angle θ about the y-axis we need to alternatively turn on and off the single-qubit quantum operations: (1) $\overline{X}_l = \exp[-i\varepsilon_{lx}\sigma_{lx}t_1/\hbar]$, with the operation time $t_1 = 3\hbar\pi/4\varepsilon_{lx}$; (2) $Z_l(\theta) = \exp[-i\theta\sigma_{lz}/2]$, with $\theta = \varepsilon_{lz}t_2/\hbar$; and (3) $X_l = \exp[-i\varepsilon_{lx}\sigma_{lx}t_3/\hbar]$, with $t_3 = \hbar\pi/4\varepsilon_{lx}$. These can be expressed as $Y_l(\theta) = X_l Z_l(\theta)\overline{X}_l = \exp[-i\theta\sigma_{1y}/2]$. Especially, we denote the rotation $\pi/2$ about the z-axis by $Z_l = \exp[-i\pi\sigma_{lz}/4]$. In principle, if the l-th qubit system has only two controllable one-qubit parameters ε_α and ε_β, then the rotation angle $\epsilon_{\alpha\beta\gamma}\theta$ about the axis γ can be obtained by first doing a rotation of $\pi/4$ about the axis α, then a rotation of θ about the axis β, and finally a rotation by $-\pi/4$ about the axis α; that is, $\exp[-i\epsilon_{\alpha\beta\gamma}\theta\sigma_\gamma/2] = e^{-i\pi\sigma_\alpha/4}e^{-i\theta\sigma_\beta/2}e^{i\pi\sigma_\alpha/4}$, where α, β, γ can be x, y, or z and the Levi-Civita tensor $\epsilon_{\alpha\beta\gamma}$ is equal to $+1$ and -1 for the even and odd permutation of its indices, respectively. To reconstruct a single-qubit state, three single-qubit measurements σ_α ($\alpha = x$, y, z) are sufficient to obtain r_z, which determines the probabilities of finding 0 and 1, as well as r_x and r_y which determine the relative phase of $|0\rangle$ and $|1\rangle$.

Two-qubit measurements. – The above discussions show that all the single-qubit measurements can be experimentally implemented by POMs $(|1\rangle\langle 1|)_l$ on the given state with or without single-qubit quantum operations. However, the implementation of multiple-qubit measurements needs non-local two-qubit operations. The basic two-qubit operation can be derived from the time evolution operator $U_{12}(t)$ of a pair of coupled qubits, labelled by 1 and 2, whose Hamiltonian H_{12} can be obtained from eq. (3) with $n = 2$. Without loss of generality, we assume: i) $\varepsilon_{1\alpha} = \varepsilon_{2\alpha} = \varepsilon_\alpha$; and ii) $J_{lm}^{\alpha\beta} = J_{lm}^{\alpha}\delta_{\alpha\beta}$ in the Hamiltonian (3), because, by applying local unitary operations, *e.g.*, [21], the Hamiltonian (3) can always be transformed into a diagonal form —which is actually used for a number of promising solid-state quantum computing models [12–19]. Then in the basis $\{|0_1, 0_2\rangle, |0_1, 1_2\rangle, |1_1, 0_2\rangle, |1_1, 1_2\rangle\}$, the time evolution operator $U_{12}(t)$ is

$$U_{12}(t) = \exp\left[-iH_{12}t/\hbar\right] = \sum_{g=1}^{4} e^{-iE_g t/\hbar}|\psi_g\rangle\langle\psi_g|, \tag{4}$$

where $|\psi_g\rangle$ ($g = 1, 2, 3, 4$) are four eigenvectors of the Hamiltonian H_{12}. The corresponding eigenvalues $E_1 = -J_{12}^x - J_{12}^y - J_{12}^z$, E_2, E_3 and E_4 are given [22] by the solutions of the cubic equation of the parameter E. Here, we only focus on two typical Hamiltonians which play an important role in the process of two-qubit operation for the most representative solid-state quantum computing models. One is that all of the one-qubit parameters are switchable, for example, quantum dots in cavities [23]. However, due to technical constraints and difficulties, it was found [24] that not all the one-qubit parameters are switchable in the two-qubit operation, for instance, for spin-coupled quantum dots [14], donor-atom nuclear or electron spins [18], and quantum Hall systems [19], two one-qubit parameters such as ε_x and ε_y are switchable, but ε_z is fixed. The basic two-qubit operation with fixed ε_z is

