Enhanced interferometry using squeezed thermal states and even or odd states

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We derive a general expression of the quantum Fisher information for a Mach-Zehnder interferometer, with the port inputs of an *arbitrary* pure state and a squeezed thermal state. We find that the standard quantum limit can be beaten, when even or odd states are applied to the pure-state port. In particular, when the squeezed thermal state becomes a thermal state, all the even or odd states have the same quantum Fisher information for given photon numbers. For a squeezed thermal state, optimal even or odd states are needed to approach the Heisenberg limit. As examples, we consider several common even or odd states: Fock states, even or odd coherent states, squeezed vacuum states, and single-photon-subtracted squeezed vacuum states. We also demonstrate that superprecision can be realized by implementing the parity measurement for these states.

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I. INTRODUCTION

Interferometers are extremely useful and precise measuring tools, which have been widely used to estimate very small phase changes in quantum metrology [1–13]. In general, the sensitivity of phase estimation within these settings crucially depends on the input states as well as the detection schemes. For a Mach-Zehnder interferometer (MZI) with classical-light inputs, the phase sensitivity is bounded by the standard quantum limit (SQL) (also called shot-noise limit), i.e., $1/\sqrt{n_T}$, where n_T is the total photon number [14,15]. To go beyond the SQL in MZI, entangled states (the states after the first beam splitter) are usually needed to carry the phase information.

It has been shown that, with parity measurements [16–25], the Heisenberg limit (HL), i.e., $1/n_T$, can be achieved in lossless interferometers by using maximally path-entangled states, such as NOON states [26–29] and entangled coherent states [17,30,31] (here the first beam splitter of the MZI should be replaced by devices to generate the entangled states). However, from the viewpoint of current experimental technology, it is a hard task to produce these entangled states involving a large number of photons. Due to restrictions in the photon numbers which can be reached, the estimation precision with maximally entangled states is even possibly worse than that obtained with high-intensity classical light sources [32,33]. Given this situation, finding optimal high-intensity states as inputs of MZIs is of practical relevance.

As an example of how to enhance the phase sensitivity with high-intensity input states, Caves [1] considered the inputs of a high-intensity coherent state and a low-intensity squeezed vacuum state. Since then, many theoretical and experimental studies have focused on this topic [23,33–36]. More recently, an alternative method to reach a sub-shot-noise phase uncertainty with high-intensity states has been studied in Ref. [37]. They [37] considered the configuration where the MZI is fed by a Fock state in one port and a high-intensity state, either a coherent state or a thermal state, in the other port. We note that, in these two cases, the separated-input states will be entangled to carry the phase by the first beam splitter of the MZI.

Here, we consider a more general case, in which the input state of the MZI is

$$\rho_{\text{in}} = |\psi\rangle_{a\,a}\langle\psi| \otimes \rho_b, \tag{1}$$

where $|\psi\rangle_a$ is an *arbitrary* pure state and ρ_b is a squeezed thermal state [38],

$$\rho_b = \sum_{n=0}^{\infty} \frac{\bar{n}_{\text{th}}^n}{(\bar{n}_{\text{th}}+1)^{n+1}} S_b(\xi)|n\rangle_{b\,b}\langle n|S_b^\dagger(\xi), \tag{2}$$

with the average thermal photon number $\bar{n}_{\text{th}}$. The squeezing operator is defined by $S_b(\xi) = \exp[(-\xi b^{\dagger 2} + \xi^* b^2)/2]$, with the squeezing factor $\xi = re^{i\theta}$ (hereafter we choose $\theta = 0$). The squeezed thermal state ρ_b can be generated by either injecting a thermal field to a squeezing device or passing a squeezed vacuum state through a thermal noise channel. Mathematically, the scenario under consideration covers several special cases of significance. (i) When $\bar{n}_{\text{th}} = 0$, ρ_b is a squeezed vacuum state. If we further choose $|\psi\rangle_a$ as a coherent state, then the state ρ_{in} is reduced to the input state in Ref. [1]. (ii) When $r = 0$, the squeezed thermal state is reduced to a thermal state; if we now choose $|\psi\rangle_a$ as a Fock state, then this produces a special case of the state discussed in Ref. [37].

We note that the quantum Fisher information (QFI), which is related to the quantum Cramér-Rao bound (CRB) [39], has been widely used in quantum metrology [20,30–37]. For example, the QFI has been used to characterize the phase sensitivity when the MZI is fed by a two-mode squeezed vacuum state [20], an entangled coherent state after the first beam splitter [30,31], and a coherent state together with a squeezed vacuum state [33,36]. In this work, we will also describe the phase sensitivity with the QFI. By calculating the QFI, we find that, if $|\psi\rangle_a$ is composed of either only-even or only-odd number states, the SQL for the phase-shift measurement can be beaten even when there is no squeezing in the thermal state. If there is squeezing in the thermal state, the HL can be approached for certain even and odd states. As examples, we consider Fock states, even or odd coherent states, squeezed vacuum states, and single-photon-subtracted squeezed vacuum states. Furthermore, we consider the photon-number parity measurement [16–25], which was introduced

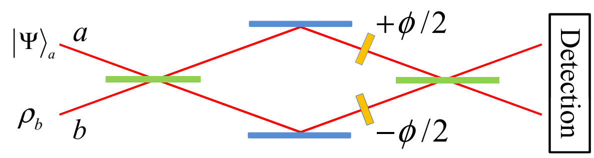

FIG. 1. (Color online) Schematic diagram of a balanced MZI, which is composed of two 50:50 beam splitters (shown in green) and two phase shifters (orange). The input state of the two ports is $|\psi\rangle_{a\,a}\langle\psi| \otimes \rho_b$, where $|\psi\rangle_a$ is an arbitrary pure state and ρ_b is a squeezed thermal state.

into optical interferometry in Refs. [16,17]. Recently, it was shown in Ref. [23] that, in two-path optical interferometry, the photon-number parity measurement achieves the quantum CRB of phase sensitivity for all proposed pure states in the field of sub-shot-noise phase sensitivity. In this work, our results indicate that, for the even and odd states considered here, the quantum CRB can also be reached by implementing the parity measurement.

