Einstein-Podolsky-Rosen steering: Its geometric quantification and witness

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We propose a measure of quantum steerability, namely, a convex steering monotone, based on the trace distance between a given assemblage and its corresponding closest assemblage admitting a local-hidden-state (LHS) model. We provide methods to estimate such a quantity, via lower and upper bounds, based on semidefinite programming. One of these upper bounds has a clear geometrical interpretation as a linear function of rescaled Euclidean distances in the Bloch sphere between the normalized quantum states of (i) a given assemblage and (ii) an LHS assemblage. For a qubit-qubit quantum state, these ideas also allow us to visualize various steerability properties of the state in the Bloch sphere via the so-called LHS surface. In particular, some steerability properties can be obtained by comparing such an LHS surface with a corresponding quantum steering ellipsoid. Thus, we propose a witness of steerability corresponding to the difference of the volumes enclosed by these two surfaces. This witness (which reveals the steerability of a quantum state) enables one to find an optimal measurement basis, which can then be used to determine the proposed steering monotone (which describes the steerability of an assemblage) optimized over all mutually unbiased bases.

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I. INTRODUCTION

Quantum entanglement [1], Einstein-Podolsky-Rosen (EPR) steering [2], and Bell nonlocality [3] are different forms of quantum nonlocality [4]. These quantum correlations are powerful resources for quantum engineering, quantum cryptography, quantum communication, and quantum information processing [5–7]. Taking an operational perspective [4], EPR steering can certify the entanglement between two systems when one of the measurements is untrusted, i.e., no assumptions are made on the functioning of the measurement device. On the other hand, Bell nonlocality can certify the entanglement with the untrusted measurements on both sides. One can also certify an entangled state by performing quantum state tomography with all-trusted measurement devices. Thus, EPR steering is a form of quantum correlation, which can be classified between entanglement and Bell nonlocality, in the following meaning: it certifies the entanglement between two systems assuming trusted measurements only on one of these [4]. Eighty years of research on EPR steering has resulted in many experimental demonstrations [8–21] and various applications [22–26], which include multipartite quantum steering [27–31], the correspondence with measurement incompatibility [32–36], one-way steering [20,37–39], one-sided device-independent processing in quantum key distribution [40], continuous-variable EPR steering [29,41–43], as well as temporal [44–49] and spatiotemporal steering [50].

In recent years, several measures of steering, such as steerable weight [38], steering robustness [51], steering fraction [52], steering cost [53], intrinsic steerability [54], as well as the relative entropy of steering [7,55] have been proposed (see also the review [39]). All these quantifiers are monotones under one-way local operations assisted by classical communication (one-way LOCCs) [7]. More recently, several works using the geometrical approaches to steering have been considered, such as depicting quantum correlations for two-qubit states [56,57] and a geometrical approach to witness steering [58,59]. Here, we would like to use the *consistent steering robustness* (CSR) introduced by Cavalcanti *et al.* [32] and the *quantum steering ellipsoid* (QSE) introduced by Jevtic *et al.* [56,57] to construct such a geometric witness. The QSE provides a visualization and geometric representation of any two-qubit state [30,31,60,61]. Specifically, the QSE for a given two-qubit state corresponds to the set of all Bloch vectors of one qubit (say Bob), which can be prepared by another qubit (say Alice) by considering all possible projective measurements on her qubit [56,57].

In this work, we propose a distance between assemblages based on the trace distance between single elements. Given an assemblage, a trace-distance measure of steerability is then proposed as the distance to the closest unsteerable assemblage. Here, we prove that the consistent trace-distance measure of steerability is a *convex steering monotone*, with respect to restricted one-way LOCCs introduced in Ref. [54]. We note that our proposal is reminiscent of other distance-based measures of various quantum phenomena. These include "nonclassical distance" for quantifying the quantumness of optical

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fields [62,63], distance-based measures of entanglement [64], trace-distance measures of coherence [65], or trace-distance measures quantifying Bell nonlocality [66].

In order to estimate the proposed steering monotone, we provide lower and upper bounds that can be efficiently computed by semidefinite programs (SDPs) [67]. Specifically, a lower bound is obtained via an operator-norm distance, whereas a few upper bounds are found by applying various known steering measures [32,38,51,68].

Moreover, we introduce the local-hidden-state (LHS) surface as a way of visualizing steerability properties of a two-qubit quantum state in the Bloch sphere. In particular, these notions connect to the QSE and provide a witness of steerability based on the different volumes enclosed by the two surfaces. This steerability witness enables finding an optimal measurement basis [60]. Thus, this is particularly important for calculating the proposed steering monotone optimized over all mutually unbiased bases. To illustrate the usefulness of LHS surfaces, we provide the explicit solution of the LHS surface for the Werner states. Moreover, we present a few upper and lower bounds of the steerability measure for the Werner, Horodecki, and rank-2 Bell-diagonal states [69,70]. Note that this approach, despite some resemblance, essentially differs from, e.g., the relative entropy of entanglement [64,70,71] and the nonclassical distance [62,63] used for quantifying the quantumness of bosonic systems.

This paper is organized as follows. In Sec. II, we summarize the basic notions concerning EPR steering. In Sec. III, we introduce a steering quantifier; we also prove that it is a monotone under restricted one-way LOCCs, and provide computable lower and upper bounds for it. In Sec. IV, we introduce a steering witness based on the notion of LHS surface and discuss its properties. In Sec. V, we apply our results to several interesting examples. Finally, in Sec. VI, we provide the conclusions and outlook for our work.

II. PRELIMINARY NOTIONS

EPR steering can be operationally defined as the success of the following task [4]: One party, say Alice, tries to convince another party, say Bob, that they share an entangled state ρ_{AB}. To accomplish this task, Bob asks Alice to perform some measurements, described by positive-operator-valued measures (POVMs) $A_{a|x}$ with $A_{a|x} \geqslant 0$, satisfying $\sum_a A_{a|x} = \mathbb{1}$, where x denotes the basis of the measurement, a is its outcome, and $\mathbb{1}$ is a unit operator. Bob's measurements are assumed to be fully characterized by quantum mechanics. Therefore, he can perform quantum state tomography and obtain the unnormalized quantum states $\sigma_{a|x} = \mathrm{tr}_A(\rho_{AB} A_{a|x} \otimes \mathbb{1})$. In particular, any $\{A_{a|x}\}_{a,x}$ gives rise to a collection of unnormalized quantum states $\{\sigma_{a|x}\}_{a,x}$, which are termed as an *assemblage*. An assemblage also includes the information of Alice's marginal statistics, $p(a|x) = \mathrm{tr}(\sigma_{a|x})$.

The assemblage $\{\sigma_{a|x}\}_{a,x}$ is unsteerable if it admits an LHS model,

$$\sigma_{a|x} = \sigma_{a|x}^{\mathrm{US}} = \sum_\lambda p(\lambda)p(a|x,\lambda)\sigma_\lambda \quad \forall a,x. \tag{1}$$

An LHS model can be understood as follows: Alice sends a preexisting quantum state σ_λ according to her input x and outcome a with a probability distribution $p(\lambda)$ and a conditional probability distribution $p(a|x,\lambda)$. In this sense, the assemblage, received by Bob, is just a classical postprocessing of the set of states $\{\sigma_\lambda\}_\lambda$, which is clearly independent of Alice's measurements. Likewise, a quantum state ρ_{AB} is called *steerable* if the given assemblage does not admit an LHS model. Such a state is necessarily entangled, but the converse is not true [4].

In the context of a resource theory of steering [7], the most general free operation for EPR steering is a stochastic one-way LOCC, defined as follows. Given an assemblage $\{\sigma_{a|x}\}_{a,x}$, Bob performs a quantum measurement on his system. The measurement is described by a completely positive trace-nonincreasing map ε defined by

$$\varepsilon(\sigma_B) := \sum_\omega K_\omega(\sigma_B)K_\omega^\dagger, \quad \text{such that} \quad \sum_\omega K_\omega^\dagger K_\omega \leqslant \mathbb{1}, \tag{2}$$

for the reduced state σ_B of Bob, where K_ω is the Kraus operator associated with a classical outcome ω. In the most general case, the set of classical outcomes is a coarse graining of the set of possible ω (quantum instruments may be defined by more than one Kraus operator), but, as we discuss below, there is no loss of generality by considering that each outcome ω is associated with a single Kraus operator K_ω.

