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General Non-Markovian Dynamics of Open Quantum Systems

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We present a general theory of non-Markovian dynamics for open systems of noninteracting fermions (bosons) linearly coupled to thermal environments of noninteracting fermions (bosons). We explore the non-Markovian dynamics by connecting the exact master equations with the nonequilibirum Green's functions. Environmental backactions are fully taken into account. The non-Markovian dynamics consists of nonexponential decays and dissipationless oscillations. Nonexponential decays are induced by the discontinuity in the imaginary part of the self-energy corrections. Dissipationless oscillations arise from band gaps or the finite band structure of spectral densities. The exact analytic solutions for various non-Markovian thermal environments show that non-Markovian dynamics can be largely understood from the environmental-modified spectra of open systems.

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Understanding the dynamics of open quantum systems is one of the most challenging topics in physics, chemistry, and biology. The environment-induced quantum dissipation and decoherence dynamics are the main concerns in the study of open quantum systems [1,2]. Decoherence control has also recently become a key task for practical implementations of nanoscale solid-state quantum information processing [3,4], where the decoherence is mainly dominated by non-Markovian dynamics due to the strong backactions from the environment. A fundamental issue is how to accurately take into account non-Markovian memory effects, which have attracted considerable attention very recently both in theory [5–13] and in experiments [14–16].

The non-Markovian dynamics of an open quantum system can be described by the master equation of the reduced density matrix $\rho(t)$. This is obtained by tracing over the environmental degrees of freedom, $\rho(t) = \mathrm{tr}[\rho_{\mathrm{tot}}(t)]$, where $\rho_{\mathrm{tot}}(t)$ is the density matrix of the total system. The standard approach to the non-Markovian dynamics uses the Nakajima-Zwanzig operator projective technique [17] where the master equation is formally written as

$$\frac{d\rho(t)}{dt} = \int_{t_0}^{t} d\tau \mathcal{K}(t-\tau)\rho(\tau). \quad (1)$$

The non-Markovian memory effects are taken into account by the time nonlocal integral kernel $\mathcal{K}(t-\tau)$. In practice, very few systems can be exactly solved from (1). Therefore, the generality of non-Markovian dynamics has not been fully understood.

In general, there are three typical time scales in an open system to characterize non-Markovian dynamics: (i) the time scale of the system $\sim 1/\varepsilon_s$, where ε_s is a typical energy scale of that system; (ii) the time scale of the environment $\sim 1/d$, where d is the bandwidth of the environmental spectral density; (iii) the mutual time scale arising from the coupling between the system and the environment $\sim 1/\Gamma$, where Γ is the dominant coupling strength. It is usually believed that non-Markovian memory effects strongly rely on the relations among these different time scales. However, such relationships have not been quantitatively established yet.

Here, we explore the non-Markovian dynamics from the analytical solution, solved by connecting the exact master equation with the nonequilibirum Green's functions. Exact master equations have been derived for open systems of noninteracting fermions (bosons) linearly coupled to thermal environments of noninteracting fermions (bosons) [8,18–20]. Establishing the connection between the master equation and the nonequilibrium Green's functions provides a new way to explore the non-Markovian dynamics even if the exact master equation of the open system is unknown.

Exact master equation and nonequilibrium Green's functions.—We begin with noninteracting fermonic (bosonic) many-body systems consisting of N single-particle energy levels ε_i ($i = 1, 2, \cdots, N$), coupled, via particle-particle exchanges [21], to a noninteracting fermionic (bosonic) environment, $H_{SB} = \sum_{\alpha k i}(V_{\alpha k i} a_i^\dagger b_{\alpha k} + V_{\alpha k i}^* b_{\alpha k}^\dagger a_i)$ [22]. The environment can contain many different reservoirs, and each reservoir is specified by its spectral density $J_{\alpha i j}(\omega) = 2\pi \sum_k V_{\alpha k i} V_{\alpha k j}^* \delta(\omega - \epsilon_k)$, where $V_{\alpha k i}$ is the coupling strength between the system and reservoir α. The operators $a_i^\dagger$ (a_i) and $b_{\alpha k}^\dagger$ ($b_{\alpha k}$) are the particle creation (annihilation) operators of the discrete energy level i of the system and the continuous level k of reservoir α, respectively. These creation-annihilation operators obey the standard anticommutation (commutation) relationship for fermions (bosons). Using the coherent-state path-integral method [23] to the Feynman-Vernon influence functional [24], the exact master equation of such an open system can be derived [8,19,20],