II. QUANTUM FISHER INFORMATION IN MZ INTERFEROMETERS

The balanced MZI considered here is formed by two 50:50 beam splitters and two phase shifters, as shown in Fig. 1. The two input ports are fed by the state ρ_{in} given in Eq. (1). If we denote the bosonic-mode annihilation operators of the two ports as a and b, then the unitary transformation associated with this interferometer can be written as

$$U(\phi) = e^{-i(\pi/2)J_x} e^{i\phi J_z} e^{i(\pi/2)J_x} = \exp(-i\phi J_y), \tag{3}$$

where ϕ is the phase to be estimated. The operators

$$J_x = \frac{1}{2}(a^\dagger b + b^\dagger a), \tag{4a}$$

$$J_y = -\frac{i}{2}(a^\dagger b - b^\dagger a), \tag{4b}$$

$$J_z = \frac{1}{2}(a^\dagger a - b^\dagger b) \tag{4c}$$

are the usual angular momentum operators in the Schwinger representation. These satisfy the commutation relations $[J_x, J_y] = iJ_z$, $[J_y, J_z] = iJ_x$, and $[J_z, J_x] = iJ_y$.

Before addressing the case of our input state ρ_{in}, we first give the QFI for a general separable-state input: $\rho_a \otimes \rho_b$, where ρ_a and ρ_b could be either pure states or mixed states. For this input state, the output state is $\rho_{\text{out}} = U(\phi)\rho_a \otimes \rho_b U^\dagger(\phi)$, and the ultimate limit of phase sensitivity is given by the quantum CRB [39],

$$\Delta\phi_{\min} = 1/\sqrt{\mathcal{F}}, \quad \mathcal{F} = \text{Tr}(\rho_{\text{out}} G^2), \tag{5}$$

where $\mathcal{F}$ is the QFI, with G the optimal phase estimator. The symmetric logarithmic derivation G of the density matrix ρ_{out} is defined by the operator relation

$$\frac{\partial \rho_{\text{out}}}{\partial \phi} = \frac{1}{2}(\rho_{\text{out}} G + G\rho_{\text{out}}). \tag{6}$$

Utilizing the spectral decompositions $\rho_a = \sum_j p_j |\psi_j\rangle_{a\,a}\langle\psi_j|$ and $\rho_b = \sum_m q_m |\varphi_m\rangle_{b\,b}\langle\varphi_m|$, the QFI can be obtained as [14,15,40,41]

$$\mathcal{F} = \sum_k 4Q_k \langle\phi_k|J_y^2|\phi_k\rangle - \sum_{kk'} \frac{8Q_k Q_{k'}}{Q_k + Q_{k'}} |\langle\phi_k|J_y|\phi_{k'}\rangle|^2, \tag{7}$$

where $Q_k = p_j q_m$, and $\{|\phi_k\rangle = |\psi_j\rangle_a \otimes |\varphi_m\rangle_b\}$ is a complete-set basis in the two-mode Hilbert space. We should point out that the $\mathcal{F}$ does not depend on the parameter ϕ. For the pure-state case, Eq. (7) is reduced to $\mathcal{F} = 4(\langle J_y^2\rangle_{\text{in}} - \langle J_y\rangle_{\text{in}}^2)$.

When the input states on ports a and b are an arbitrary pure state $|\psi\rangle_a$ and a squeezed thermal state ρ_b, respectively, the QFI can be obtained as

$$\mathcal{F}_{|\psi\rangle_a} = \bar{n}_a + \bar{n}_b + 2\bar{n}_a\bar{n}_b + \Theta_{|\psi\rangle_a}, \tag{8}$$

where $\bar{n}_a = {}_a\langle\psi|a^\dagger a|\psi\rangle_a$ and

$$\bar{n}_b = (2\bar{n}_{\text{th}} + 1)\sinh^2(r) + \bar{n}_{\text{th}}, \tag{9}$$

are the average photon numbers for modes a and b, respectively. The $\Theta_{|\psi\rangle_a}$ in Eq. (8) is given by

$$\Theta_{|\psi\rangle_a} = \sinh(2r)(2\bar{n}_{\text{th}} + 1)\text{Re}[\langle a^2\rangle] - \frac{4\bar{n}_{\text{th}}(\bar{n}_{\text{th}} + 1)}{2\bar{n}_{\text{th}} + 1} \times [\cosh(2r) + \cos(2\varphi_0)\sinh(2r)]|\langle a\rangle|^2, \tag{10}$$

where φ_0 is defined by $\langle a\rangle = |\langle a\rangle|e^{i\varphi_0}$, and the expectation values $\langle a\rangle$ and $\langle a^2\rangle$ are taken over the state $|\psi\rangle_a$. Equations (8) and (10) show that the QFI depends not only on the average photon numbers of the two modes, but also on the statistical properties of the annihilation operator: $\langle a\rangle$ and $\langle a^2\rangle$.

Based on Eqs. (5) and (8), we can determine the QFIs corresponding to the SQL and HL. For an ideal MZI, the total photon number operator $a^\dagger a + b^\dagger b$ is a conserved quantity. If we denote the total photon number as $n_T \equiv \bar{n}_a + \bar{n}_b$, then the SQL and HL are defined by

$$\Delta\phi_{\text{SQL}} = 1/\sqrt{n_T}, \quad \Delta\phi_{\text{HL}} = 1/n_T. \tag{11}$$

In these two limits, the corresponding QFIs are

$$\mathcal{F}_{\text{SQL}} = n_T, \quad \mathcal{F}_{\text{HL}} = n_T^2. \tag{12}$$

Comparing $\mathcal{F}_{\text{SQL}}$ and $\mathcal{F}_{\text{HL}}$ with Eq. (8), we can obtain these results: To surpass the SQL, the condition $\Theta_{|\psi\rangle_a} > -2\bar{n}_a\bar{n}_b$ needs to be satisfied; while to approach the HL, the input state should impose that $\Theta_{|\psi\rangle_a} \to \bar{n}_a^2 + \bar{n}_b^2 - (\bar{n}_a + \bar{n}_b)$.