After such an operation, Bob communicates with Alice, obtaining a classical result ω prior to her measurement. She applies a local deterministic wiring map W_ω, defined explicitly below, described by the normalized conditional probability distributions: $p(x|x',\omega)$, describing the generation of any initial input x from final input x' and Bob's result ω, and $p(a'|a,x,x',\omega)$, describing the generation of Alice's final outcome a' from a, x, x', and ω. The final assemblage with input x' and outcomes ω,a' becomes

$$\left\{\sigma_{a'|x'}^\omega\right\}_{a',x'} := M_\omega(\{\sigma_{a|x}\}_{a,x}). \tag{3}$$

Here, $M_\omega(\{\sigma_{a|x}\}_{a,x}) := K_\omega W_\omega(\{\sigma_{a|x}\}_{a,x})K_\omega^\dagger$ is a subchannel of the map $M := \sum_\omega M_\omega$ when Bob postselects the ωth outcome with probability $p(\omega) = \mathrm{Tr}[M_\omega(\sum_a\{\sigma_{a|x}\}_{a,x})]$, while

$$W_\omega(\{\sigma_{a|x}\}_{a,x}) = \left\{\sum_{a,x} p(x|x',\omega)p(a'|a,x,x',\omega)\sigma_{a|x}\right\}_{a',x'} \tag{4}$$

is a deterministic wiring map. We recall that a function S is a steering monotone (see Ref. [7]) if it is zero for unsteerable assemblages and it is a monotone, i.e., it does not increase (on average), under one-way LOCCs, i.e.,

$$\sum_\omega p(\omega)S\left(\frac{M_\omega(\{\sigma_{a|x}\}_{a,x})}{p(\omega)}\right) \leqslant S(\{\sigma_{a|x}\}_{a,x}), \tag{5}$$

for a given assemblage $\{\sigma_{a|x}\}_{a,x}$. Note that the particular case when $\sum_\omega p(\omega) = 1$ or $\sum_\omega K_\omega^\dagger K_\omega = \mathbb{1}$ is called a deterministic one-way LOCC. Otherwise, this is a stochastic one-way LOCC. Finally, we note that the use of a coarse-grained set of classical outcomes simply implies the equality of some of Alice wirings, i.e., $W_\omega = W'_\omega$, if ω and ω' are coarse grained into the same classical outcome.

In the following, we consider a *restricted* set of one-way LOCC, which has been proposed in Ref. [54]. This restriction consists of requiring that Alice's choice of wiring does not depend on the classical outcome ω obtained by Bob via local operations. This restriction can be motivated by practical reasons [54]: Given a spatial separation between Alice and Bob, the protocol may be more efficient if Alice directly applies her operation on her system instead of waiting for communication with Bob. This "restriction hypothesis" translates into the condition $p(x|x',\omega) = p(x|x')$, and hence into the condition

$$W_\omega(\{\sigma_{a|x}\}_{a,x}) = \left\{\sum_{a,x} p(x|x')p(a'|a,x,x',\omega)\sigma_{a|x}\right\}_{a',x'}.$$

III. GEOMETRIC QUANTIFIERS OF STEERABILITY

A. Trace-distance steerability measure

The (quantum) trace distance is a metric to distinguish two density operators ρ and ρ', i.e., $D_Q(\rho,\rho') := \frac{1}{2}\|\rho - \rho'\|$, where $\|X\| := \mathrm{tr}[|X|]$ is the trace norm. When ρ and ρ' commute, the trace distance reduces to the classical trace distance, i.e., the Kolmogorov distance [72], which can be defined as $D_C(P,P') := \frac{1}{2}\sum_x |P(x) - P'(x)|$ for two probability distributions P and P'.

One can easily prove the properties of a metric, i.e., it is (i) non-negative, (ii) symmetric, (iii) vanishes if and only if $\rho = \rho'$, and (iv) satisfies the triangle inequality. Similarly, we can define the distance between two assemblages as

$$D_A(\{\sigma_{a|x}\}_{a,x},\{\sigma'_{a|x}\}_{a,x}) = \sum_{a,x} p(x)D(\sigma_{a|x},\sigma'_{a|x}), \tag{6}$$

where $D(a,b) := \frac{1}{2}\|a - b\|$, and $\{\sigma_{a|x}\}_{a,x}$ and $\{\sigma'_{a|x}\}_{a,x}$ are two assemblages with the same number of inputs N_x. In general, $p(x)$ can be chosen to be uniform with respect to the number of measurement settings, i.e., $\frac{1}{N_x}$. As for D_Q, one can easily prove that D_A satisfies all the properties of a metric (cf. Appendix A). Note that trace distance between two assemblages is first introduced by Kaur *et al.* [55]. Nevertheless, our definition is different from theirs.

In the following, we want to introduce a measure of steerability based on the distance of a given assemblage from the set of unsteerable states. Several convex steering monotones have been introduced, with different properties and different interpretations. Our goal here is to introduce a different measure based on the trace distance between assemblages. We introduce a quantifier, called *consistent trace-distance measure of steerability*, defined as the minimal trace distance to the "consistent" unsteerable assemblage [32], namely,

$$S_{\mathrm{TD}}(\{\sigma_{a|x}\}_{a,x}) := \min\left\{D_A(\{\sigma_{a|x}\}_{a,x},\{\rho_{a|x}\}_{a,x})\right| \\ \left.\{\rho_{a|x}\}_{a,x}\in \mathrm{LHS},\ \sum_a \rho_{a|x} = \sum_a \sigma_{a|x},\ \forall x\right\}, \tag{7}$$

where LHS denotes the set of unsteerable assemblages, i.e., those admitting an LHS model given by Eq. (1). In Appendix B, we prove that S_{TD} is a restricted *convex steering monotone*.

Unfortunately, it is quite hard to calculate such a monotone without knowing the structure of LHSs. Instead, we find a way to derive lower and upper bounds based on SDPs.

B. Upper bound based on the restricted-noise consistent steering robustness

An upper bound of S_{TD} can be obtained via the notion of *steering robustness* [51], i.e., the amount of noise that can be added to an assemblage to make it unsteerable, and the notion of CSR [32], i.e., with the requirement that the noise assemblage has the same reduced state. We introduce a robustness measure based on this kind of mixing with a reduced state, which can be summarized as follows: Given an assemblage $\{\sigma_{a|x}\}_{a,x}$ and the associated reduced state $\sigma_B = \sum_a \sigma_{a|x}$, we define a steering monotone as

$$S^{\mathrm{R}}_{\mathrm{CSR}}(\{\sigma_{a|x}\}_{a,x}) \\ = \min\left\{t\left|\ \frac{\sigma_{a|x} + t[p(a|x)\sigma_B]}{1+t} \text{ is unsteerable}\right.\right\}, \tag{8}$$

which can be referred to as a *restricted-noise consistent steering robustness* (RNCSR), where $p(a|x) = \mathrm{tr}(\sigma_{a|x})$. The quantity $S^{\mathrm{R}}_{\mathrm{CSR}}$ can be efficiently computed as an SDP (see Appendix C).

Given an assemblage $\{\sigma_{a|x}\}_{a,x}$, the unsteerable assemblage, which is obtained as the solution of the SDP for calculating $\mathrm{S}^{\mathrm{R}}_{\mathrm{CSR}}$, is denoted by $\{\sigma^{\mathrm{R}}_{a|x}\}_{a,x}$. We can then easily compute the distance between these two assemblages as follows:

$$D_A\big(\{\sigma_{a|x}\}_{a,x},\{\sigma^{\mathrm{R}}_{a|x}\}_{a,x}\big) \\ = \frac{1}{N_x}\frac{t_{\min}}{1+t_{\min}}\sum_{a,x} p(a|x)D(\tilde{\sigma}_{a|x},\sigma_B), \tag{9}$$

where $t_{\min}$ is the optimal parameter t obtained from Eq. (8) and the tilde denotes normalized states, e.g., $\tilde{\sigma}_{a|x} = \sigma_{a|x}/\mathrm{tr}(\sigma_{a|x})$. As a consequence, the minimal trace distance between an assemblage and the restricted set of unsteerable assemblage obtained via mixing with noise σ_B corresponds to substituting $t_{\min}$ into the solution of an SDP for the $S^{\mathrm{R}}_{\mathrm{CSR}}$ in Eq. (9). Thus, an upper bound on S_{TD} can be simply given by

$$S^{\mathrm{R}}_{\max}(\{\sigma_{a|x}\}_{a,x}) = D_A\big(\{\sigma_{a|x}\}_{a,x},\{\sigma^{\mathrm{R}}_{a|x}\}_{a,x}\big). \tag{10}$$