$$\frac{d\rho(t)}{dt} = \frac{1}{i}[\tilde{H}_S(t), \rho(t)] + \sum_{ij}\{\gamma_{ij}(t)[2a_j\rho(t)a_i^\dagger - a_i^\dagger a_j\rho(t) - \rho(t)a_i^\dagger a_j] + \tilde{\gamma}_{ij}(t)[a_i^\dagger\rho(t)a_j \pm a_j\rho(t)a_i^\dagger - a_i^\dagger a_j\rho(t) \mp \rho(t)a_ja_i^\dagger]\}. \tag{2}$$

The first term in (2) is the unitary term with the renormalized Hamiltonian $\tilde{H}_S(t) = \sum_{ij}\tilde{\varepsilon}_{sij}(t)a_i^\dagger a_j$. The second and third terms give the nonunitary dissipation and fluctuations, respectively. The ± and ∓ signs in the third term correspond to the system being bosonic or fermionic. The renormalized energy levels $\tilde{\boldsymbol{\varepsilon}}_s(t)$, the time-dependent dissipation coefficient $\boldsymbol{\gamma}(t)$ and the fluctuation coefficient $\tilde{\boldsymbol{\gamma}}(t)$ in (2) are given by

$$\tilde{\boldsymbol{\varepsilon}}_s(t) = \frac{i}{2}[\dot{\boldsymbol{u}}(t,t_0)\boldsymbol{u}^{-1}(t,t_0) - \text{H.c.}], \tag{3a}$$

$$\boldsymbol{\gamma}(t) = -\frac{1}{2}[\dot{\boldsymbol{u}}(t,t_0)\boldsymbol{u}^{-1}(t,t_0) + \text{H.c.}], \tag{3b}$$

$$\tilde{\boldsymbol{\gamma}}(t) = \dot{\boldsymbol{v}}(t,t) - [\dot{\boldsymbol{u}}(t,t_0)\boldsymbol{u}^{-1}(t,t_0)\boldsymbol{v}(t,t) + \text{H.c.}]. \tag{3c}$$

In Eqs. (3), the $N \times N$ matrix functions $\boldsymbol{u}(t,t_0)$ and $\boldsymbol{v}(t,t)$ are related to the nonequilibrium Green's functions of the system in the Schwinger-Keldysh nonequilibrium theory [25,26], $\boldsymbol{u}_{ij}(t,t_0) = \langle[a_i(t), a_j^\dagger(t_0)]_\mp\rangle$, and $\boldsymbol{v}_{ij}(t,t) = \langle a_j^\dagger(t)a_i(t)\rangle$ subtracting an initial-state dependent part [27]. These Green's functions obey the Dyson equations,

$$\frac{d}{d\tau}\boldsymbol{u}(\tau,t_0) + i\boldsymbol{\varepsilon}_s\boldsymbol{u}(\tau,t_0) + \int_{t_0}^{\tau}d\tau'\boldsymbol{g}(\tau,\tau')\boldsymbol{u}(\tau',t_0) = 0, \tag{4a}$$

$$\frac{d}{d\tau}\boldsymbol{v}(\tau,t) + i\boldsymbol{\varepsilon}_s\boldsymbol{v}(\tau,t) + \int_{t_0}^{\tau}d\tau'\boldsymbol{g}(\tau,\tau')\boldsymbol{v}(\tau',t) = \int_{t_0}^{t}d\tau'\tilde{\boldsymbol{g}}(\tau,\tau')\boldsymbol{u}^\dagger(\tau',t_0), \tag{4b}$$

