Supplemental Material for *Ground State Electroluminescence*

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In this supplemental material we first describe in detail our model of the weak system-environment interaction described by transition rates between system eigenstates. We then show how to derive analytical expressions for these transition rates perturbative in the normalized light-matter coupling. We use these system transition rates to derive analytical formula for the photonic emission rates for both ground state electroluminescence and standard electroluminescence processes. We conclude by studying the dependency of the photonic emission on the chemical potential.

System-environment interaction

The quantum system analysed in this work is intrinsically open: electrons are allowed to enter and leave via couplings with two external reservoirs, and photons can escape the cavity through the finite mirror reflectivity. In order to model the coupling of the system with its environment, we introduce Hamiltonians that describe the coupling with the source (in) and drain (out) electronic reservoirs and with the continuum of extra-cavity electromagnetic modes

$$H_{\rm in}=\sum_{k,j=e,g}\lambda_{\rm in}^{k}c_{k}^{\rm in}\left|j\right\rangle\left\langle s\right|+\text{h.c.},\quad H_{\rm out}=\sum_{k,j=e,g}\lambda_{\rm out}^{k}c_{k}^{\rm out}\left|j\right\rangle\left\langle s\right|+\text{h.c.},\quad H_{\rm cav}=\sum_{k}\lambda_{\rm cav}^{k}(a+a^{\dagger})(\alpha_{k}+\alpha_{k}^{\dagger}),\tag{1}$$

where $c_k^{\rm in}$, $c_k^{\rm out}$, and α_k are respectively fermionic annihilation operators for electrons in the in and out reservoirs and bosonic annihilation operators for free space photons, each one indexed by a generic index k, whose dimensionality and physical interpretation will depend upon the specific implementation considered. The theory of open quantum systems [1–3] teaches us that, under suitable weak couplings and Markovian approximations, we can integrate out the reservoirs, leading to a master equation for the system density matrix ρ in the Lindblad form

$$\dot{\rho}=-\frac{i}{\hbar}[H,\rho]+\mathcal{L}_{\rm in}(\rho)+\mathcal{L}_{\rm out}(\rho)+\mathcal{L}_{\rm cav}(\rho),\tag{2}$$

where $\mathcal{L}_j(\rho)=\sum_{\alpha\neq\beta}\Gamma_j^{\alpha\to\beta}\mathcal{D}[|\alpha\rangle\langle\beta|](\rho)$, $\mathcal{D}[O](\rho)=\frac{1}{2}(2O\rho O^{\dagger}-\rho O^{\dagger}O-O^{\dagger}O\rho)$, and the coefficients $\Gamma_j^{\alpha\to\beta}$ are just the transition rates between (closed) system states $|\alpha\rangle$ and $|\beta\rangle$ calculated using the Fermi golden rule for the interaction Hamiltonian H_j. Explicitly we can thus write

$$\Gamma_{j}^{\alpha\to\beta}\;=\;\frac{2\pi}{\hbar}\sum_{f}|\langle\alpha,i|\,H_{j}\,|\beta,f\rangle\,|^{2}\delta(\Delta),\tag{3}$$

where i is the initial state of the reservoir, the sum is over all its final states, and Δ is the energy difference between the initial and final states of the system and reservoir. Note that while in the weak coupling regime, we recover, for opportune values of the source chemical potential, state-independent transition rates

$$\begin{aligned}\Gamma_{\rm in}\;&\equiv\;\lim_{\eta\to0}\Gamma_{\rm in}^{s\to g}=\lim_{\eta\to0}\Gamma_{\rm in}^{s\to e}\\ \Gamma_{\rm out}\;&\equiv\;\lim_{\eta\to0}\Gamma_{\rm out}^{g\to s}=\lim_{\eta\to0}\Gamma_{\rm out}^{e\to s}\\ \Gamma_{\rm cav}\;&\equiv\;\lim_{\eta\to0}\Gamma_{\rm cav}^{1\to0}.\end{aligned}\tag{4}$$

In the strong coupling regime this is not possible anymore: as the states are mixed light-matter excitations, the transition rates will in general depend on both the initial and final states. In order to obtain numerical estimates for the populations and the photonic emission rates, we numerically solved Eq. (2) over a truncated bosonic Hilbert space for the cavity. The following sections of this Supplemental Material will detail how we used the transition rates calculated in Eq. (3) to obtain the perturbative analytical estimates for the emission rates shown in Fig. 3 of the main text.