Note that if Bob's system is a qubit, then $S^{\mathrm{R}}_{\max}(\{\sigma_{a|x}\}_{a,x})$ corresponds to a half of the sum of all the Euclidean distances between the Bloch vectors $\tilde{\sigma}_{a|x}$ and σ_B in the Bloch sphere multiplied by the probability distribution $p(a|x)$ and the scaling factor $t_{\min}/[N_x(1+t_{\min})]$. Mathematically, this can be expressed as

$$S^{\mathrm{R}}_{\max}(\{\sigma_{a|x}\}_{a,x}) = \sum_{a,x}\frac{p(a|x)\,t_{\min}}{N_x(1+t_{\min})}\frac{|\vec{p}_{a|x} - \vec{q}_b|}{2}, \tag{11}$$

where $|\vec{a} - \vec{b}|$ denotes the Euclidean distance between vectors $\vec{a}$ and $\vec{b}$. Moreover, $\vec{p}_{a|x,i} = \mathrm{tr}(\tilde{\sigma}_{a|x}\sigma_i)$ and $\vec{q}_{b,i} = \mathrm{tr}(\sigma_B\sigma_i)$ (for $i = 1,2,3$) are the components of the Bloch vectors of $\tilde{\sigma}_{a|x}$ and σ_B, respectively, and $\{\sigma_1,\sigma_2,\sigma_3\} \equiv \{X,Y,Z\}$ denote the Pauli operators.

C. Upper bound based on the consistent steering robustness

Here we provide another upper bound on the steering monotone S_{TD}, which is also based on the CSR. We show that this new bound is even tighter than that of $S^{\mathrm{R}}_{\max}(\{\sigma_{a|x}\}_{a,x})$, as defined in Eq. (10).

The CSR is defined as follows [32]:

$$S_{\mathrm{CSR}}(\{\sigma_{a|x}\}_{a,x}) = \min\left\{t \,\middle|\, \frac{\sigma_{a|x}+t\tau_{a|x}}{1+t}\right.$$
$$\left.\text{is unsteerable, and } \sum_a \tau_{a|x} = \sigma_B,\ \forall x\right\}, \quad (12)$$

where $\{\tau_{a|x}\}_{a,x}$ is an arbitrary noise assemblage with the same reduced state.

Similarly to $S^{\mathrm{R}}_{\max}$, an upper bound of S_{TD} can be obtained from the optimal solution $\{\sigma^{\mathrm{CSR}}_{a|x}\}_{a,x}$ of an SDP for the CSR. Note that although the $\{\sigma^{\mathrm{CSR}}_{a|x}\}_{a,x}$ is an optimal unsteerable assemblage, it may not be closest according to the trace distance. Thus, an upper bound based on the CSR can be defined as

$$S_{\max}(\{\sigma_{a|x}\}_{a,x}) = D_A\big(\{\sigma_{a|x}\}_{a,x},\{\sigma^{\mathrm{CSR}}_{a|x}\}_{a,x}\big). \quad (13)$$

Because $S^{\mathrm{R}}_{\max}$ was obtained for a restricted noise, it is obvious that the following inequality holds in general:

$$S_{\max} \leqslant S^{\mathrm{R}}_{\max}. \quad (14)$$

D. Lower bound based on operator norm

In this section, we show how to compute a lower bound of S_{TD} as an SDP. Without loss of generality, we can write the assemblage $\{\rho_{a|x}\}_{a,x} \in$ LHS, as $\rho_{a|x} = \sum_\lambda \delta_{a,\lambda_x}\sigma_\lambda$, where λ is a vector $(\lambda_x)_x$ and δ_{a,λ_x} represent the deterministic strategy for choosing the assemblage element σ_λ [23]. For consistency, we assume the condition $\sum_\lambda \sigma_\lambda = \sum_a \rho_{a|x} = \sigma_B$. Then we note that the trace norm can be lower bounded by the operator norm

$$\|A\|_\infty := \min\{\mu| -\mu\mathbb{1} \leqslant A \leqslant \mu\mathbb{1}\}, \quad (15)$$

i.e., $\|A\|_\infty \leqslant \|A\|$ for all operators A. Combining the lower bound based on this norm with the definition of the unsteerable assemblage, we obtain the following lower bound for $S_{\mathrm{TD}}(\{\sigma_{a|x}\}_{a,x})$:

$$\begin{aligned} S_{\min} := \min_{\sigma_\lambda} &: \frac{1}{2N_x}\sum_{a,x}\Big\|\sigma_{a|x} - \sum_\lambda \delta_{a,\lambda_x}\sigma_\lambda\Big\|_\infty \\ \text{subject to:} &\quad \sum_\lambda \sigma_\lambda = \sigma_B; \\ &\sigma_\lambda \geqslant 0,\quad \forall\lambda; \end{aligned} \quad (16)$$

where δ_{a,λ_x} is the usual deterministic strategy for the LHS model. Now, we can rewrite the above problem as the following SDP:

$$\begin{aligned} S_{\min} = \min_{\mu_{a,x},\sigma_\lambda} &: \frac{1}{2N_x}\sum_{a,x}\mu_{a,x}, \\ \text{subject to:} &\quad -\mu_{a,x}\mathbb{1} \leqslant \sigma_{a|x} - \sum_\lambda \delta_{a,\lambda_x}\sigma_\lambda \leqslant \mu_{a,x}\mathbb{1}, \\ &\sum_\lambda \sigma_\lambda = \sigma_B; \\ &\sigma_\lambda \geqslant 0,\quad \forall\lambda. \end{aligned} \quad (17)$$

By definition, we have $0 \leqslant \mu_{a,x} \leqslant 1$, so the same holds for the solution of the SDP. This implies that the primal SDP problem is bounded. Moreover, it is also strictly feasible, e.g., just take any strictly positive assemblage σ_λ, consistent with the reduced state σ_B, and $\mu_{a,x} = 1$ for all a,x. This implies the strong duality condition, i.e., the primal and dual SDPs have the same optimal value.

Note, however, that the operator-norm quantifier $S_{\min}(\{\sigma_{a|x}\}_{a,x})$ is *not* a convex steering monotone since the operator-norm distance is, in general, not contractive under completely positive trace-nonincreasing maps.

Finally, we have the following lower and upper bounds:

$$\begin{aligned} S_{\min}(\{\sigma_{a|x}\}_{a,x}) \leqslant S_{\mathrm{TD}}(\{\sigma_{a|x}\}_{a,x}) &\leqslant S_{\max}(\{\sigma_{a|x}\}_{a,x}) \\ &\leqslant S^{\mathrm{R}}_{\max}(\{\sigma_{a|x}\}_{a,x}), \end{aligned} \quad (18)$$

which can be efficiently computed via our SDPs. A clear comparison of these three upper bounds and lower bound for some states is shown in Fig. 1.

IV. GEOMETRIC WITNESS OF STEERABILITY

In Sec. III, we concentrated on assemblages, but in this section we focus on steerability of a quantum state rather than assemblages.

In addition to the geometrical picture introduced in Sec. III, we provide a way to visualize two-qubit steering properties through the notion of a LHS surface and a QSE. We first recall that the QSE [56] is defined as the surface of normalized assemblages $\tilde{\sigma}_{a|x} = \sigma_{a|x}/\mathrm{tr}(\sigma_{a|x})$, obtained by Bob for all possible projective measurements of Alice. All projective measurements on Alice's side form the surface of the QSE, while the POVMs correspond to the points in the interior. The QSE centers at $\tilde{c} = (\tilde{b} - T^{\mathrm{T}}\tilde{a})/(1-\tilde{a}^2)$ with the orientation and semiaxes lengths $s_i = \sqrt{q_i}$ given by the eigenvectors and eigenvalues q_i of the ellipsoid matrix,

$$Q = \frac{1}{1-\tilde{a}^2}(T^{\mathrm{T}} - \tilde{b}\tilde{a}^{\mathrm{T}})\left(\mathbb{1} + \frac{\tilde{a}\tilde{a}^{\mathrm{T}}}{1-\tilde{a}^2}\right)(T - \tilde{a}\tilde{b}^{\mathrm{T}}), \quad (19)$$

where $\tilde{a}$ and $\tilde{b}$ are the Bloch vectors of the reduced states of Alice and Bob, respectively. Here, T is the correlation matrix with elements $T_{jk} = \mathrm{tr}[\rho_{AB}\sigma_j \otimes \sigma_k]$ (for $j,k = 1,2,3$), where ρ_{AB} is the bipartite state shared by Alice and Bob.