For a fixed n_T, we expect to obtain a large $\mathcal{F}_{|\psi\rangle_a}$, namely a small $\Delta\phi_{\min}$, by choosing a proper state $|\psi\rangle_a$. When $\bar{n}_a$ and $\bar{n}_b$ are fixed, this means that we need to find some input states to make $\Theta_{|\psi\rangle_a}$ as large as possible. In general, it is difficult to know how the value of $\Theta_{|\psi\rangle_a}$ depends on the statistics of mode a. However, in the following special case, we can obtain a non-negative $\Theta_{|\psi\rangle_a}$: under the condition of either $\langle a\rangle = 0$ or $\bar{n}_{\text{th}} = 0$, the second term in Eq. (10) disappears, and then we always have $\Theta_{|\psi\rangle_a} \geqslant 0$. We can check that all even or odd states satisfy this condition $\langle a\rangle = 0$. This means that even or odd states can be used as a resource to enhance the QFI.

III. PHASE SENSITIVITY AND PARITY MEASUREMENT FOR EVEN OR ODD STATES

The state ρ_b in Eq. (2) has two variables: the squeezing factor r and the thermal photon number $\bar{n}_{\text{th}}$. When $r = 0$, the squeezed thermal state is reduced to a thermal state. In this case, to surpass the SQL, a quantum state on port a is needed. Recall that for the quasiclassical (coherent) state $|\psi\rangle_a = |\alpha_0\rangle_a$, we always have $\Theta_{|\alpha_0\rangle_a} = -2\bar{n}_a\bar{n}_b(\bar{n}_b+1)(\bar{n}_b+1/2)^{-1} \leqslant -2\bar{n}_a\bar{n}_b$. In particular, when $|\psi\rangle_a$ is an even or odd state, regardless of its form, the QFI is

$$\mathcal{F}_{\mathrm{e/o}} = \bar{n}_a + \bar{n}_b + 2\bar{n}_a\bar{n}_b, \tag{13}$$

where $\bar{n}_b = \bar{n}_{\mathrm{th}}$. This result means that we can obtain a sub-shot-noise uncertainty just by mixing a thermal light with an *arbitrary* even or odd state in a MZI. In particular, when $\bar{n}_a \sim \bar{n}_b \sim n_T/2 \gg 1$, we have the approximate relation $\mathcal{F}_{\mathrm{e/o}} \propto n_T^2/2$, which is of the same scale of $\mathcal{F}_{\mathrm{HL}} = n_T^2$. The situation for $r > 0$ is more complicated. The QFI in this case depends on the form of $|\psi\rangle_a$. Below we will consider several common even or odd states: the Fock state $|N\rangle_a$, even or odd coherent states $|\alpha_{0\pm}\rangle_a$, the squeezed vacuum state $|\xi_0\rangle_a$, and the single-photon-subtracted squeezed vacuum state $|\zeta(1)\rangle_a$. To see the advantages of even or odd states, we first consider the coherent state $|\alpha_0\rangle_a$ as a reference.

A. Coherent state $|\alpha_0\rangle_a$

Suppose that the port a is fed by a coherent state $|\alpha_0\rangle_a$ with $\alpha_0 = |\alpha_0|e^{i\theta_c}$. When $\theta_c = 0$ the QFI can be obtained from Eqs. (8) and (10) as

$$\mathcal{F}_{|\alpha_0\rangle_a} = \frac{e^{2r}}{2\bar{n}_{\mathrm{th}}+1}\bar{n}_a + \bar{n}_b, \tag{14}$$

where $\bar{n}_a = \bar{n}_{|\alpha_0\rangle_a} = |\alpha_0|^2$ and $\bar{n}_b = (2\bar{n}_{\mathrm{th}}+1)\sinh^2(r) + \bar{n}_{\mathrm{th}}$. We note that Eq. (14) has been used to analyze the effects of linear photon losses with inputs of coherent states and squeezed vacuum states [1,33,36], and the quantum CRB can be obtained by measuring a symmetric logarithmic derivative [34]. When $0 \leqslant \bar{n}_{\mathrm{th}} < (e^{2r}-1)/2$, we have $\mathcal{F}_{|\alpha_0\rangle_a} > \mathcal{F}_{\mathrm{SQL}}$, then the SQL is surpassed. By analyzing the function

$$\Theta_{|\alpha_0\rangle_a} = |\alpha_0|^2\left[\frac{\sinh(2r) - 4\bar{n}_{\mathrm{th}}(\bar{n}_{\mathrm{th}}+1)\cosh(2r)}{2\bar{n}_{\mathrm{th}}+1}\right], \tag{15}$$

we find $\Theta_{|\alpha_0\rangle_a} > 0$ under the condition $0 \leqslant \bar{n}_{\mathrm{th}} < (\sqrt{1+\tanh(2r)}-1)/2$. This relation shows that, in the small thermal photon regime, the input with coherent state and squeezed thermal state can surpass the QFI $\mathcal{F}_{\mathrm{e/o}}$. However, a disadvantage in this case is that we cannot increase the total photon number by adding the thermal photon number.