subjected to the boundary conditions $\boldsymbol{u}(t_0,t_0) = 1$ and $\boldsymbol{v}(t_0,t) = 0$ with $t_0 \le \tau \le t$, where $\boldsymbol{\varepsilon}_s$ is a $N \times N$ matrix given by the bare single-particle energy levels of the system. The self-energy corrections, $\boldsymbol{g}(\tau,\tau')$ and $\tilde{\boldsymbol{g}}(\tau,\tau')$, which take into account all the backactions from the environment, are expressed explicitly by

$$\boldsymbol{g}(\tau,\tau') = \sum_\alpha\int\frac{d\omega}{2\pi}\boldsymbol{J}_\alpha(\omega)e^{-i\omega(\tau-\tau')}, \tag{5a}$$

$$\tilde{\boldsymbol{g}}(\tau,\tau') = \sum_\alpha\int\frac{d\omega}{2\pi}\boldsymbol{J}_\alpha(\omega)f_\alpha(\omega)e^{-i\omega(\tau-\tau')}, \tag{5b}$$

where the function $f_\alpha(\omega) = [e^{\beta_\alpha(\omega-\mu_\alpha)} \mp 1]^{-1}$ is the Bose-Einstein (Fermi-Dirac) distribution of bosonic (fermionic) reservoir α at the initial time t_0. Equations (2)–(5) establish a rigorous connection between the known exact master equation and the nonequilibrium Green's functions for open systems we concern.

General solutions of non-Markovian dynamics.—Different from the Nakajima-Zwanzig master equation, the exact master equation (2) is local in time, characterized by the dissipation and the fluctuation coefficients, $\boldsymbol{\gamma}(t)$ and $\tilde{\boldsymbol{\gamma}}(t)$. Non-Markovian memory effects are manifested as follows: (i) The coefficients $\boldsymbol{\gamma}(t)$ and $\tilde{\boldsymbol{\gamma}}(t)$ are microscopically and nonperturbatively determined by the nonequilibrium Green's functions from the Dyson equations (4). The non-Markovian memory effect is fully coded into the homogenous nonlocal time integrals in (4) with the integral kernel $\boldsymbol{g}(\tau,\tau')$. In other words, the self-energy correction $\boldsymbol{g}(\tau,\tau')$ serves as a memory kernel that count all the backactions from the environment. (ii) The coefficients $\boldsymbol{\gamma}(t)$ and $\tilde{\boldsymbol{\gamma}}(t)$ are constrained by the nonequilibrium fluctuation-dissipation theorem. The inhomogenous nonlocal time integral in (4b) with the integral kernel $\tilde{\boldsymbol{g}}(\tau,\tau')$, depicts the fluctuation arisen from the environment. Because $\boldsymbol{v}(t_0,t) = 0$, we can analytically solve Eq. (4b),

$$\boldsymbol{v}(\tau,t) = \int_{t_0}^{\tau}d\tau_1\int_{t_0}^{t}d\tau_2\boldsymbol{u}(\tau,\tau_1)\tilde{\boldsymbol{g}}(\tau_1,\tau_2)\boldsymbol{u}^\dagger(t,\tau_2). \tag{6}$$

This solution shows that Eq. (3c) is a generalized nonequilibrium fluctuation-dissipation theorem in the time domain (the reduction to the equilibrium fluctuation-dissipation theorem is given in the Supplemental Material, Ref. [22]). The fluctuation-dissipation theorem is a consequence of the unitarity of the whole system. It guarantees the positivity of the reduced density matrix during the non-Markovian time evolution.

Based on the above intrinsic features of open quantum systems, we can now explore the general properties of non-Markovian dynamics. From Eqs. (3), we can express the Green's function $\boldsymbol{u}(t,t_0)$ in terms of the dissipation coefficient $\boldsymbol{\gamma}(t)$ as

$$\boldsymbol{u}(t,t_0) = \mathcal{T}\exp\left\{-\int_{t_0}^{t}d\tau[i\tilde{\boldsymbol{\varepsilon}}(\tau) + \boldsymbol{\gamma}(\tau)]\right\}, \tag{7}$$

where $\mathcal{T}$ is the time-ordering operator. This solution indicates that $\boldsymbol{u}(t,t_0)$ fully determines the dissipation dynamics of the system. However, due to the time dependence of the dissipation coefficients, the detailed dissipation dynamics can vary significantly for different environments.