We can analogously define the corresponding LHS surface. Instead of considering all possible single measurements, however, we need to fix a measurement assemblage for Alice. In this case, we assume that Alice can perform three mutually unbiased measurements [73] on her side with outcomes ± 1. Consequently, Bob obtains the assemblage $\{\sigma_{a|x}\}_{a,x}$, consisting of six terms. To compute the closest unsteerable assemblage, $\{\sigma^{\mathrm{US}}_{a|x}\}_{a,x}$, we restrict to the RNCSR case which can be computed as an SDP. By normalizing such an assemblage, $\tilde{\sigma}^{\mathrm{US}}_{a|x} = \sigma^{\mathrm{US}}_{a|x}/\mathrm{tr}(\sigma^{\mathrm{US}}_{a|x})$, Bob obtains six vectors in the Bloch sphere. The LHS surface is then obtained when Alice performs all possible rotations of her mutually unbiased measurement bases.

Intuitively, the bipartite state ρ_{AB} is unsteerable if its LHS surface and QSE are identical because $\{\tilde{\sigma}_{a|x}\}_{a,x} = \{\tilde{\sigma}^{\mathrm{US}}_{a|x}\}_{a,x}$. Moreover, it is clear that conv (LHS surface) is always contained in conv (QSE), where we denoted with conv the convex hull of the points in the corresponding surface. In fact, given $\{\sigma_{a|x}\}_{a,x}$, the corresponding solution $\{\sigma^{\mathrm{US}}_{a|x}\}_{a,x}$, computed via an SDP for the RNCSR, satisfies $\mathrm{tr}(\sigma_{a|x}) = \mathrm{tr}(\sigma^{\mathrm{US}}_{a|x})$, and

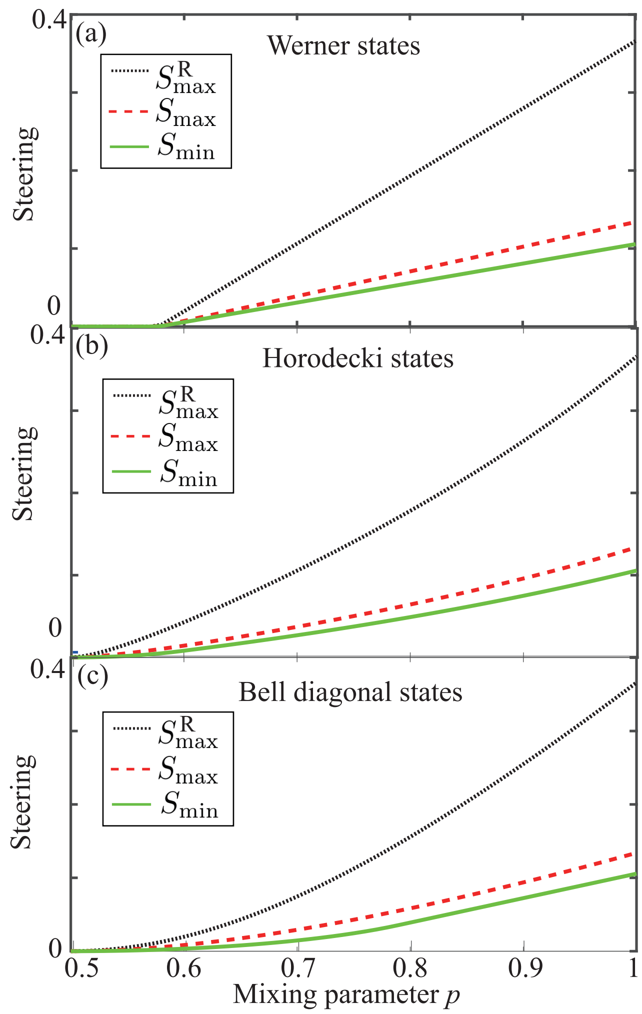

FIG. 1. Upper (S^{R}_{max} and S_{max}) and lower (S_{min}) bounds of the steerability measure S_{TD} for (a) the Werner states (rank-4 Bell-diagonal states), (b) Horodecki states, and (c) rank-2 Bell-diagonal states vs their mixing parameter p. A given assemblage is created when Alice performs three mutually unbiased measurements (eigenvectors of the Pauli spin matrices X,Y, and Z). Here, we compute the unsteerable assemblage obtained from the restricted-noise consistent steering robustness (RNCSR, S^{R}_{CSR}) bounded by S^{R}_{max} and the consistent steering robustness (CSR, S_{CSR}) bounded by S_{max}. As can be seen, the steerability monotonically increases with increasing parameter for (b), (c) $p \geqslant 1/2$ and (a) $p \geqslant 1/\sqrt{3}$. Note that the meaning of the mixing parameter p is completely different in (a), (b), and (c) as given by the definitions of the corresponding states in Secs. V A, V B, and V C.

hence $\tilde{\sigma}^{US}_{a|x}$ is a convex hull of points inside the QSE, namely,

$$\tilde{\sigma}^{US}_{a|x} = \frac{\tilde{\sigma}_{a|x} + t\sigma_B}{1+t}. \tag{20}$$

Therefore, we can geometrically witness steering when

$$\Delta V \equiv V_{QSE} - V_{LHS\ surface} > 0, \tag{21}$$

where V_{QSE} and $V_{LHS\ surface}$ are the volumes of the QSE and LHS surface, respectively.

Note that the steering witness ΔV focuses on the steerability of a *quantum state*, while the proposed steering monotone S_{TD} describes the steerability of an *assemblage*. Thus, one could think that it is rather hard, in general, to compare these approaches and to show which of these is more useful. Now we would like to explain an important relation between the witness ΔV and the steering monotone S_{TD}. Note that S_{TD} is defined on an assemblage and hence it requires the measurement settings to be fixed. In contrast to this, the calculation of ΔV involves looking at all possible mutually unbiased bases; hence it provides a more complete information about the steerability of a given state. In addition, McCloskey *et al.* [60] showed that the geometric information encoded in the QSE often provides the optimal measurement directions, corresponding to the three ellipsoid semiaxes. Similarly, the LHS surface provides the information about the measurement directions giving usually the highest steering monotone S_{TD}.

The concept of the LHS surface can be generalized to include different SDP characterizations of the "closest" unsteerable assemblages, e.g., via the steering robustness [51] or other quantifiers [39]. Moreover, such a notion can also be generalized beyond the qubit case. The interest for the present approach is motivated by the possibility of visualizing the steering properties of a state onto the Bloch sphere and its relations with the QSE.

V. APPLICATIONS

In Sec. III, we provided examples of lower and upper bounds for our steering monotone S_{TD}. In what follows, we demonstrate the usefulness of the LHS surface and the trace-distance measures of steerability in several related examples.

We analyze three important prototype classes of states (i.e., the Werner, Horodecki, and Bell-diagonal states), which are formed by the singlet state $|S\rangle$ mixed with three different states. Thus, the meaning of the mixing parameter is different in these states although denoted, for simplicity, by the same symbol p. Specifically, (a) for the Werner states, the singlet state $|S\rangle$ is mixed with a (separable) completely mixed state, which is not orthogonal to $|S\rangle$; (b) for the Horodecki states, $|S\rangle$ is mixed with a separable state, which is orthogonal to $|S\rangle$; and (c) for the Bell-diagonal states, $|S\rangle$ is mixed with another maximally entangled state (i.e., the entangled triplet state), which is orthogonal to $|S\rangle$.