B. Fock state $|N\rangle_a$

In the case of the Fock state $|N\rangle_a$, the average photon number in mode a is N. In this case, we have

$$\Theta_{|N\rangle_a} = 0, \tag{16}$$

and the QFI is [37]

$$\mathcal{F}_{|N\rangle_a} = N + \bar{n}_b + 2N\bar{n}_b, \tag{17}$$

which is independent of the values of r, for a fixed average photon number $\bar{n}_b$. This means that the Fock state input can naturally surpass the SQL. In particular, we have $\mathcal{F}_{|N\rangle_a} > \mathcal{F}_{|\alpha_0\rangle_a}$ when $\bar{n}_{\mathrm{th}} > (\sqrt{1+\tanh(2r)}-1)/2$. Therefore, for a sufficiently large thermal photon number $\bar{n}_{\mathrm{th}}$, the Fock state is better than coherent states for the estimation of phase uncertainty in our case.

The quantum CRB $\Delta\phi_{\min}$ can be reached in this case by detecting the photon number parity on one of the output modes. For mode a, the photon-number parity operator is

$$\Pi_a = (-1)^{a^\dagger a}. \tag{18}$$

We can obtain the expectation value of the parity operator by calculating the Wigner function of the output state [23]. For the input state $|N\rangle_{a\,a}\langle N| \otimes \rho_b$, the Wigner function is

$$W_{\mathrm{in}}(\alpha,\beta) = W_{|N\rangle_a}(\alpha)\, W_{\rho_b}(\beta). \tag{19}$$

Here $W_{|N\rangle_a}(\alpha)$ and $W_{\rho_b}(\beta)$ are, respectively, the Wigner functions for the Fock state and the squeezed thermal state ($\theta = 0$) [42]:

$$W_{|N\rangle_a}(\alpha) = \frac{2}{\pi}(-1)^N \exp(-2|\alpha|^2)L_N(4|\alpha|^2), \tag{20a}$$

$$W_{\rho_b}(\beta) = \frac{2}{\pi(2\bar{n}_{\mathrm{th}}+1)}\exp\left[-\frac{2\left(e^{2r}\beta_r^2 + e^{-2r}\beta_i^2\right)}{2\bar{n}_{\mathrm{th}}+1}\right], \tag{20b}$$

where $L_N(x)$ is the Laguerre polynomial of the Nth order, β_r and β_i are the real and imaginary parts of β, respectively. By making the replacements

$$\alpha \to \tilde{\alpha} = \alpha\cos(\phi/2) + \beta\sin(\phi/2), \tag{21a}$$

$$\beta \to \tilde{\beta} = -\alpha\sin(\phi/2) + \beta\cos(\phi/2), \tag{21b}$$

in $W_{\mathrm{in}}(\alpha,\beta)$, we obtain the Wigner function of the output state as (see the Appendix for details)

$$W_{\mathrm{out}}(\alpha,\beta) = W_{\mathrm{in}}(\tilde{\alpha},\tilde{\beta}). \tag{22}$$

The expectation value of the parity operator can be written as

$$\langle\Pi_a\rangle_{|N\rangle_a} = \frac{\pi}{2}\int_{-\infty}^{\infty} W_{\mathrm{out}}(0,\beta)\, d^2\beta, \tag{23}$$

where the Wigner function at the origin of the phase space for mode a is found to be

$$W_{\mathrm{out}}(0,\beta) = \frac{4(-1)^N}{\pi^2(2\bar{n}_{\mathrm{th}}+1)}\exp\left(-A\beta_r^2 - B\beta_i^2\right) \times L_N(4\sin^2(\phi/2)|\beta|^2) \tag{24}$$

with

$$A = 2\left[\frac{e^{2r}}{2\bar{n}_{\mathrm{th}}+1}\cos^2(\phi/2) + \sin^2(\phi/2)\right], \tag{25a}$$

$$B = 2\left[\frac{e^{-2r}}{2\bar{n}_{\mathrm{th}}+1}\cos^2(\phi/2) + \sin^2(\phi/2)\right]. \tag{25b}$$

It is a difficult task to write out the explicit form of Eq. (23) for general N. However, for small N, the explicit form is accessible. When $N = 0,1,2$, we have

$$\langle\Pi_a\rangle_{|0\rangle_a} = \frac{2}{(2\bar{n}_{\mathrm{th}}+1)\sqrt{AB}}, \tag{26a}$$

$$\langle\Pi_a\rangle_{|1\rangle_a} = \frac{2\left[(A+B)(1-\cos\phi) - AB\right]}{(2\bar{n}_{\mathrm{th}}+1)(AB)^{3/2}}, \tag{26b}$$

$$\langle\Pi_a\rangle_{|2\rangle_a} = \frac{2\Xi}{(2\bar{n}_{\mathrm{th}}+1)(AB)^{5/2}}, \tag{26c}$$

where

$$\Xi = AB[AB - 2(A+B)(1-\cos\phi)] + 2\sin^4(\phi/2)(3A^2 + 2AB + 3B^2). \tag{27}$$

Using the result of $\langle \Pi_a \rangle$, we can obtain the fluctuation of the parity operator as $\Delta\Pi_a = \sqrt{\langle \Pi_a^2 \rangle - \langle \Pi_a \rangle^2} = \sqrt{1 - \langle \Pi_a \rangle^2}$. According to the error propagation formula

$$\Delta\phi_{\min} = \min\left[\frac{\Delta\Pi_a}{|d\langle \Pi_a \rangle / d\phi|}\right], \tag{28}$$

we can analytically show that the quantum CRB $\Delta\phi_{\min} = 1/\sqrt{\mathcal{F}_{|N\rangle_a}}$ (for $N = 0,1,2$) can be reached in the limit $\phi \to 0$. This result indicates that the quantum CRB can be reached by implementing the parity measurement. We also numerically checked that the quantum CRB can be reached with parity measurement for larger values of N.