Explicitly, Eq. (5) show that $\boldsymbol{g}(\tau,\tau') = \boldsymbol{g}(\tau-\tau')$ and $\tilde{\boldsymbol{g}}(\tau,\tau') = \tilde{\boldsymbol{g}}(\tau-\tau')$. Thus we can write $\boldsymbol{u}(t,t_0) = \boldsymbol{u}(t-t_0)$. Using the modified Laplace transform $\boldsymbol{U}(z) = \int_{t_0}^{\infty}dt\boldsymbol{u}(t)e^{iz(t-t_0)}$, it is easy to obtain

$$\boldsymbol{U}(z) = \frac{i}{z\mathbf{I} - \boldsymbol{\varepsilon}_s - \boldsymbol{\Sigma}(z)}, \tag{8}$$

where $\mathbf{I}$ is the identity, $\boldsymbol{\Sigma}(z)$ is the Laplace transform of the self-energy correction,

$$\boldsymbol{\Sigma}(z) = \sum_\alpha\int\frac{d\omega}{2\pi}\frac{\boldsymbol{J}_\alpha(\omega)}{z-\omega} \xrightarrow{z=\omega\pm i0^+} \boldsymbol{\Delta}(\omega) \mp i\sum_\alpha\frac{\boldsymbol{J}_\alpha(\omega)}{2}, \tag{9}$$

and $\boldsymbol{\Delta}(\omega) = \sum_\alpha \mathcal{P} \int \frac{d\omega'}{2\pi} \frac{\boldsymbol{J}_\alpha(\omega')}{\omega - \omega'}$ is the principal value of the integral. It can be shown that the general solution of $\boldsymbol{u}(t, t_0)$ is given by

$$\boldsymbol{u}(t - t_0) = \sum_i Z_i e^{-i\omega_i(t-t_0)} + \sum_k \int_{B_k} \frac{d\omega}{2\pi} [\boldsymbol{U}(\omega + i0^+) - \boldsymbol{U}(\omega - i0^+)] e^{-i\omega(t-t_0)}. \tag{10}$$

The first term in (10) corresponds to localized modes with poles $\{\omega_i\}$ located at the real z axis with $\sum_\alpha \boldsymbol{J}_\alpha(\omega) = 0$. The coefficients $\{Z_i\}$ are the corresponding residues. The localized modes exist only when the environmental spectral density has band gaps or a finite band; i.e., $\sum_\alpha \boldsymbol{J}_\alpha(\omega)$ vanishes in some frequency regions; see Fig. 1. These localized modes do not decay, and give dissipationless non-Markovian dynamics. The second term in (10) is the contribution from the branch cuts $\{B_k\}$, due to the discontinuity of $\boldsymbol{\Sigma}(z)$, so does $\boldsymbol{U}(z)$, across the real axis on the complex space z; see Eq. (9). The branch cuts usually generate nonexponential decays [28], which is another significance of the non-Markovian dynamics. When the system is weakly coupled to the environment, the nonexponential decays are reduced to exponential-like decays.

Equation (10) provides indeed a general solution of the non-Markovian dissipation dynamics. It shows that the non-Markovian dissipation dynamics consists of nonexponential decays plus dissipationless localized modes. Such a solution for the two-point Green's function $\boldsymbol{u}(t, t_0)$ is generic and can be proven from the quantum field theory [29], even if particle-particle interactions are included.