A. Steerability of Werner states

We analytically show the solution of the LHS surface for the Werner states [69], which are mixtures of the singlet state and the maximally mixed state, i.e.,

$$\rho^{W}(p) = p|S\rangle\langle S| + (1-p)\frac{\mathbb{1}}{4}, \tag{22}$$

where $|S\rangle = 1/\sqrt{2}(|10\rangle - |01\rangle)$ and $0 \leqslant p \leqslant 1$ is the mixing weight. It is clear that the Werner states are rank-4 Bell-diagonal states for $p < 1$. When Alice applies the three Pauli operators (X, Y, and Z) to measure a given Werner state, the corresponding Bloch vectors of Bob's normalized assemblage are $(\pm p,0,0)$, $(0,\pm p,0)$, and $(0,0,\pm p)$. The simplest solution of the preexisted quantum states $\{\sigma_\lambda\}_\lambda$ is located at $(\pm p, \pm p, \pm p)$ in the Bloch sphere. On the other hand, the LHSs are the mixtures of four preexisted states according

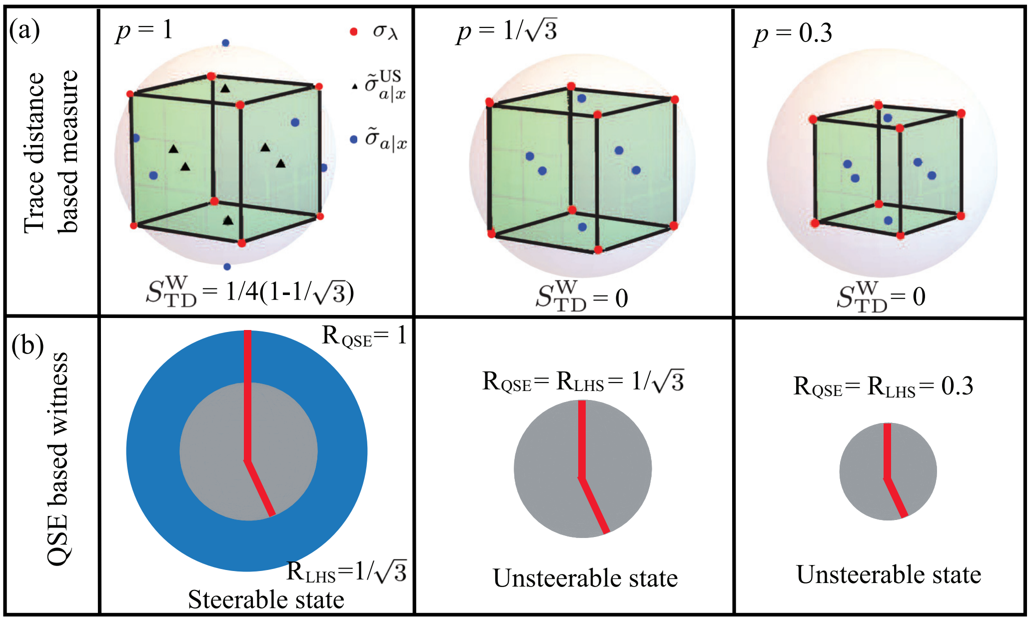

FIG. 2. EPR steerability of the Werner states: (a) steerability measure S^{W}_{TD}, which is based on trace distance, and (b) steerability witness given by Eq. (21), which is based on the volume of the QSE and LHS surface. For (a), we use the three Pauli bases to obtain the preexisted states $\{\sigma_\lambda\}_\lambda$ (red circles), steered states $\{\tilde{\sigma}_{a|x}\}_{a,x}$ (black triangles), and the unsteerable assemblage $\{\tilde{\sigma}^{US}_{a|x}\}_{a,x}$ (blue circles), respectively. The auxiliary green cube helps one to see the relative positions of the LHS and the steered states. As expected, when $p \geqslant 1/\sqrt{3}$, we find that the positions of the steered states are outside the LHSs. On the other hand, the Bloch vectors of $\{\tilde{\sigma}^{US}_{a|x}\}_{a,x}$ remain as $(\pm 1/\sqrt{3},0,0)$, $(0,\pm 1/\sqrt{3},0)$, and $(0,0,\pm 1/\sqrt{3})$, independent of p. This is because the maximal length of a Bloch vector is equal to unity. When $p \leqslant 1/\sqrt{3}$, the unsteerable states $\{\tilde{\sigma}^{US}_{a|x}\}_{a,x}$ are exactly identical to the steered states $\{\tilde{\sigma}_{a|x}\}_{a,x}$. For (b), we show the QSE and LHS surface for different values of p. Both surfaces are spheres so it is sufficient to show the two-dimensional projection. The outer and inner circles are the QSE and LHS surface, respectively. These circles are centered at $\tilde{c} = (0,0,0)$. The three semiaxes of the QSE lie along x, y, and z. However, the orientations of the three LHS axes are the same as those of the QSE, but the length is p only when $0 < p < 1/\sqrt{3}$. When $p > 1/\sqrt{3}$, the lengths are fixed at $1/\sqrt{3}$. In other words, the QSE and LHS surfaces are identical when $p \leqslant 1/\sqrt{3}$. Once $p \geqslant 1/\sqrt{3}$, the LHS surface is fixed, but the QSE expands with p.

to the strategy $p(\lambda)$ with probability $p(a|x,\lambda) = 1/4$. When $p \leqslant 1/\sqrt{3}$, the LHSs $\tilde{\sigma}^{US}_{a|x}$ are identical to the steered states $\tilde{\sigma}_{a|x}$. As $p \geqslant 1/\sqrt{3}$, the Bloch vectors of $\tilde{\sigma}^{US}_{a|x}$ are fixed at $(\pm 1/\sqrt{3},0,0)$, $(0,\pm 1/\sqrt{3},0)$, and $(0,0,\pm 1/\sqrt{3})$, as shown in Fig. 2. This is because the maximal length of a Bloch vector is equal to unity. The optimal value of p for the LHS and preexisted states is $1/\sqrt{3}$, coinciding with the upper bound S^{R}_{max} on the steering inequality $\langle XX\rangle + \langle YY\rangle + \langle ZZ\rangle \leqslant \sqrt{3}$ [21]. However, the set of steered states $\tilde{\sigma}_{a|x}$, i.e., the QSE, gradually expands with p. One can also rotate the measurement settings on Alice's side, but keep them mutually unbiased. Once all sets of the three measurements are performed, Bob obtains the LHS surface, which is the set of all $\tilde{\sigma}^{US}_{a|x}$ (see Fig. 2). One can solve analytically this simple case and show that the LHS surface of the Werner state is actually a sphere, centered at $\tilde{c} = (0,0,0)$ as the QSE, with radius $1/\sqrt{3}$ for $p \geqslant 1/\sqrt{3}$, and radius p otherwise (see Fig. 2 and Appendix D). The trace distance between them is equal to

$$D^{W}_{A}(\{\sigma_{a|x}\}_{a,x}) = \frac{1}{4}\left(p - \frac{1}{\sqrt{3}}\right) \qquad (23)$$

for $p \geqslant 1/\sqrt{3}$, and 0 otherwise, which is identical to a quarter of the Euclidean distance between the $\tilde{\sigma}^{US}_{a|x}$ and $\tilde{\sigma}_{a|x}$. Interestingly, the $D^{W}_{A}(\{\sigma_{a|x}\}_{a,x})$, which can be computed by our analytical solution for the Werner states, is smaller than S_{max} and the same as S_{min}. Thus, we conclude that $S^{W}_{TD} = D^{W}_{A}(\{\sigma_{a|x}\}_{a,x})$, where the superscript W indicates that the results are for the Werner states. Note that in this example, we considered only three Pauli measurement bases. It can be constructive to compare Fig. 2 for the Werner states with Figs. 3 and 4 for other special states.

Another comparison of various upper and lower bounds for the Werner states is shown in Fig. 1(a). It is seen that the upper bounds of S^{W}_{TD} for the Werner states are vanishing for the mixing parameter $p \leqslant 1/\sqrt{3}$ and are linearly increasing with $p > 1/\sqrt{3}$. This is contrary to the behavior of the same bounds for the states analyzed in Figs. 1(b) and 1(c).