$$\begin{aligned} U_{12}(t) \;=\; & \frac{1}{2}\big(e^{i\phi}\cos\gamma + e^{-i\phi}\cos\beta\big)I + i\frac{(1-a^2)c}{2}e^{-i\phi}\sin\beta(\sigma_{1z}+\sigma_{2z}) + \\ & + \frac{1}{2}\big(e^{-i\phi}\cos\beta - e^{i\phi}\cos\gamma\big)\ \sigma_{1z}\otimes\sigma_{2z} - i\frac{1}{2}\big(e^{i\phi}\sin\gamma + 2ac\,e^{-i\phi}\sin\beta\big)\ \sigma_{1x}\otimes\sigma_{2x} - \\ & - i\frac{1}{2}\big(e^{i\phi}\sin\gamma - 2ac\,e^{-i\phi}\sin\beta\big)\ \sigma_{1y}\otimes\sigma_{2y}\,, \end{aligned} \tag{5}$$

where $\gamma = \frac{t}{\hbar}(J_{12}^x + J_{12}^y)$, $\beta = \frac{t}{\hbar}\sqrt{4\varepsilon_z^2 + (J_{12}^x - J_{12}^y)^2}$, $\phi = \frac{t}{\hbar}J_{12}^z$, $a = 2b + \sqrt{4b^2+1}$, with $b = \varepsilon_z/(J_{12}^x - J_{12}^y)$, and $c = 1/(1+a^2)$. We also assume in eq. (5) that the parameters satisfy conditions $[2J_{12}^z \pm (J_{12}^x + J_{12}^y)]^2 \neq 4\varepsilon_z^2 + (J_{12}^x - J_{12}^y)^2$ and $2J_{12}^z \pm \sqrt{4\varepsilon_z^2 + (J_{12}^x - J_{12}^y)^2} \neq (J_{12}^x + J_{12}^y)^2$.

Examples of two-qubit measurements. – Using eq. (5), we can obtain the two-qubit operations by choosing system-specific parameters. For example, the two-qubit operation with fixed ε_z for the Heisenberg model, XXZ model, and the XY model can be obtained from eq. (5) by setting parameters $J_{mn}^x = J_{mn}^y = J_{mn}^z$, $J_{mn}^x = J_{mn}^y \neq J_{mn}^z$ and $J_{mn}^x = J_{mn}^y$, $J_{mn}^z = 0$, respectively. If all one-qubit parameters are switchable, then the two-qubit operation can be obtained from eq. (5) by only setting $\varepsilon_z = 0$. Other effective spin quantum computing models presented up to now can be reduced by single-qubit operations to eq. (5). For instance, i) the two-qubit operations of the superconducting charge qubit [12] can be reduced to eq. (5) with $J_{12}^x = J_{12}^z = 0$ by a conjugation by $(\pi/4)(\sigma_{1y}+\sigma_{2y})$ [25] on the Hamiltonian; ii) the two-qubit operation for the models in refs. [17] and [26] can be reduced to eq. (5) with $J_{12}^x = J_{12}^y = 0$ and model [12] by the conjugation by $(\pi/4)(\sigma_{1y}+\sigma_{2y})$ and conjugation by $(\pi/4)(\sigma_{1x}+\sigma_{2x})$ on the Hamiltonian of the system. Combining the basic two-qubit operations $U_{12}(t)$ with the single-qubit operations, we can obtain any desired two-qubit operation by choosing the evolution time t and the system-specific parameters $\{\varepsilon_\alpha, J_{12}^\alpha : \alpha = x, y, z\}$.