The Green's function $\boldsymbol{u}(t, t_0)$ reveals the general non-Markovian dissipation dynamics. The non-Markovian fluctuation dynamics are constrained by the fluctuation-dissipation theorem via the Green's function $\boldsymbol{v}(t, t)$ of (6). Thus, the whole picture of non-Markovian dynamics is fully characterized by the dissipation and fluctuation coefficients of (3). The nonexponential decay part of (10) makes the dissipation coefficient $\boldsymbol{\gamma}(t)$ oscillate between positive and negative values, representing the backflow of information from the system to the environment [9,10]. Nonexponential decays alone give $\boldsymbol{\gamma}(t)$ a nonzero asymptotical value. If there are localized modes, $\boldsymbol{\gamma}(t)$ will vanish in the steady-state limit, resulting in dissipationless oscillations. In the weak coupling region, $\boldsymbol{\gamma}(t)$ can still be time dependent but keeps positive, the corresponding dynamics give simple exponential-like decays, observed mainly in the Markovian limit. Furthermore, Eqs. (3c) and (6) together show that except for the initial environmental temperature dependence, the time dependence of the fluctuation coefficient $\tilde{\boldsymbol{\gamma}}(t)$ behaves similar to $\boldsymbol{\gamma}(t)$, due to the fluctuation-dissipation theorem. In conclusion, non-Markovian dynamics can be fully understood from the solution of the Green's function $\boldsymbol{u}(t, t_0)$.

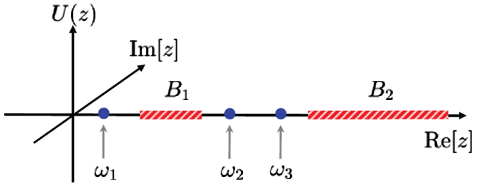

FIG. 1 (color online). A schematic pole structure of the Green's function $\boldsymbol{U}(z)$. The red-shaded regimes on the real z axis correspond to $\sum_\alpha \boldsymbol{J}_\alpha(z) \neq 0$.

Examples and discussion.—To be more specific, let us first examine the non-Markovian dynamics of a single-mode bosonic nanosystem, such as a nanophotonic or optomechanical resonator, coupled to a general non-Markovian environment with spectral density

$$J(\omega) = 2\pi\eta\omega\left(\frac{\omega}{\omega_c}\right)^{s-1} \exp\left(-\frac{\omega}{\omega_c}\right), \tag{11}$$

where η is the coupling constant between the system and the environment, and ω_c is the frequency cutoff. When $s = 1$, <1, and >1, the corresponding environments are Ohmic, sub-Ohmic, and super-Ohmic, respectively [30]. Following the above general procedure, the analytical solution of the non-Markovian dissipation dynamics is given by (setting $t_0 = 0$ for simplicity)

$$u(t) = Ze^{-i\omega' t} + \frac{2}{\pi} \int_0^\infty d\omega \frac{J(\omega) e^{-i\omega t}}{4[\omega - \varepsilon_s - \Delta(\omega)]^2 + J^2(\omega)}, \tag{12}$$

where $\Delta(\omega) = \frac{1}{2}[\Sigma(\omega + i0^+) + \Sigma(\omega - i0^+)]$ and the Laplace transform of the self-energy correction

$$\Sigma(\omega) = \begin{cases} \eta\omega_c[\pi\sqrt{-\tilde{\omega}} e^{-\tilde{\omega}} \mathrm{erfc}(\sqrt{-\tilde{\omega}}) - \sqrt{\pi}] & s = 1/2 \\ \eta\omega_c[\tilde{\omega} \exp(-\tilde{\omega}) Ei(\tilde{\omega}) - 1] & s = 1 \\ \eta\omega_c[\tilde{\omega}^3 e^{-\tilde{\omega}} Ei(\tilde{\omega}) - \tilde{\omega}^2 - \tilde{\omega} - 2] & s = 3 \end{cases}, \tag{13}$$

with $\tilde{\omega} = \omega/\omega_c$. Due to the vanishing spectral density for $\omega < 0$, a localized mode at $\omega' = \varepsilon_s - \Sigma(\omega') < 0$ occurs when $\eta\omega_c\Gamma(s) > \varepsilon_s$, here $\Gamma(s)$ is a gamma function. The localized mode leads to the dissipationless process. The corresponding residue is $Z = [1 - \Sigma'(\omega')]^{-1}$. This analytical solution precisely reproduces the exact numerical solution in the previous work [11]. Figure 2 shows that for a small η, the dissipation dynamics is an exponential-like decay, The corresponding $\gamma(t)$ and $\tilde{\gamma}(t)$ are time dependent but positive (corresponding to Markovian dynamics). When $\eta \gtrsim 0.3$, the nonexponential decay dominates, and $\gamma(t)$ and $\tilde{\gamma}(t)$ oscillate in positive and negative values with nonzero asymptotical values. When $\eta \gtrsim 0.6$, the localized state occurs, and $u(t)$ does not decay to zero. Correspondingly, $\gamma(t)$ and $\tilde{\gamma}(t)$ asymptotically approach to zero.