C. Even or odd coherent states $|\alpha_{0\pm}\rangle_a$

We now turn to the case of even or odd coherent states (also called Schrödinger's cat states). The definition of even or odd coherent states is

$$|\alpha_{0\pm}\rangle_a = \mathcal{N}_\pm^{\alpha_0}(|\alpha_0\rangle \pm |-\alpha_0\rangle), \tag{29}$$

where $\mathcal{N}_\pm^{\alpha_0} = 1/[2(1 \pm e^{-2|\alpha_0|^2})]^{1/2}$ are the normalization constants. Without loss of generality, hereafter we assume that α_0 is real. According to Eq. (10), we obtain

$$\Theta_{|\alpha_{0\pm}\rangle_a} = \alpha_0^2(2\bar{n}_{\text{th}} + 1)\sinh(2r). \tag{30}$$

The corresponding QFIs are then given by

$$\mathcal{F}_{|\alpha_{0\pm}\rangle_a} = (2\bar{n}_{\text{th}} + 1)\left[\alpha_0^2\sinh(2r) + \bar{n}_{|\alpha_{0\pm}\rangle_a}\cosh(2r)\right] + \bar{n}_b, \tag{31}$$

where the average photon numbers are

$$\bar{n}_{|\alpha_{0+}\rangle_a} = \alpha_0^2 \tanh\left(\alpha_0^2\right), \tag{32a}$$

$$\bar{n}_{|\alpha_{0-}\rangle_a} = \alpha_0^2 \coth\left(\alpha_0^2\right). \tag{32b}$$

When $\alpha_0 \geqslant 2$, we have the approximate relation $\bar{n}_{|\alpha_{0+}\rangle_a} \simeq \bar{n}_{|\alpha_{0-}\rangle_a} \simeq \alpha_0^2$, and then the QFIs are approximately reduced to

$$\mathcal{F}'_{|\alpha_{0+}\rangle_a} \simeq \mathcal{F}'_{|\alpha_{0-}\rangle_a} \simeq e^{2r}(2\bar{n}_{\text{th}} + 1)\bar{n}_{|\alpha_{0\pm}\rangle_a} + \bar{n}_b, \tag{33}$$

We see from $\mathcal{F}'_{|\alpha_{0\pm}\rangle_a}$ that a sub-shot-noise uncertainty can be obtained as long as r and $\bar{n}_{\text{th}}$ are not simultaneously zero. Furthermore, the HL can be approached, if $\bar{n}_{\text{th}}$ satisfies the relation

$$\bar{n}_a(2\bar{n}_{\text{th}} + 1)\sinh(2r) = \bar{n}_a^2 + \bar{n}_b^2 - (\bar{n}_a + \bar{n}_b). \tag{34}$$

According to $\mathcal{F}'_{|\alpha_{0\pm}\rangle_a}$, we can obtain a large QFI by increasing the average thermal photon number $\bar{n}_{\text{th}}$ for a fixed squeezing parameter r. This point is different from the coherent-state case [Eq. (14)], in which a large $\bar{n}_{\text{th}}$ may lead to $\mathcal{F}_{|\alpha_0\rangle_a} < \mathcal{F}_{\text{SQL}}$. It should be pointed out that these states are difficult to be created with high photon numbers under current experimental conditions, but this might be possible in the future. The Wigner function of the even or odd coherent state is [43]

$$W_{|\alpha_{0\pm}\rangle_a}(\alpha) = \frac{e^{-2|\alpha|^2}}{\pi(1 \pm e^{-2\alpha_0^2})}[e^{-2\alpha_0^2 + 4\alpha_r\alpha_0} + e^{-2\alpha_0^2 - 4\alpha_r\alpha_0} \pm 2\cos(4\alpha_i\alpha_0)], \tag{35}$$

where α_r and α_i are the real and imaginary parts of α, respectively. In terms of Eq. (35), we can obtain the expectation value of the parity operator, with the same method in the above section, as

$$\langle \Pi_a \rangle_{|\alpha_{0\pm}\rangle_a} = \frac{2\left[e^{-2\alpha_0^2}\exp\left(\frac{C^2}{4A}\right) \pm \exp\left(-\frac{C^2}{4B}\right)\right]}{(2\bar{n}_{\text{th}} + 1)(1 \pm e^{-2\alpha_0^2})\sqrt{AB}}, \tag{36}$$

where A and B have been given in Eq. (25), and

$$C = 4\alpha_0 \sin(\phi/2). \tag{37}$$

Using Eqs. (28) and (36), we can check that the quantum CRB can be achieved when $\phi \to 0$, i.e., $\Delta\phi_{\min} = 1/\sqrt{\mathcal{F}_{|\alpha_{0\pm}\rangle_a}}$. Similar analysis can be done for finite-dimensional even and odd coherent states [44].

D. Squeezed vacuum state $|\xi_0\rangle_a$

When the input state on port a is a squeezed vacuum state $|\xi_0\rangle_a = S_a(\xi_0)|0\rangle_a$, with $\xi_0 = Re^{i\theta_0}$, which is an even state. The average photon number for this state is $\bar{n}_a = \bar{n}_{|\xi_0\rangle_a} = \sinh^2(R)$. Based on the optimal phase-matching condition $\theta_0 = \pi$ [41], we obtain

$$\Theta_{|\xi_0\rangle_a} = (\bar{n}_{\text{th}} + 1/2)\sinh(2R)\sinh(2r). \tag{38}$$

The QFI in this case is

$$\mathcal{F}_{|\xi_0\rangle_a} = (\bar{n}_{\text{th}} + 1/2)\cosh[2(R + r)] - 1/2. \tag{39}$$

When $\bar{n}_a \gg 1$, Eq. (39) is approximately reduced to

$$\mathcal{F}'_{|\xi_0\rangle_a} \simeq e^{2r}(2\bar{n}_{\text{th}} + 1)\bar{n}_{|\xi_0\rangle_a} + \bar{n}_b, \tag{40}$$

which has a similar form as in Eq. (33).