Perturbative transition rates

In order to calculate the transition rates in Eq. (3), we need matrix elements of the form $\langle\alpha|\, O\,|\beta\rangle$, where $|\alpha\rangle$ and $|\beta\rangle$ are initial and final states $\{|s,0\rangle, |s,1\rangle, |G\rangle, |\pm\rangle\}$, and the operator O is either one of $\{|e\rangle\langle s|, |g\rangle\langle s|, a\}$ or one of their hermitic conjugates. We thus need to express the coupled states of the Rabi system, $|G\rangle$ and $|\pm\rangle$, in terms of the uncoupled states, $|g,n\rangle$ and $|e,n\rangle$, where n is the number of cavity photons. To accomplish this, introducing the Pauli matrices $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ and $\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|$, we start by rewriting the one electron sector of the system Hamiltonian, Eq. (1) of the main text, in the form of a dimensionless Rabi Hamiltonian [4]

$$\frac{H_{\rm Rabi}}{\hbar\omega_C} = a^\dagger a + \frac{1}{2}\sigma_z + \eta\sigma_x(a+a^\dagger). \tag{5}$$

Using standard perturbation theory it is easy to find the required expressions for the coupled states that, to the second order in η, read

$$\begin{aligned} |G\rangle &= (1-\frac{\eta^2}{8})\,|g,0\rangle + \frac{\eta}{2}\,|e,1\rangle + \frac{\eta^2}{2\sqrt{2}}\,|g,2\rangle \\ |\pm\rangle &= \epsilon_g\,|g,1\rangle + \epsilon_e\,|e,0\rangle + \gamma_g\,|g,3\rangle + \gamma_e\,|e,2\rangle \quad , \end{aligned} \tag{6}$$

where $\epsilon_g = \frac{1}{\sqrt{2}}\left(1 \mp \frac{\eta}{4} - \frac{9\eta^2}{32}\right)$, $\epsilon_e = \frac{1}{\sqrt{2}}\left(1 \pm \frac{\eta}{4} - \frac{\eta^2}{32}\right)$, $\gamma_g = -\frac{\eta}{2} + \frac{\sqrt{3}\eta^2}{4}$, and $\gamma_e = \mp\frac{\eta^2}{8}$. Plugging Eq. (6) into Eq. (3), and exploiting the fact that we know the unperturbed transition rates from Eq. (4), under the hypothesis of white reservoirs we are now able to express transition rates between coupled states as functions of η and the bare transition rates $\Gamma_{\rm in/out/cav}$

$$\begin{array}{lll} \Gamma_{\rm in}^{0\to G} = \Gamma_{\rm in}(1-\frac{\eta^2}{4})\theta(\mu-\hbar\omega_G) & \Gamma_{\rm out}^{G\to 0} = \Gamma_{\rm out}(1-\frac{\eta^2}{4}) & \Gamma_{\rm cav}^{1\to 0} = \Gamma_{\rm cav} \\ \Gamma_{\rm in}^{1\to G} = \Gamma_{\rm in}\frac{\eta^2}{4}\theta(\mu+\hbar\omega_C-\hbar\omega_G) & \Gamma_{\rm out}^{G\to 1} = \Gamma_{\rm out}\frac{\eta^2}{4} & \Gamma_{\rm cav}^{\pm\to G} = \Gamma_{\rm cav}(\frac{1}{2}\mp\frac{3}{4}+\frac{3\eta^2}{4}) \\ \Gamma_{\rm in}^{0\to\pm} = \Gamma_{\rm in}(\frac{1}{2}\pm\frac{\eta}{4})\theta(\mu-\hbar\omega_\pm-\hbar\omega_G) & \Gamma_{\rm out}^{\pm\to 0} = \Gamma_{\rm out}(\frac{1}{2}\pm\frac{\eta}{4}) & \Gamma_{\rm cav}^{+\to -} = 0 \\ \Gamma_{\rm in}^{1\to\pm} = \Gamma_{\rm in}(\frac{1}{2}\mp\frac{\eta}{4}-\frac{\eta^2}{4})\theta(\mu+\hbar\omega_C-\hbar\omega_\pm-\hbar\omega_G) & \Gamma_{\rm out}^{\pm\to 1} = \Gamma_{\rm out}(\frac{1}{2}\mp\frac{\eta}{4}-\frac{\eta^2}{4}) & \Gamma_{\rm cav}^{-\to +} = 0 \ . \end{array} \tag{7}$$