Now, let us consider the XY exchange coupling system with switchable one-qubit parameters as an example to answer how to obtain the expectation values of two-qubit measurements. If we want to obtain, for example, the expectation value r_{zy} of $\sigma_{1z}\otimes\sigma_{2y}$ in such system, we can first switch off all the one-qubit parameters $\varepsilon_{l\,\alpha}$, then let the two-qubit system evolve during the time $\tau = \hbar\pi/8J_{12}^x$ with the evolution operator $U_{12}(\tau)$, then switch off the exchange coupling and only make the first qubit have a $\pi/2$ rotation $Y_1 = \exp[-i\pi\sigma_{1y}/4]$ around the y-axis, that is

$$\rho \xrightarrow{U_{12}(\tau)} U_{12}(\tau)\rho\, U_{12}^\dagger(\tau) \xrightarrow{Y_1} Y_1 U_{12}(\tau)\rho\, U_{12}^\dagger(\tau) Y_1^\dagger = \widetilde{\rho}. \tag{6}$$

Afterwards, we make the measurement $(|1\rangle\langle 1|)_1$ on the rotated state $\widetilde{\rho}$ obtaining the probability $p = \mathrm{Tr}[\widetilde{\rho}\,(|1\rangle\langle 1|)_1] = (\sqrt{2} + r_{x0} + r_{zx})/2\sqrt{2}$, corresponding to the equivalent two-qubit measurement $-\sigma_{1x} - \sigma_{1z}\otimes\sigma_{2x}$. Because r_{x0} has been obtained by the equivalent single-qubit measurement σ_{1x}, r_{zx} is completely determined by the above result. Eight other values of

TABLE I – *Quantum operation and two-qubit measurements.*

XY model		Heisenberg model	
EM	Operations	Equivalent measurement (EM)	Operations
$\sigma_{1y}+\sigma_{1x}\sigma_{1x}$	$X_1U_1Y_1$	$\sigma_{1z}+\sigma_{2z}+\sigma_{1y}\sigma_{2x}-\sigma_{1x}\sigma_{2y}$	U_2
$-\sigma_{1z}+\sigma_{1x}\sigma_{2y}$	$Y_1U_1Y_1$	$\sigma_{1y}+\sigma_{2z}-\sigma_{1z}\sigma_{2x}-\sigma_{1x}\sigma_{2y}$	U_2X_1
$-\sigma_{1z}-\sigma_{1x}\sigma_{2z}$	$Y_1U_1Y_1X_2$	$\sigma_{2z}-\sigma_{1x}+\sigma_{1y}\sigma_{2x}-\sigma_{1z}\sigma_{2y}$	U_2Y_1
$-\sigma_{1z}-\sigma_{1y}\sigma_{2x}$	$X_1U_1X_1$	$\sigma_{1z}+\sigma_{2z}+\sigma_{1x}\sigma_{2x}+\sigma_{1y}\sigma_{2y}$	U_2Z_1
$-\sigma_{1x}-\sigma_{1y}\sigma_{2y}$	$Y_1U_1X_1$	$\sigma_{1z}-\sigma_{2x}+\sigma_{1y}\sigma_{2z}-\sigma_{1x}\sigma_{2y}$	U_2Y_2
$-\sigma_{1x}+\sigma_{1y}\sigma_{2z}$	$Y_1U_1X_1X_2$	$-\sigma_{1x}-\sigma_{2x}-\sigma_{1z}\sigma_{2y}+\sigma_{1y}\sigma_{2z}$	Y_1U_2
$\sigma_{1y}-\sigma_{1z}\sigma_{2x}$	X_1U_1	$\sigma_{1y}+\sigma_{2y}-\sigma_{1z}\sigma_{2x}+\sigma_{1x}\sigma_{2z}$	X_1U_2
$-\sigma_{1x}-\sigma_{1z}\sigma_{2y}$	Y_1U_1	$\sigma_{1y}+\sigma_{2z}+\sigma_{1z}\sigma_{2y}-\sigma_{1x}\sigma_{2x}$	$U_2X_1Z_2$
$-\sigma_{1x}+\sigma_{1z}\sigma_{2z}$	$Y_1U_1X_2$	$\sigma_{1z}-\sigma_{2x}+\sigma_{1x}\sigma_{2z}+\sigma_{1y}\sigma_{2y}$	$U_2Z_1Y_2$