Using the Wigner function of the squeezed vacuum state [42]

$$W_{|\xi_0\rangle_a}(\alpha) = \frac{2}{\pi}\exp\left[-2\left(e^{-2R}\alpha_r^2 + e^{2R}\alpha_i^2\right)\right], \tag{41}$$

the expected signal of the parity measurement can be obtained as

$$\langle \Pi_a \rangle_{|\xi_0\rangle_a} = \frac{2}{(2\bar{n}_{\text{th}} + 1)\sqrt{A_1 B_1}}, \tag{42}$$

where we introduce

$$A_1 = \frac{2e^{2r}}{2\bar{n}_{\text{th}} + 1}\cos^2(\phi/2) + 2e^{-2R}\sin^2(\phi/2), \tag{43a}$$

$$B_1 = \frac{2e^{-2r}}{2\bar{n}_{\text{th}} + 1}\cos^2(\phi/2) + 2e^{2R}\sin^2(\phi/2). \tag{43b}$$

Based on Eqs. (28) and (42), we obtain the quantum CRB for phase estimation $\Delta\phi_{\min} = 1/\sqrt{\mathcal{F}_{|\xi_0\rangle_a}}$ in the limit $\phi \to 0$.

E. Single-photon-subtracted squeezed vacuum state $|\zeta(1)\rangle_a$

The single-photon-subtracted squeezed vacuum state is defined by

$$|\zeta(1)\rangle_a = \mathcal{N}_1 a S_a(\zeta)|0\rangle_a, \quad (44)$$

where $\zeta = R' e^{i\theta'}$, and $\mathcal{N}_1 = 1/\sinh(R')$ is the normalization constant. Up to a trivial phase factor, the state $|\zeta(1)\rangle_a$ is equivalent to the squeezed single-photon state $S_a(\zeta)|1\rangle_a$, which has almost unit fidelity to a superposed coherent state of small amplitude [45]. We note that a state-input with squeezed single-photon state and coherent state has been studied in Ref. [46]. When $\theta' = \pi$, we have

$$\Theta_{|\zeta(1)\rangle_a} = 3(\bar{n}_{\rm th} + 1/2)\sinh(2R')\sinh(2r), \quad (45)$$

and the average photon number $\bar{n}_a = \bar{n}_{|\zeta(1)\rangle_a} = 1 + 3\sinh^2(R')$. According to Eq. (8), the QFI can be obtained as

$$\mathcal{F}_{|\zeta(1)\rangle_a} = 3(\bar{n}_{\rm th} + 1/2)\cosh[2(R'+r)] - 1/2, \quad (46)$$

and the quantum CRB is $\Delta\phi_{\min} = 1/\sqrt{\mathcal{F}_{|\zeta(1)\rangle_a}}$. When implementing the parity detection, we obtain

$$\langle\Pi_a\rangle_{|\zeta(1)\rangle_a} = \frac{2}{(2\bar{n}_{\rm th}+1)\sqrt{A_1B_1}}\left(\frac{A_2}{2A_1} + \frac{B_2}{2B_1} - 1\right), \quad (47)$$

where A_1 and B_1 are given by Eq. (43) with the replacement $R \to R'$, A_2 and B_2 are defined by

$$A_2 = 4e^{-2R'}\sin^2(\phi/2), \qquad B_2 = 4e^{2R'}\sin^2(\phi/2). \quad (48)$$

In the derivation of Eq. (47), we have used the Wigner function of the single-photon-subtracted squeezed vacuum state [47]

$$W_{|\zeta(1)\rangle_a}(\alpha) = \frac{2}{\pi}\exp\left[-2\left(e^{-2R'}\alpha_r^2 + e^{2R'}\alpha_i^2\right)\right] \times\left[4\left(e^{-2R'}\alpha_r^2 + e^{2R'}\alpha_i^2\right) - 1\right]. \quad (49)$$

In terms of Eqs. (28) and (47), we find that the best phase sensitivity, i.e., the quantum CRB $\Delta\phi_{\min} = 1/\sqrt{\mathcal{F}_{|\zeta(1)\rangle_a}}$, can be reached in the limit $\phi \to 0$.

F. Quantum CRB versus the thermal photon number $\bar{n}_{\rm th}$

We now have obtained the phase sensitivity for a MZI, which is fed by various even or odd states in mode a and a squeezed thermal state in mode b. To better show these results. Table 1 lists the average photon number, the function $\Theta_{|\psi\rangle_a}$, and the quantum Fisher information $\mathcal{F}_{|\psi\rangle_a}$ for various even or odd states: Fock states, even or odd coherent states, squeezed vacuum states, and single-photon-subtracted squeezed vacuum states. The phase uncertainties can be obtained by $\Delta\phi_{\min} = 1/\sqrt{\mathcal{F}_{|\psi\rangle_a}}$. For the states considered here, the SQL can be surpassed because of $\mathcal{F}_{|\psi\rangle_a} > n_T$, and the phase sensitivity $\Delta\phi_{\min}$ can be reached with the parity measurement.

To clearly see the behaviors of the phase sensitivity for various input states, in Fig. 2 we plot the quantum CRB $\Delta\phi_{\min} = 1/\sqrt{\mathcal{F}_{|\psi\rangle_a}}$ as a function of $\bar{n}_{\rm th}$. Here we fix the average photon number in mode a: $\bar{n}_a \simeq 4$. Even or odd states used here with this photon number are accessible with current or near-future experiments. For the Fock state $|N\rangle_a$, $\bar{n}_a \simeq 4$ means $N = 4$. In experiments, Fock states with up to two [48] and three [49] running photons in optics have recently been generated. For even or odd coherent states $|\alpha_{0\pm}\rangle_a$, $\bar{n}_a \simeq 4$ corresponds to $\alpha_0 \simeq 2$, which is very close to current experimental realizations. A recent experiment [50] reported the generation of an optical coherent-state superposition with $\alpha_0 \simeq \sqrt{2.6}$. For squeezed vacuum and single-photon states, we have $R \simeq 1.45$ and $R' \simeq 0.94$, satisfying the relation $\sinh^2(R) = 1 + 3\sinh^2(R')$ or $\cosh(2R) = 3\cosh(2R')$. The squeezing factor $R = 1.45$ corresponds to a squeezing of 12.6 dB, which can be realized with current experimental techniques [51].