Photonic emission rates for pure GSE

When the chemical potential of the source is such that $\hbar\omega_G < \mu < \hbar\omega_G + \hbar\omega_\pm$, then electrons can only populate the system via the dressed ground state $|G\rangle$. Limiting ourselves to states with up to one photon we can then describe the dynamics of the system through the rate equation

$$\begin{aligned} \dot{P}_{s,0} &= -P_{s,0}\Gamma_{\rm in}^{0\to G} + P_G\Gamma_{\rm out}^{G\to 0} + P_{s,1}\Gamma_{\rm cav} + P_+\Gamma_{\rm out}^{+\to 0} + P_-\Gamma_{\rm out}^{-\to 0} \\ \dot{P}_{s,1} &= -P_{s,1}(\Gamma_{\rm cav} + \Gamma_{\rm in}^{1\to +} + \Gamma_{\rm in}^{1\to -} + \Gamma_{\rm in}^{1\to G}) + P_G\Gamma_{\rm out}^{G\to 1} + P_+\Gamma_{\rm out}^{+\to 1} + P_-\Gamma_{\rm out}^{-\to 1} \\ \dot{P}_G &= -P_G(\Gamma_{\rm out}^{G\to 0} + \Gamma_{\rm out}^{G\to 1}) + P_{s,0}\Gamma_{\rm in}^{0\to G} + P_{s,1}\Gamma_{\rm in}^{1\to G} + P_+\Gamma_{\rm cav}^{+} + P_-\Gamma_{\rm cav}^{-} \\ \dot{P}_+ &= -P_+(\Gamma_{\rm cav}^{+} + \Gamma_{\rm out}^{+\to 0} + \Gamma_{\rm out}^{+\to 1}) + P_{s,1}\Gamma_{\rm in}^{1\to +} \\ \dot{P}_- &= -P_-(\Gamma_{\rm cav}^{-} + \Gamma_{\rm out}^{-\to 0} + \Gamma_{\rm out}^{-\to 1}) + P_{s,1}\Gamma_{\rm in}^{1\to -} \quad , \end{aligned} \tag{8}$$

that, together with the normalization condition $P_{s,0} + P_{s,1} + P_G + P_+ + P_- = 1$, can be solved analytically yielding the steady state populations of the different states. The emission rates for the different GSE processes can then be written, as explained in the main text, as

$$f_C = P_{s,1}\Gamma_{\rm cav}, \quad f_+ = P_+\Gamma_{\rm cav}^{+}, \quad f_- = P_-\Gamma_{\rm cav}^{-}. \tag{9}$$

Using the perturbative expression for the states in Eq. (6), and assuming for simplicity $\Gamma \equiv \Gamma_{\rm in} = \Gamma_{\rm out}$, we finally obtain

$$f_C = \frac{\eta^2\Gamma}{8}(1-\xi\frac{\Gamma}{\Gamma_{\rm cav}}), \quad f_\pm = \frac{\eta^2\Gamma}{16}\frac{\Gamma}{\Gamma_{\rm cav}} \quad , \tag{10}$$

where $\xi = \frac{1+\theta(\mu-\mu'_+)}{2}$ with $\mu'_+ = \hbar\omega_+ - \hbar\omega_C$ These analytical results are plotted (for $\mu > \mu'_+$) in Figure 3(a) of the main text.

Photonic emission rates for GSE plus standard electroluminescence

When the chemical potential is such that $\mu > \hbar\omega_G + \hbar\omega_+$, electrons can also jump from the source directly into excited states. In this regime the rate equation takes the following form

$$\begin{aligned}
\dot{P}_{s,0} &= -P_{s,0}(\Gamma_{\rm in}^{0\to G} + \Gamma_{\rm in}^{0\to -} + \Gamma_{\rm in}^{0\to +}) + P_G\Gamma_{\rm out}^{G\to 0} + P_{s,1}\Gamma_{\rm cav} + P_+\Gamma_{\rm out}^{+\to 0} + P_-\Gamma_{\rm out}^{-\to 0} \\
\dot{P}_{s,1} &= -P_{s,1}(\Gamma_{\rm cav} + \Gamma_{\rm in}^{1\to +} + \Gamma_{\rm in}^{1\to -} + \Gamma_{\rm in}^{1\to G}) + P_G\Gamma_{\rm out}^{G\to 1} + P_+\Gamma_{\rm out}^{+\to 1} + P_-\Gamma_{\rm out}^{-\to 1} \\
\dot{P}_G &= -P_G(\Gamma_{\rm out}^{G\to 0} + \Gamma_{\rm out}^{G\to 1}) + P_{s,0}\Gamma_{\rm in}^{0\to G} + P_{s,1}\Gamma_{\rm in}^{1\to G} + P_+\Gamma_{\rm cav}^{+} + P_-\Gamma_{\rm cav}^{-} \\
\dot{P}_+ &= -P_+(\Gamma_{\rm cav}^{+} + \Gamma_{\rm out}^{+\to 0} + \Gamma_{\rm out}^{+\to 1}) + P_{s,1}\Gamma_{\rm in}^{1\to +} + P_{s,0}\Gamma_{\rm in}^{0\to +} \\
\dot{P}_- &= -P_-(\Gamma_{\rm cav}^{-} + \Gamma_{\rm out}^{-\to 0} + \Gamma_{\rm out}^{-\to 1}) + P_{s,1}\Gamma_{\rm in}^{1\to -} + P_{s,0}\Gamma_{\rm in}^{0\to -}.
\end{aligned} \tag{11}$$