equivalent two-qubit measurements for this pair can also be obtained by projecting $(|1\rangle\langle 1|)_1$ on the measured state with the quantum operations summarized in table I. Each measured value is related to the expectation values of a single-qubit and a two-qubit measurements. For a two-qubit state in this system, the above 9 two-qubit and 6 single-qubit measurements are enough to obtain 15 unknown parameters $r_{l_1l_2}$ $(l_1,l_2=0,x,y,z)$, where l_1, l_2 are not simultaneously taken as zero. The 16 matrix elements of the two-qubit state are obtained by combining the 15 parameters $r_{l_1l_2}$ with the normalization condition and finally the two-qubit state can be completely reconstructed.

The implementation of equivalent two-qubit measurements with a well-chosen two-qubit operation for a pair of coupled two-qubit systems plays a significant role in the reconstruction of a state. For the XY and Heisenberg models with switchable one-qubit parameters, the equivalent measurements $\sqrt{2}W^\dagger\sigma_{1z}W$ and $2W^\dagger\sigma_{1z}W$, to obtain the expectation values of 9 two-qubit measurements, are summarized in table I, where the non-local two-qubit operation operators U_1 and U_2 for the XY and Heisenberg models are chosen by eq. (5) with the system-specific parameters and the evolution time $\tau=\hbar\pi/8J_{12}^x$ as

$$2\sqrt{2}U_1 = \left(\sqrt{2}+1\right)I+\left(\sqrt{2}-1\right)\sigma_z^1\otimes\sigma_z^2-i\sigma_y^1\otimes\sigma_y^2-i\sigma_x^1\otimes\sigma_x^2\,, \tag{7}$$

$$2\sqrt{2}U_2 = (2-i)\,I-i\sigma_z^1\otimes\sigma_z^2-i\sigma_y^1\otimes\sigma_y^2-i\sigma_x^1\otimes\sigma_x^2\,. \tag{8}$$

The reconstructions of the qubit states in these models with fixed ε_z are the same with switchable one-qubit parameters, if the ratios $\varepsilon_z/J_{12}^x=4m/(2n-1)$ $(m,n=1,2,\cdots)$ are appropriately chosen and the operation time is $\tau=(n\hbar\pi)/(2\varepsilon_z)$.

We also find that the tomographic measurement steps for most systems can be reduced to the same steps needed for the XY model. For example, i) by choosing appropriate values of J_{12}^z, J_{12}^x (J_{12}^z, J_{12}^x, and ε_z) and operation time τ for the XXZ model with the switchable one-qubit parameters (fixed ε_z) such that $J_{12}^z\tau/\hbar=2n\pi$, $J_{12}^x\tau/\hbar=(2m-1)\pi/8$ ($J_{12}^z\tau/\hbar=2n\pi$, $J_{12}^x\tau/\hbar=(2m-1)\pi/8$, $\varepsilon_z\tau/\hbar=l\pi/2$) with $l,m,n=1,2\cdots$, then we can obtain the same two-qubit operation as for the XY model and the qubit state can be reconstructed by using the same steps as the XY model, ii) the qubit state of the superconducting charge-qubit model can also be reconstructed by using the same steps as the XY model when the parameters and evolution time are appropriately chosen [27], iii) the qubit states in the systems modelled in refs. [17] and [26] can also be reconstructed by using the same steps used for the XY model.