As shown in Fig. 2, when increasing $\bar{n}_{\rm th}$, the phase uncertainties for even- or odd-state inputs decrease monotonically. In some parameters, the phase uncertainties can approach the HL. However, for the coherent-state case, the phase uncertainty first increases and then gradually decreases, and it ultimately approaches the SQL when increasing $\bar{n}_{\rm th}$. When $r = 0$ (ρ_b becomes the thermal state), we can see from Fig. 2(a) that the values of the optimal phase estimation (corresponding to the quantum CRB) satisfy $\Delta\phi_{|\alpha_{0\pm}\rangle_a} = \Delta\phi_{|\xi_0\rangle_a} = \Delta\phi_{|\zeta(1)\rangle_a} \leqslant \Delta\phi_{|\alpha\rangle_a}$, where the last equality takes place if and only if $\bar{n}_{\rm th} = 0$ (i.e., ρ_b is a vacuum state). When ρ_b is a thermal state, the values of the optimal phase estimation for all even or odd states are the same, which can beat the SQL. This point can be seen from the expression of $\mathcal{F}_{\rm e/o}$. However, the phase sensitivity for a coherent-state input cannot beat the SQL, because two classical states cannot be entangled by a beam splitter [52].

When $r > 0$, as shown in Figs. 2(b) and 2(c), the phase sensitivities for all even or odd states always beat the SQL, and approach the HL when $\bar{n}_b \sim \bar{n}_a$. In the coherent state case, the phase sensitivity can beat the SQL when $\bar{n}_{\rm th}$ is not too large. This is because $\mathcal{F}_{|\alpha_0\rangle_a} > \mathcal{F}_{\rm SQL}$ when $0 \leqslant \bar{n}_{\rm th} < (e^{2r} - 1)/2$. In addition, for a given state in Fig. 2 panels, from top to bottom, the values of the corresponding points with the same $\bar{n}_{\rm th}$ decrease because the squeezing part will increase the total photon number.

TABLE I. The average photon number $\bar{n}_a$ in mode a, the function $\Theta_{|\psi\rangle_a}$, and the QFI $\mathcal{F}_{|\psi\rangle_a}$ for the MZI, when the two input ports are fed by an even or odd state $|\psi\rangle_a$ in mode a and a squeezed thermal state ρ_b in mode b. Here, the $|\psi\rangle_a$ could be either a Fock state $|N\rangle_a$, even or odd coherent states $|\alpha_{0\pm}\rangle_a$ (here we assume α_0 is real), squeezed vacuum state $|\xi_0\rangle_a$ with $\xi_0 = -R$, or a single-photon-subtracted squeezed vacuum state $|\zeta(1)\rangle_a$ with $\zeta = -R'$. The average photon number in mode b is $\bar{n}_b = (2\bar{n}_{\rm th}+1)\sinh^2(r) + \bar{n}_{\rm th}$, and the total photon number is $n_T = \bar{n}_a + \bar{n}_b$. The phase uncertainties can be obtained by $\Delta\phi_{\min} = 1/\sqrt{\mathcal{F}_{|\psi\rangle_a}}$. For the states considered here, the SQL can be surpassed because of $\mathcal{F}_{|\psi\rangle_a} > n_T$, and the phase sensitivity $\Delta\phi_{\min}$ can be reached with the parity measurement.

Input states $\vert\psi\rangle_a$	$\bar{n}_a$	$\Theta_{\vert\psi\rangle_a}$	$\mathcal{F}_{\vert\psi\rangle_a}$
$\vert N\rangle_a$	N	0	$N + (2N+1)[(2\bar{n}_{\rm th}+1)\sinh^2(r) + \bar{n}_{\rm th}] > n_T$
$\vert\alpha_{0\pm}\rangle_a$	α_0^2 for $\alpha_0 \geqslant 2$	$\alpha_0^2(2\bar{n}_{\rm th}+1)\sinh(2r)$	$\alpha_0^2(2\bar{n}_{\rm th}+1)\sinh(2r) > n_T$
$\vert\xi_0\rangle_a$ with $\xi_0 = -R$	$\sinh^2(R)$	$(\bar{n}_{\rm th}+1/2)\sinh(2R)\sinh(2r)$	$(\bar{n}_{\rm th}+1/2)\cosh[2(R+r)] - 1/2 > n_T$
$\vert\zeta(1)\rangle_a$ with $\zeta = -R'$	$1 + 3\sinh^2(R')$	$3(\bar{n}_{\rm th}+1/2)\sinh(2R')\sinh(2r)$	$3(\bar{n}_{\rm th}+1/2)\cosh[2(R'+r)] - 1/2 > n_T$