Solving exactly as in the previous section, we obtain the emission rates

$$f'_C = \frac{\Gamma}{6}\left(\frac{2\Gamma}{\Gamma_{\rm cav}} + \eta^2\left(1 - \frac{11}{6}\frac{\Gamma}{\Gamma_{\rm cav}}\right)\right), \quad f'_\pm = \frac{\Gamma}{6}\left(1 - \frac{2\Gamma}{\Gamma_{\rm cav}} \pm \frac{\eta}{2}\left(1 - \frac{8\Gamma}{\Gamma_{\rm cav}}\right) + \frac{\eta^2}{12}\left(1 - \frac{32\Gamma}{\Gamma_{\rm cav}}\right)\right). \tag{12}$$

These analytical results are plotted in Figure 3(b) of the main text.
When the chemical potential is such that $\mu > \hbar\omega_G + \hbar\omega_-$ but $\mu < \hbar\omega_G + \hbar\omega_+$ emission at $\hbar\omega_+$ is suppressed to the value $f'_+ = \frac{3}{40}\frac{\Gamma}{\Gamma_{\rm cav}}\eta^2$ while we similarly get

$$f'_C = \frac{\Gamma}{5}\left((1-7\eta)\frac{\Gamma}{\Gamma_{\rm cav}} + \frac{3}{2}\eta^2\left(\frac{1}{2} - \frac{43}{50}\frac{\Gamma}{\Gamma_{\rm cav}}\right)\right) \quad f'_- = \frac{\Gamma}{5}\left(1 - \frac{2\Gamma}{\Gamma_{\rm cav}} - \frac{\eta}{5}\left(2 - \frac{19\Gamma}{\Gamma_{\rm cav}}\right) + \frac{\eta^2}{50}\left(3 - \frac{469}{4}\frac{\Gamma}{\Gamma_{\rm cav}}\right)\right). \tag{13}$$

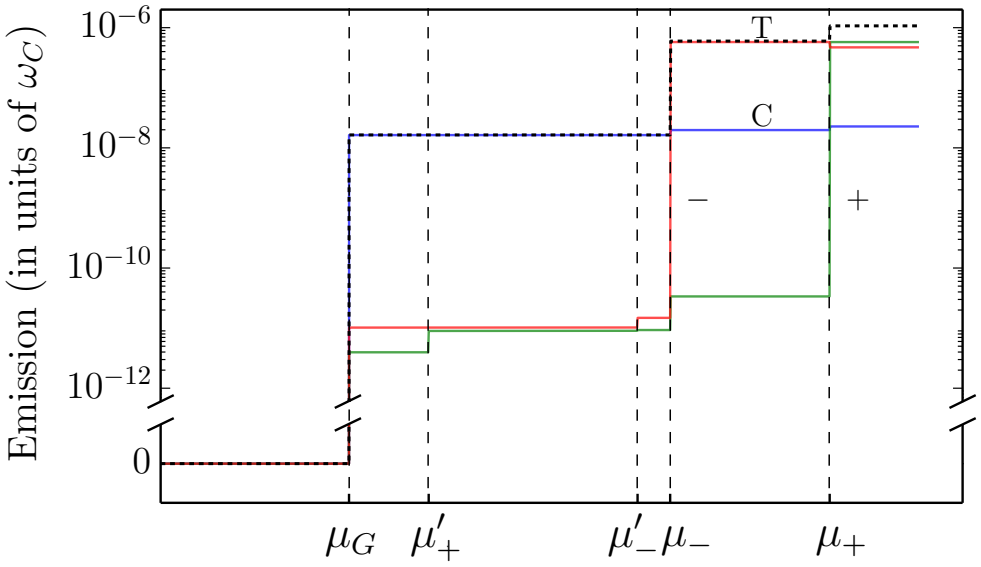

FIG. 1: Integrated emission spectrum $S(\omega)$ for the total emission (black dashed line), central peak (blue line, $\omega = \omega_C$) and both satellites channels (red line, $\omega = \omega_-$ and green line, $\omega - \omega_+$), as a function of the potential difference between drain and source electronic reservoirs. Parameters: $\eta = 0.2$, $\Gamma = \Gamma_{\rm in} = \Gamma_{\rm out} = 0.5 \times 10^{-6}\omega_C$, $\Gamma_{\rm cav} = 7 \times 10^{-4}\omega_C$.