B. Steerability of Horodecki states

The Horodecki states are the mixtures of a maximally entangled state, say the singlet state $|S\rangle$, and a separable state, say $|00\rangle$, i.e.,

$$\rho^{H}(p) = p|S\rangle\langle S| + (1-p)|00\rangle\langle 00|. \qquad (24)$$

In Fig. 3, we show that the LHS surfaces, which are computed by the RNCSR for the Horodecki states, are similar to those for the Werner states. When $0 \leqslant p \leqslant 1/2$, the LHS surface and QSE are identical. Therefore, the trace distance between a given assemblage and unsteerable assemblage, which we consider X, Y, and Z, is 0 when $0 \leqslant p \leqslant 1/2$. As $p \geqslant 1/2$, the QSE and LHS surfaces gradually expand, but the QSE expands more rapidly than the LHS surface. The trace distance of the

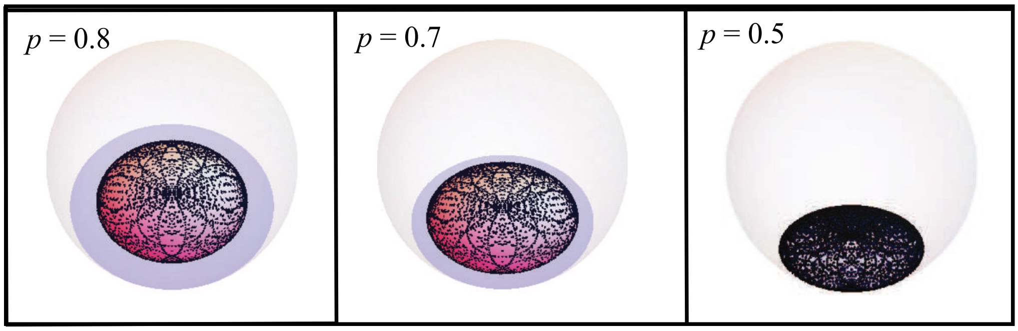

FIG. 3. QSE-based witness of the steerability of the Horodecki states. As can be seen, a given state is steerable if the LHS surface and QSE are not identical. Horodecki states are unsteerable if the parameter $p = 1/2$. Otherwise these states are steerable. Note that one of the lower bounds of S_{TD} corresponds to the distance between the inner and outer surfaces (i.e., the Euclidean distance between $\tilde{\sigma}^{\mathrm{US}}_{a|x}$ and $\tilde{\sigma}_{a|x}$ in the Bloch sphere) multiplied by the probability distribution $p(a|x)$ and some scaling factor defined in the text.

assemblages also increases when $1/2 \leqslant p \leqslant 1$ [see Fig. 1(b)]. Via numerical fitting of the computed points, we find that the LHS surface associated with the Horodecki states is consistent with the corresponding QSE. Another comparison of various upper and lower bounds for the Horodecki states is shown in Fig. 1(b).

C. Steerability of rank-2 Bell-diagonal states

In general, Bell-diagonal states of two qubits are mixtures of the four maximally entangled quantum states. Here, for simplicity, we consider special rank-2 Bell-diagonal states, i.e., mixtures of the singlet state $|S\rangle$ and a triplet state $|T\rangle = 1/\sqrt{2}(|10\rangle + |01\rangle)$, i.e.,

$$\rho^{\mathrm{B}}(p) = p|S\rangle\langle S| + (1-p)|T\rangle\langle T|, \tag{25}$$

where p is the mixing weight. In Fig. 4, we show that the LHS surface, which was computed by the RNCSR, is identical to the QSE only when $p = 1/2$; otherwise, these are different. The distance between a given assemblage and unsteerable assemblage, which we consider X, Y, and Z, also reveals the same behavior, as shown in Fig. 1(c). However, the LHS surface of these Bell-diagonal states cannot be fitted by an ellipsoid.

The upper bounds of S_{TD} for the Horodecki states [as shown in Fig. 1(b)] and the Bell-diagonal states [Fig. 1(c)] are vanishing for the mixing parameter $p \in [0,1/2]$ and $p = 1/2$, respectively. Moreover, these bounds are *nonlinearly* increasing with $p \geqslant 1/2$. This is in contrast to those upper bounds for the Werner states [shown in Fig. 1(a)], which vanish for $p \in [0,1/\sqrt{3}]$ and are *linearly* increasing with $p > 1/\sqrt{3}$. As already mentioned in the introduction to Sec. V, the meaning of the mixing parameter p for these three classes of states is completely different. By analyzing the Werner states, we see that by mixing the singlet state $|S\rangle$ with the maximally mixed state $\mathbb{1}/4$, the steerability of such "noisy" singlet states is completely destroyed for a wider range of the values of the mixing parameter p in comparison to the other two cases, i.e., to the mixing of the singlet state $|S\rangle$ with a state $|\psi\rangle$ orthogonal to $|S\rangle$ both for the separable state $|\psi\rangle = |00\rangle$ (which results in the Horodecki states) and for the maximally entangled state $|\psi\rangle = |T\rangle$ (which results in the Bell-diagonal states).

VI. CONCLUSIONS AND OUTLOOK

In this work, we defined the trace distance between two assemblages and the corresponding measure of steerability based on this distance. We have shown that this measure of steerability is indeed a *convex steering monotone* under restricted one-way LOCCs. We provided a way of estimating such a quantity via lower and upper bounds based on SDPs. Specifically, a lower bound is based on the operator norm, while a few upper bounds are found by applying various steering measures, including the CSR [32] and a restricted version of the CSR. Using the latter bound, we proposed a way of visualizing the steerability property of a quantum state in the Bloch sphere via the notion of a LHS surface, which relates the steerability

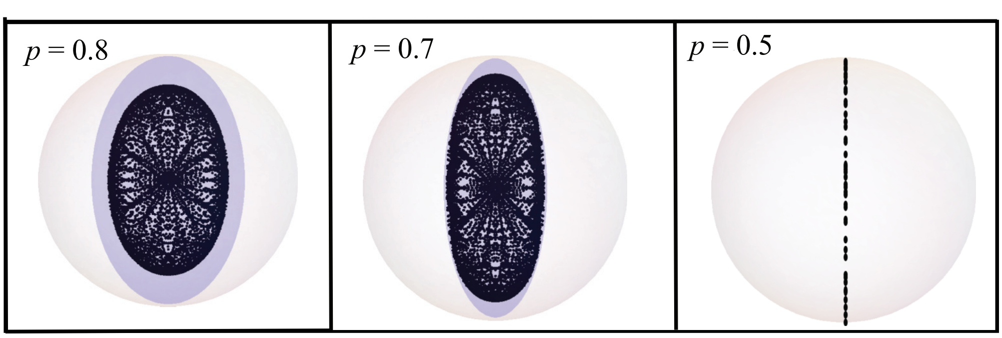

FIG. 4. QSE-based witness of the steerability of the rank-2 Bell-diagonal states. As can be seen, a given state is steerable if its LHS surface and QSE are not identical. The Bell-diagonal states are unsteerable if the parameter $p = 1/2$. Otherwise these are steerable.

problem, in the sense of the existence of a LHS model, with the notion of QSEs.

We computed EPR steerability by describing a set of states in the Bloch sphere. We did not construct a space of assemblages. Thus, we defined, in particular, the upper bound $S^{\rm R}_{\rm max}$, which can be directly computed by summing up (with some coefficients) all the "Euclidean distances" between Bloch vectors. Therefore, $S^{\rm R}_{\rm max}$ has a clear geometrical meaning. Moreover, in Sec. V A, we also pointed out that $S_{\rm TD}$ for the Werner states (as denoted by $S^{\rm W}_{\rm TD}$) has a direct relation to the distance between the set of states in the Bloch sphere. The monotone $S^{\rm W}_{\rm TD}$ is a linear function of the distance from the points $(x,0,0)$, $(0,x,0)$, and $(0,0,x)$ (where $x=\pm 1/\sqrt{3}$) of the Bloch sphere to a normalized quantum state of its assemblage in the Bloch sphere, when the mixing parameter $p>1/\sqrt{3}$. Thus, by referring to a "geometrical" interpretation, we mean the Euclidean distance between quantum states in the Bloch sphere.

We defined the witness ΔV of steerability corresponding to the difference of the volumes enclosed by the QSE and the LHS surface. The LHS surfaces enable calculation of the proposed steering monotone $S_{\rm TD}$ optimized over all mutually unbiased bases. We remark that this observation relates the different concepts of (i) the steering witness ΔV, which reveals the steerability of a *quantum state*, and (ii) the steering monotone $S_{\rm TD}$, which describes the steerability of an *assemblage*.

Our study stimulates some further investigations. First, it is known that the QSE has an analytical representation. Therefore, it is natural to ask if the LHS surface also has an analytical formulation, at least, for some specific states. Our analysis shows that this is the case for the Werner states and we have numerical evidence that the same could be possible for the Horodecki states. Second, since we have obtained the witness of steerability for a given quantum state, given in Eq. (21), it is interesting to investigate whether such a difference of volumes, i.e., between the QSE and LHS surface, has some physical meaning and can be used to obtain a new steering monotone for a quantum state.

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APPENDIX A: METRIC PROPERTIES

Here we define the trace distance between two assemblages as

$$D_A(\{\sigma_{a|x}\}_{a,x},\{\rho_{a|x}\}_{a,x}) = \frac{1}{N_x}\sum_{a,x} D(\sigma_{a|x},\rho_{a|x}), \tag{A1}$$

where $D(\rho,\rho') := \frac{1}{2}\|\rho-\rho'\|$ and $\|X\| := \mathrm{tr}[|X|]$ is the trace norm. Now we show that D_A satisfies the following three basic properties required for a true metric:

(1) It is obvious that $D_A(\{\sigma_{a|x}\}_{a,x},\{\rho_{a|x}\}_{a,x})=0$ because $\{\sigma_{a|x}\}_{a,x}$ and $\{\rho_{a|x}\}_{a,x}$ are the same.