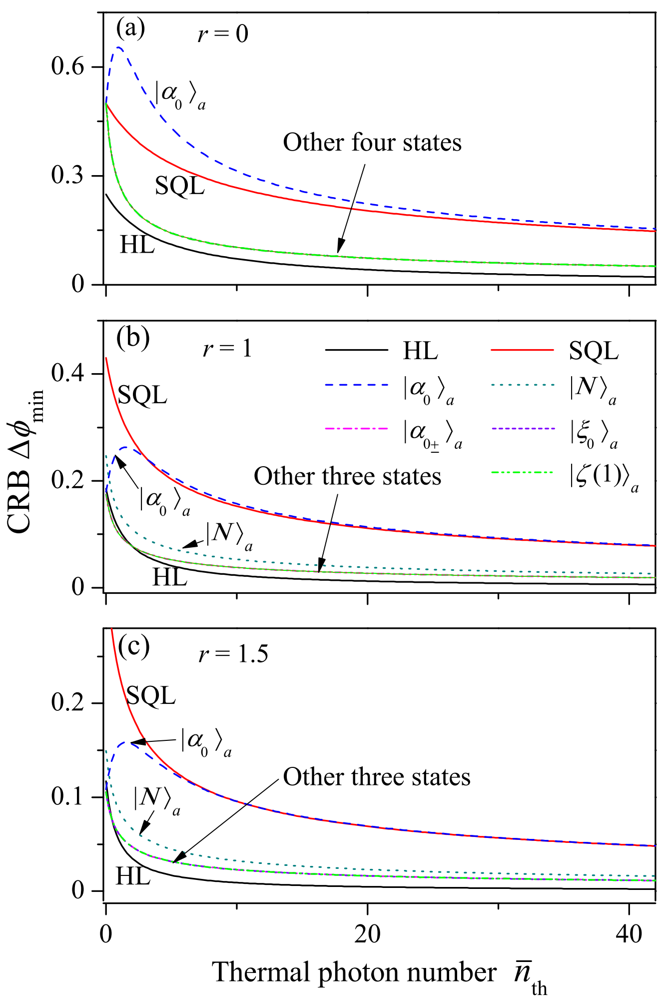

FIG. 2. (Color online) The quantum CRB $\Delta\phi_{\min} \equiv (\mathcal{F}_{|\psi\rangle_a})^{-1/2}$ as a function of the thermal photon number $\bar{n}_{\text{th}}$ for various input states in mode a: coherent state $|\alpha_0\rangle_a$ (blue dashed curve), Fock state $|N\rangle_a$ (cadet blue dotted curve), even or odd coherent states $|\alpha_{0\pm}\rangle_a$ (magenta dash-dotted curve), squeezed vacuum state $|\xi_0\rangle_a$ (purple short-dashed curve), and single-photon-subtracted squeezed vacuum state $|\zeta(1)\rangle_a$ (green dash-dot-dotted curve). Here, the average photon number for mode a is $\bar{n}_a \simeq 4$, the mode b is in the squeezed thermal state ρ_b with the squeezing factor (a) $r = 0$, (b) $r = 1$, (c) $r = 1.5$. The HL (black solid curve) and SQL (red solid curve) are shown for comparison. Note that the curves for the last four states exactly overlap with each other in panel (a), and the curves for the last three states approximately overlap with each other in panels (b) and (c).

IV. CONCLUSION

In summary, we have studied the QFI for a MZI, which is fed by an arbitrary pure state and a squeezed thermal state. We have shown that, when the input pure state is an even or odd state, the phase sensitivity can be drastically improved. By mixing the even or odd states and a high-intensity thermal light, a sub-shot-noise phase uncertainty can be obtained, and this uncertainty only depends on the total photon number, regardless of the form of the even or odd states. For the case of a squeezed thermal state, the sensitivity can be further improved even approaching the HL when the pure-state port is fed by an even or odd state. As examples, we considered Fock states, even or odd coherent states, squeezed vacuum states, and single-photon-subtracted squeezed vacuum states. Furthermore, we have demonstrated that the superprecision given by the quantum CRB can be realized by implementing the parity measurement.

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APPENDIX: DERIVATION OF EQ. (22)

In this Appendix, we present a detailed derivation of Eq. (22). The Wigner function of the input state ρ_{in} is defined by

$$W_{\text{in}}(\alpha,\beta) = \frac{4}{\pi^2}\text{Tr}[\rho_{\text{in}} D_b(\beta) D_a(\alpha)(-1)^{a^\dagger a + b^\dagger b} D_a^\dagger(\alpha) D_b^\dagger(\beta)], \tag{A1}$$

where the displacement operators are defined by $D_a(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$ and $D_b(\beta) = e^{\beta b^\dagger - \beta^* b}$.

For the input state ρ_{in}, the output state is $\rho_{\text{out}} = U(\phi)\rho_{\text{in}} U^\dagger(\phi)$, where $U(\phi) = e^{-i\phi J_y}$ is the unitary evolution operator of the MZI. The Wigner function of the output state is

$$\begin{aligned} W_{\text{out}}(\alpha,\beta) &= \frac{4}{\pi^2}\text{Tr}[\rho_{\text{out}} D_b(\beta) D_a(\alpha)(-1)^{a^\dagger a + b^\dagger b} D_a^\dagger(\alpha) D_b^\dagger(\beta)] \\ &= \frac{4}{\pi^2}\text{Tr}[\rho_{\text{in}} \Lambda(\phi,\alpha,\beta)(-1)^{a^\dagger a + b^\dagger b} \Lambda^\dagger(\phi,\alpha,\beta)] \end{aligned} \tag{A2}$$

with

$$\Lambda(\phi,\alpha,\beta) = U^\dagger(\phi) D_b(\beta) D_a(\alpha) U(\phi). \tag{A3}$$

In Eq. (A2), we have used the commutation relation $[a^\dagger a + b^\dagger b, J_y] = 0$.

In terms of the relations

$$U^\dagger(\phi) a U(\phi) = a\cos(\phi/2) - b\sin(\phi/2), \tag{A4a}$$

$$U^\dagger(\phi) b U(\phi) = a\sin(\phi/2) + b\cos(\phi/2), \tag{A4b}$$

we obtain

$$\Lambda(\phi,\alpha,\beta) = D_a(\tilde{\alpha}) D_b(\tilde{\beta}), \tag{A5}$$

where we introduce

$$\tilde{\alpha} = \alpha\cos(\phi/2) + \beta\sin(\phi/2), \tag{A6a}$$

$$\tilde{\beta} = -\alpha\sin(\phi/2) + \beta\cos(\phi/2). \tag{A6b}$$

Therefore, the Wigner function of the output state can be expressed as $W_{\text{out}}(\alpha,\beta) = W_{\text{in}}(\tilde{\alpha},\tilde{\beta})$.

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