Dependency of the photonic emission on the chemical potential

In this section we give a more detailed analysis of the dependency of the photonic emission on the chemical potential difference between drain and source electronic reservoirs. For concreteness, we will follow and describe the specific example shown in Fig. 1.
To start, we notice that, as the chemical potential is increased, the photonic emission goes through several discontinuities reflecting the energetic landscape of the system. This behaviour can be traced back to the analytical form of the transition rates in Eq. (7), which describes how electrons entering the system at different values for the chemical potential can induce different transitions. We are now going to describe the origin of the different discontinuities in more detail. We are now going to describe the behaviour at the critical values of the chemical potential

$$\begin{array}{rcl} \mu_G &=& \hbar\omega_G \\ \mu'_+ &=& \hbar\omega_G + \hbar\omega_+ - \hbar\omega_C \\ \mu'_- &=& E_{|2,-\rangle} - \hbar\omega_- \quad \text{where} \quad |2,-\rangle \equiv \frac{|g,2\rangle - |e,1\rangle}{\sqrt{2}} \\ \mu_- &=& \hbar\omega_G + \hbar\omega_- \\ \mu_+ &=& \hbar\omega_G + \hbar\omega_+ \end{array} \tag{14}$$

where different discontinuities in the emission take place and can be understood as follows. For clarity we also identify when the processes derive from Ground State Electroluminescence (GSE) or GSE and Regular Electroluminescence (GSE+RE).

- $\mu < \mu_G$.
  - No emission is possible as the energy of electrons flowing through the system cannot induce any transition.

GSE

- $\mu_G < \mu < \mu'_+$
  - Blue Line. The condition $\mu > \mu_G$ implies that the Ground state of the system can be populated and photons can be emitted at the energy $\hbar\omega = \hbar\omega_C$ through the GSE channel $|s,0\rangle \to |G\rangle \to |s,1\rangle \xrightarrow[\downarrow \hbar\omega_C]{} |s,0\rangle$.
  - Red Line. The condition $\mu > \mu_G$ also implies that $\mu > E_{|-\rangle} - E_{|s,1\rangle} = \hbar\omega_- + \hbar\omega_G - \hbar\omega_C$ (at second order in $\eta < 1$, we have that $\omega_G > \omega_- - \omega_C$). This allows for the ancillary channel $|s,1\rangle \to |-\rangle \xrightarrow[\downarrow \hbar\omega_-]{} |G\rangle$ to emit at energy $\hbar\omega = \hbar\omega_-$.
  - Green Line. Emission at energy $\hbar\omega = \hbar\omega_+$ is due only to residual emission from the main GSE channel.
- $\mu'_+ < \mu < \mu'_-$
  - Green Line At $\mu = \mu'_+$ the ancillary channel $|s,1\rangle \to |+\rangle \xrightarrow[\downarrow \hbar\omega_+]{} |G\rangle$ opens allowing for increased emission at energy $\hbar\omega = \hbar\omega_+$.
- $\mu'_- < \mu < \mu_-$
  - Red Line. At $\mu = \mu'_- \equiv E_{|2,-\rangle} - E_{|-\rangle}$ a process involving higher energy states become possible. More precisely, this regime opens the new channel $|s,1\rangle \to |2,-\rangle \to |-\rangle \xrightarrow[\downarrow \hbar\omega_-]{} |G\rangle$ with an increment of the emission at the energy $\hbar\omega_-$.

GSE + RE

- $\mu_- < \mu < \mu_+$ In this regime electrons can directly populate the higher excited state $|-\rangle$ through the channel $|s,0\rangle \to |-\rangle \xrightarrow[\downarrow \hbar\omega_-]{} |G\rangle$ which characterize regular electroluminescence with main emission at the energy $\hbar\omega = \hbar\omega_-$.
- $\mu > \mu_+$ In this regime electrons can directly populate the higher excited states $|\pm\rangle$ through the channel $|s,0\rangle \to |\pm\rangle \xrightarrow[\downarrow \hbar\omega_\pm]{} |G\rangle$ which characterize regular electroluminescence with main emission at energies $\hbar\omega = \hbar\omega_\pm$.

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