The second example is a fermionic system, a single electron transistor made of a quantum dot coupled to a source and a drain. The source and the drain are treated as two reservoirs of the environment. Their spectral densities take a Lorentzian form with a sharp cutoff,

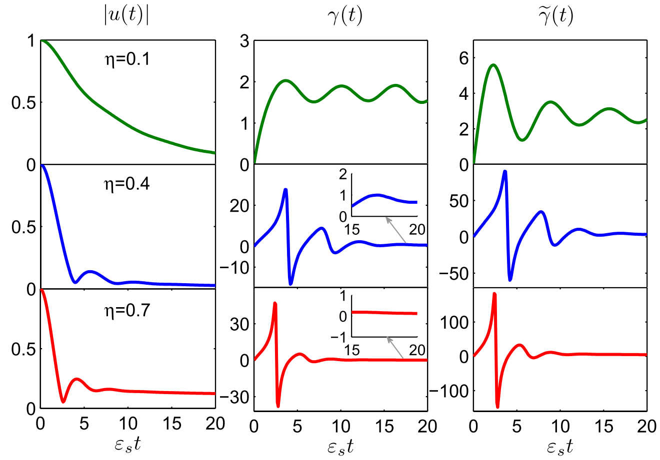

FIG. 2 (color online). The time evolution of the Green's function $u(t)$, the dissipation and the fluctuation coefficients, $\gamma(t)$ and $\tilde{\gamma}(t)$, in a sub-Ohmic bath, for several different values of the coupling constant η. Here we take other parameters $\omega_c = \varepsilon_s$, and $k_B T = \varepsilon_s$.

$$J_\alpha(\omega) = \frac{\Gamma_\alpha d_\alpha^2}{(\omega - \omega_c)^2 + d_\alpha^2}\Theta(\Omega - |\omega - \omega_c|), \tag{14}$$

with $\alpha = L(R)$ for the source (drain), where d_α is the half-width of the spectral density and Γ_α is the coupling strength between the system and reservoir α. We add a sharp cutoff to simulate a finite band for the environmental density of states. When $\Omega \to \infty$, the above spectral density is reduced to the usual Lorentzian spectral density that has been used in various studies of nanoelectronics [8,31–34]. We consider the symmetric case, $(\Gamma_L, d_L) = (\Gamma_R, d_R) = (\Gamma, d)$. Then the analytical solution of the Green's function $u(t)$ becomes

$$\begin{aligned} u(t) =& \sum_{j=1}^{2} Z_j e^{-i\omega'_j t} \\ &+ \frac{1}{\pi}\int_{\omega_c-\Omega}^{\omega_c+\Omega} d\omega \frac{J(\omega)e^{-i\omega t}}{[\omega - \varepsilon_s - \Delta(\omega)]^2 + J^2(\omega)}, \end{aligned} \tag{15}$$

where $J(\omega) = J_L(\omega) = J_R(\omega)$ and $\Delta(\omega)$ is the real part of the self-energy $\Sigma(\omega)$,

$$\Sigma(\omega) = \frac{J(\omega)}{\pi}\left[\log\frac{\omega_c - \Omega - \omega}{\omega_c + \Omega - \omega} + \frac{2(\omega - \omega_c)}{d}\tan^{-1}\frac{\Omega}{d}\right]. \tag{16}$$

The two localized states are located outside of the band, i.e., $\omega'_j = \varepsilon_s + \Sigma(\omega'_j)$, with $\omega'_1 < \omega_c - \Omega$, and $\omega'_2 > \omega_c + \Omega$. The corresponding residue is given by $Z_j = [1 - \Sigma'(\omega'_j)]^{-1}$. Again, the localized modes lead to a dissipationless process and the integral term shows a nonexponential decay. Taking $\Omega \to \infty$, the two localized modes are excluded, and the solution of $u(t)$ reproduces the exact non-Markovian dynamics of the usual Lorentzian spectral density (for detailed derivation, see Ref. [22]).