(2) Here, we prove that the trace distance between two assemblages is symmetric:

$$\begin{aligned} D_A(\{\sigma_{a|x}\}_{a,x},\{\rho_{a|x}\}_{a,x}) &= \frac{1}{N_x}\sum_{a,x} D(\{\sigma_{a|x}\}_{a,x},\{\rho_{a|x}\}_{a,x}) \\ &= \frac{1}{N_x}\sum_{a,x} D(\{\rho_{a|x}\}_{a,x},\{\sigma_{a|x}\}_{a,x}) \\ &= D_A(\{\rho_{a|x}\}_{a,x},\{\sigma_{a|x}\}_{a,x}). \end{aligned} \tag{A2}$$

The second equality is based on a property of the matrix norm.

(3) We now show that the trace distance between two assemblages satisfies the triangle inequality,

$$\begin{aligned} &D_A(\{\sigma_{a|x}\}_{a,x},\{\rho_{a|x}\}_{a,x}) \\ &= \frac{1}{N_x}\sum_{a,x} D(\{\sigma_{a|x}\}_{a,x},\{\rho_{a|x}\}_{a,x}) \\ &\leqslant \frac{1}{N_x}\sum_{a,x}[D(\{\sigma_{a|x}\}_{a,x},\{\theta_{a|x}\}_{a,x}) + D(\{\theta_{a|x}\}_{a,x},\{\rho_{a|x}\}_{a,x})] \\ &= D_A(\{\sigma_{a|x}\}_{a,x},\{\theta_{a|x}\}_{a,x}) + D_A(\{\theta_{a|x}\}_{a,x},\{\rho_{a|x}\}_{a,x}). \end{aligned} \tag{A3}$$

The first inequality follows from the property of the trace norm. This completes our proof.

APPENDIX B: RESTRICTED CONVEX STEERING MONOTONE

First, we recall the definition of a convex steering monotone introduced in Ref. [7] with the restrictive assumption from Ref. [54], namely, the independence of Alice's choice from Bob's outcome ω.

A function S, relating assemblages with non-negative real numbers, is a convex steering monotone if it satisfies the following:

(i) It vanishes for unsteerable assemblages,

$$S(\{\sigma_{a|x}\}_{a,x}) = 0 \quad \text{for all} \quad \{\sigma_{a|x}\}_{a,x}\in \mathrm{LHS}. \tag{B1}$$

(ii) (Monotonicity) S is nonincreasing, on average, under restricted one-way LOCCs, i.e.,

$$\sum_{\omega} P(\omega)\, S\left(\left\{\frac{\sigma^{\omega}_{a'|x'}}{P(\omega)}\right\}_{a',x'}\right) \leqslant S(\{\sigma_{a|x}\}_{a,x}), \tag{B2}$$

where

$$\sigma^{\omega}_{a'|x'} := \sum_{a,x} p(x|x')p(a'|a,x,x',\omega)K_\omega\sigma_{a|x}K^\dagger_\omega \tag{B3}$$

is an assemblage obtained from the initial assemblage $\{\sigma_{a|x}\}_{a,x}$ by performing restricted one-way LOCCs. Here, K_ω is a Kraus operator with outcome ω and $\sum_i K_i^\dagger K_i = \mathbb{1}$, while $p(x|x')$ and $p(a'|a,x,x',\omega)$ are classical postprocessing, i.e., deterministic wiring maps, on Alice's side.

(iii) (Convexity) Given a real number $0 \leqslant \mu \leqslant 1$ and two assemblages $\{\sigma_{a|x}\}_{a,x}$ and $\{\sigma'_{a|x}\}_{a,x}$, the steering function S satisfies the inequality

$$\begin{aligned}&S(\mu\{\sigma_{a|x}\}_{a,x} + (1-\mu)\{\sigma'_{a|x}\}_{a,x})\\&\quad\leqslant \mu S(\{\sigma_{a|x}\}_{a,x}) + (1-\mu)S(\{\sigma'_{a|x}\}_{a,x}).\end{aligned}$$

Given an assemblage, we recall that the consistent trace-distance measure of steerability is defined as

$$\begin{aligned}S_{\mathrm{TD}}(\{\sigma_{a|x}\}_{a,x}) := \min\Bigg\{&D_A(\{\sigma_{a|x}\}_{a,x},\{\rho_{a|x}\}_{a,x})|\\&\{\rho_{a|x}\}_{a,x} \in \mathrm{LHS},\ \sum_a \rho_{a|x} = \sum_a \sigma_{a|x},\ \forall x\Bigg\},\end{aligned} \tag{B4}$$

where $D_A(\{\sigma_{a|x}\}_{a,x},\{\sigma'_{a|x}\}_{a,x}) = \sum_{a,x} p(x) D(\sigma_{a|x},\sigma'_{a|x})$. First, it is obvious that the trace-distance measure of steerability satisfies condition (i). Before we prove that the trace-distance measure of steerability satisfies condition (ii), we prove the following Lemma.

Lemma 1. Let $\{\mathcal{I}_\omega\}_\omega$ be a collection of positive trace-nonincreasing maps, summing up to a trace-nonincreasing map $\mathcal{I} := \sum_\omega \mathcal{I}_\omega$. Then, for any Hermitian operators T and S, we have

$$\sum_\omega \mathrm{tr}[|\mathcal{I}_\omega(T) - \mathcal{I}_\omega(S)|] \leqslant \mathrm{tr}[|T-S|]. \tag{B5}$$

Proof. The proof is a slight modification of the one by Ruskai [74]. Let us define $X := T - S$. Since X is Hermitian, by spectral decomposition, we can write $X = X^+ - X^-$, with $X^+, X^- \geqslant 0$. We then have

$$\begin{aligned}&\sum_\omega \mathrm{tr}[|\mathcal{I}_\omega(T) - \mathcal{I}_\omega(S)|]\\&= \sum_\omega \mathrm{tr}[|\mathcal{I}_\omega(X)|] = \sum_\omega \mathrm{tr}[|\mathcal{I}_\omega(X^+) - \mathcal{I}_\omega(X^-)|]\\&\leqslant \sum_\omega \mathrm{tr}[|\mathcal{I}_\omega(X^+)|] + \mathrm{tr}[|\mathcal{I}_\omega(X^-)|]\\&= \sum_\omega \mathrm{tr}[\mathcal{I}_\omega(X^+)] + \mathrm{tr}[\mathcal{I}_\omega(X^-)]\\&= \mathrm{tr}\left[\sum_\omega \mathcal{I}_\omega(X^+ + X^-)\right]\\&\leqslant \mathrm{tr}[X^+ + X^-] = \mathrm{tr}[|X^+ - X^-|] = \mathrm{tr}[|X|],\end{aligned} \tag{B6}$$

where $\mathrm{tr}[|X|]$ and $\mathrm{tr}[X]$ denote the trace norm and trace, respectively, andw we used, in order, the triangle inequality, positivity, linearity, and trace-nonincreasing property. ■

Lemma 2. The trace distance between two assemblages does not increase under deterministic wiring maps on Alice's side, under the restricted hypothesis $p(x|x',\omega) = p(x|x')$ of Eq. (B3).