It should be emphasized that the different qubit measurements on the quantum state with fixed quantum operations produce different results and the quantum operations are not unique to obtain each expectation value of the measurement. In table I, we only discuss the procedure for the first-qubit POM, if we can make all single-qubit measurements, then the operation steps to obtain some of the expectation values of the multiple-qubit measurements can be decreased. For example, if we can experimentally perform the second-qubit projection $(|1\rangle\langle 1|)_2$ in the XY model, the expectation value of $\sigma_{1y} \otimes \sigma_{2z}$ can be obtained by using two steps of operations $U_{12}(\tau)$ and Y_2, that is $U^{\dagger}_{12}(\tau)Y^{\dagger}_{2}\,\sigma_{2z}\,Y_2 U_{12}(\tau) = -(\sigma_{2x} + \sigma_{1y} \otimes \sigma_{2z})/\sqrt{2}$. But four steps are needed for the first-qubit measurement. The price paid is that noise may be increased because the system is in contact with more probes.

Multi-qubit measurements. – These measurements can also be obtained by designing step-by-step single- and two-qubit operations. In principle, to obtain the expectation value of an l-qubit $(2 < l \leq n)$ measurement, we need at least $l-1$ two-qubit operations for different pairs of the l-qubit. For example, let us obtain the expectation value r_{zzx} corresponding to the three-qubit measurement $\sigma_{1z} \otimes \sigma_{2z} \otimes \sigma_{3x}$ for the XY interaction system with switchable one-qubit parameters. We can replace the first qubit by the third qubit in the two-qubit operation $U_{12}(\tau)$ and perform an operation $U_{23}(\tau)$ on the second and third qubits, followed by another operation $U_{12}(\tau)$ on the first and second qubits, followed by a $\pi/2$ rotation Y_1 about the y-axis for the first qubit, followed finally by the measurement $(|1\rangle\langle 1|)_1$. That is, an equivalent three-qubit measurement can be obtained as

$$U^{\dagger}_{23}(\tau)U^{\dagger}_{12}(\tau)Y^{\dagger}_{1}\sigma_{1z}Y_1U_{12}(\tau)U_{23}(\tau) = -\frac{1}{2\sqrt{2}}\sigma_{1x} - \frac{1}{4}\sigma_{1z} \otimes \sigma_{2y} + \frac{1}{4}\sigma_{1z} \otimes \sigma_{2z} \otimes \sigma_{3x}\,, \qquad (9)$$

where the assumption of exchange couplings are the same for all qubit pairs used. The probability of measuring $(|1\rangle\langle 1|)_1$ on the above rotated three-qubit state is $p' = (2\sqrt{2}+2r_{x00}+\sqrt{2}r_{zy0} - \sqrt{2}r_{zzx})/4\sqrt{2}$. Then r_{zzx} can be determined by the p, r_{x00}, and r_{zy0}, the latter two parameters have been obtained by single- and two-qubit measurements. We can also obtain other probabilities of the equivalent three-qubit measurements related to the expectation values of the three-qubit measurements by projecting $(|1\rangle\langle 1|)_l$ on the final operated state. For a three-qubit state, we can solve the equations for all probabilities of equivalent one-, two-, and three-qubit measurements to obtain expectation values of all measurements; finally all matrix elements of a three-qubit state are obtained by these expectation values, and the state is reconstructed. Any n-qubit measurement can be obtained in a similar way to the three-qubit measurement, then the n-qubit state can also be reconstructed.

Discussion. – In summary, we have proposed a scheme for tomographic reconstruction of qubit states for a class of promising solid-state quantum computing models. We find that elemental logic gates, such as CNOT gate, control phase gate, etc., are not necessary in this process. An appropriate non-local two-qubit operation is enough to realize this purpose. The generalization of the above discussion to the spin measurement [16, 17] is straightforward because of the equivalence between $|1\rangle\langle 1| = \frac{1}{2}(\sigma_0 - \sigma_z)$ and σ_x (σ_y) by a $\pi/2$ $(-\pi/2)$ rotation about the y (x) axes. Using the present technology, our proposal is experimentally feasible in these solid-state qubit systems. Ideally, the reconstructed qubit state ρ should satisfy the properties of the normalization, positivity, and hermiticity. However, we always deal with a limited amount of experimental data, which are also affected by noise and imperfect quantum measurements. To overcome problems due to unavoidable statistical errors, we can use the maximum-likelihood estimation of the density matrix [28] to obtain a more accurate reconstructed qubit.

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