The third example is a two-level system with single-photon processes under the rotating wave approximation (spontaneous emission). In general, a multilevel atomic open system does not obey the master equation (2). However, the Schrödinger equation of a two-level atomic system with only spontaneous single-photon emission processes (at zero temperature) can be reduced to the Dyson equation of (4a) [35–37]. For a two-level artificial atom, such as a quantum dot, embedded in photonic crystals, because of the photonic band gap it was shown [36] that the corresponding solution contains exponential decays, nonexponential decays, and localized bound modes all together. We find analytically [38] that the complex pole with exponential decay shown in Ref. [36] has been included in the branch-cut integral of (10). Explicitly, the spectral density of the photonic crystals is $J(\omega) = \frac{2C}{\sqrt{\omega - \omega_e}}\Theta(\omega - \omega_e)$ [35,36]. From Eq. (10), we directly obtain the analytical solution of the spontaneous-emission dynamics

$$\begin{aligned} u(t) =& \frac{2\omega_r}{3\omega_r + \Delta} e^{i(\omega_r - \omega_e)t} \\ &+ \frac{C}{\pi}\int_{\omega_e}^{\infty} d\omega \frac{\sqrt{\omega - \omega_e}\, e^{-i\omega t}}{(\omega - \varepsilon_s)^2(\omega - \omega_e) + C^2}, \end{aligned} \tag{17}$$

where ω_r is the real root given by $(\omega_r + \Delta)\sqrt{\omega_r} = C$, and $\Delta = \varepsilon_s - \omega_e$ is the detuning. This analytical solution recovers both the exact analytical and numerical solutions given in Refs. [35,36].

The above examples show that very different open systems coupled to very different environments obey the same solution, Eq. (10), of the non-Markovian dynamics. The solutions of these examples can further be written in general as $u(t - t_0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\mathcal{D}(\omega)\exp\{-i\omega(t - t_0)\}$ with

$$\mathcal{D}(\omega) = 2\pi\sum_j Z_j \delta(\omega - \omega'_j) + \frac{J(\omega)}{[\omega - \varepsilon_s - \Delta(\omega)]^2 + J^2(\omega)/4}. \tag{18}$$

Equation (18) shows that the environment modifies the system spectrum as a combination of localized modes (dissipationless process) plus a continuum spectrum part (nonexponential decays). Remarkably, the result obtained from these simple examples gives indeed the underlying structure of two-point correlation functions in arbitrary complicated systems [29]. This indicates that alternatively, non-Markovian dynamics can be fully characterized by the environmental-modified spectrum of the system. If the spectrum of an open system can be measured, its non-Markovian dynamics can be extracted from its Fourier transform. This largely simplifies the exploration of the general properties of non-Markovian" dynamics for more complicated open systems.

Conclusion.—By connecting the exact master equation with the nonequilibrium Green's functions, we derive a general analytical solution of the non-Markovian dynamics for open systems of noninteracting fermions (bosons) linearly coupled to thermal environments of noninteracting fermions (bosons), i.e., Eq. (10) or (18). From the analytical solution, we show that the underlying non-Markovian dynamics consist of nonexponential decays and dissipationless oscillations. The dissipationless processes arise from band gaps or finite band structures of the environmental spectral densities. The nonexponential decays are induced by the discontinuity in the imaginary part of the self-energy corrections from the environment. The exponential decays observed in the Markovian limit are a special case in the weak coupling limit. Since the nonequilibrium Green's functions are well defined for arbitrary quantum systems, this theory may also provide a new approach to explore non-Markovian dynamics for more complicated open systems whose exact master equation may be unknown.

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[22] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.109.170402 for a general discussion of the system-reservoir coupling Hamiltonians, a reduction from nonequilibrium to equilibrium function-dissipation theorem, and a derivation of the spectral Green's function.

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