Proof. A wiring map W_ω, depending on a parameter ω, is a transformation of assemblages into assemblages given (component-wise) by Eq. (4). Note that given two assemblages $\{\sigma^1_{a|x}\}_{a,x}, \{\sigma^2_{a|x}\}_{a,x}$, we can write

$$\begin{aligned}D_A\big(W_\omega\big(\{\sigma^1_{a|x}\}_{a,x}\big), W_\omega\big(\{\sigma^2_{a|x}\}_{a,x}\big)\big) &= D_A\left(\left\{\sum_{a,x} p(x|x')p(a'|a,x,x',\omega)\sigma^1_{a|x}\right\}_{a',x'}, \left\{\sum_{a,x} p(x|x')p(a'|a,x,x',\omega)\sigma^2_{a|x}\right\}_{a',x'}\right)\\&= \sum_{a',x'} p(x') D\left(\sum_{a,x} p(x|x')p(a'|a,x,x',\omega)\sigma^1_{a|x}, \sum_{a,x} p(x|x')p(a'|a,x,x',\omega)\sigma^2_{a|x}\right)\\&\leqslant \sum_{a',x',a,x} p(x|x')p(a'|a,x,x',\omega)p(x')D\big(\sigma^1_{a|x},\sigma^2_{a|x}\big) = \sum_{a,x,x'} p(x,x')D\big(\sigma^1_{a|x},\sigma^2_{a|x}\big)\\&= \sum_{a,x} p(x) D\big(\sigma^1_{a|x},\sigma^2_{a|x}\big) = D_A\big(\{\sigma^1_{a|x}\}_{a,x},\{\sigma^2_{a|x}\}_{a,x}\big),\end{aligned} \tag{B7}$$

where the inequality holds since, for $\lambda_i \geqslant 0$ (not necessarily summing up to one),

$$D\left(\sum_i \lambda_i\rho_i, \sum_i \lambda_i\rho'_i\right) = \frac{1}{2}\mathrm{tr}\left|\sum_i \lambda_i(\rho_i - \rho'_i)\right| \leqslant \frac{1}{2}\sum_i \mathrm{tr}\,|\lambda_i(\rho_i - \rho'_i)| = \frac{1}{2}\sum_i \lambda_i\,\mathrm{tr}\,|\rho_i - \rho'_i|. \tag{B8}$$

This concludes the proof. ■

Lemma 3. The quantifier S_{TD} does not increase, on average, by performing local operations on Bob's side defined by a collection of completely positive trace-nonincreasing maps $\{\mathcal{I}_\omega\}_\omega$, which sum up to a trace-preserving map $\mathcal{I} = \sum_\omega \mathcal{I}_\omega$.

Proof. Let $\{\tilde{\rho}^{*\omega}_{a|x}\}_{a,x}$ be the optimal unsteerable consistent assemblage giving the minimum trace distance for $\mathcal{I}_\omega(\{\sigma_{a|x}\}_{a,x})/P(\omega)$, and $\{\rho^*_{a|x}\}_{a,x}$ the unsteerable consistent assemblage giving the minimum trace distance for $\{\sigma_{a|x}\}_{a,x}$. We can then write

$$\begin{aligned}\sum_{\omega} P(\omega) S_{\rm TD}\left(\frac{\mathcal{I}_\omega(\{\sigma_{a|x}\}_{a,x})}{P(\omega)}\right) &= \sum_{\omega} P(\omega) D_A\left(\frac{\mathcal{I}_\omega(\{\sigma_{a|x}\}_{a,x})}{P(\omega)},\{\tilde{\rho}^*_{a|x}\}\right) \leqslant \sum_{\omega} P(\omega) D_A\left(\frac{\mathcal{I}_\omega(\{\sigma_{a|x}\}_{a,x})}{P(\omega)},\frac{\mathcal{I}_\omega(\{\rho^*_{a|x}\}_{a,x})}{P(\omega)}\right)\\ &= \sum_{\omega} D_A(\mathcal{I}_\omega(\{\sigma_{a|x}\}_{a,x}),\mathcal{I}_\omega(\{\rho^*_{a|x}\}_{a,x})) = \sum_{\omega,a,x} \mathrm{tr}[|\mathcal{I}_\omega(\sigma_{a|x}) - \mathcal{I}_\omega(\rho^*_{a|x})|] \leqslant \sum_{a,x} \mathrm{tr}[|\sigma_{a|x} - \rho^*_{a|x}|],\end{aligned} \tag{B9}$$

where we used for the first inequality the fact that $\tilde{\rho}^*_{a|x}$ is the minimum, linearity of the trace distance for non-negative $P(\omega)$, and Lemma 1 for the last inequality. ■

Theorem 1. The consistent trace-distance measure of steerability $S_{\rm TD}$ does not increase on average under restricted one-way LOCCs, namely,

$$\sum_{\omega} P(\omega)\; S_{\rm TD}\left(\left\{\frac{\sigma^{\omega}_{a'|x'}}{P(\omega)}\right\}_{a',x'}\right) \leqslant S_{\rm TD}(\{\sigma_{a|x}\}_{a,x}). \tag{B10}$$

Proof. The proof is simply given by an application of Lemmas 2 and 3, namely,

$$\begin{aligned}\sum_{\omega} P(\omega)\; S_{\rm TD}\left(\left\{\frac{\sigma^{\omega}_{a'|x'}}{P(\omega)}\right\}_{a',x'}\right) &= \sum_{\omega} P(\omega)\; S_{\rm TD}\left(\left\{\frac{W_\omega(\{K_\omega\sigma_{a|x}K_\omega^\dagger\}_{a,x})_{a'|x'}}{P(\omega)}\right\}_{a',x'}\right)\\ &\leqslant \sum_{\omega} P(\omega)\; S_{\rm TD}\left(\left\{\frac{K_\omega\sigma_{a|x}K_\omega^\dagger}{P(\omega)}\right\}_{a,x}\right) \leqslant S_{\rm TD}(\{\sigma_{a|x}\}_{a,x}).\end{aligned} \tag{B11}$$

■

Finally, we prove convexity. Given the assemblage $\{K_{a|x}\}_{a|x}$, obtained as a convex mixture $K_{a|x} = \mu\sigma_{a|x} + (1-\mu)\rho_{a|x}$, we have

$$\begin{aligned}S_{\rm TD}(\{K_{a|x}\}_{a|x}) :&= \min_{\{K^{\rm US}_{a|x}\}_{a,x}\in{\rm LHS}} D_A\big(\{K_{a|x}\}_{a,x},\big\{K^{\rm US}_{a|x}\big\}_{a,x}\big)\\ :&= \min_{\{K^{\rm US}_{a|x}\}_{a,x}\in{\rm LHS}} D_A\big(\{\mu\sigma_{a|x} + (1-\mu)\rho_{a|x}\}_{a,x},\big\{K^{\rm US}_{a|x}\big\}_{a,x}\big)\\ &\leqslant D_A\big(\{\mu\sigma_{a|x} + (1-\mu)\rho_{a|x}\}_{a,x},\big\{\mu\tilde{\sigma}^{\rm US}_{a|x} + (1-\mu)\tilde{\rho}^{\rm US}_{a|x}\big\}_{a,x}\big)\\ &= \sum_{a,x} p(x)\frac{1}{2}\big\|\mu\sigma_{a|x} + (1-\mu)\rho_{a|x} - \mu\tilde{\sigma}^{\rm US}_{a|x} - (1-\mu)\tilde{\rho}^{\rm US}_{a|x}\big\|\\ &= \sum_{a,x} p(x)\frac{1}{2}\big\|\mu\sigma_{a|x} - \mu\tilde{\sigma}^{\rm US}_{a|x} + (1-\mu)\rho_{a|x} - (1-\mu)\tilde{\rho}^{\rm US}_{a|x}\big\|\\ &\leqslant \sum_{a,x} p(x)\left\{\frac{1}{2}\big\|\mu\sigma_{a|x} - \mu\tilde{\sigma}^{\rm US}_{a|x}\big\| + \frac{1}{2}\big\|(1-\mu)\rho_{a|x} - (1-\mu)\tilde{\rho}^{\rm US}_{a|x}\big\|\right\}\\ &= \mu S_{\rm TD}(\{\sigma_{a|x}\}_{a,x}) + (1-\mu)S_{\rm TD}(\{\rho_{a|x}\}_{a,x}).\end{aligned} \tag{B12}$$

The first inequality is that the convex combination of the other two optimal LHS assemblages is not necessarily the optimal assemblage for the convex combination assemblages (but it is still consistent in the sense of the total reduced state). The final inequality is due to the property of the trace norm. This completes our proof of convexity.

APPENDIX C: SEMIDEFINITE PROGRAMMING FORMULATION OF $S^{\rm R}_{\rm CSR}$

The RNCSR, $S^{\rm R}_{\rm CSR}$ defined by Eq. (8), can be computed by the following SDP:

$$\begin{aligned}&\text{min:}\; \sum_{\lambda} \mathrm{tr}(\sigma_\lambda) - 1,\\ &\text{subject to:}\; \sum_{\lambda} p(a|x,\lambda)\sigma_\lambda - \sigma_{a|x}\\ &\qquad \geqslant \left[\sum_{\lambda}\mathrm{tr}(\sigma_\lambda) - 1\right]\mathrm{tr}(\sigma_{a|x})\sigma_B,\\ &\qquad \text{for all } a,x;\\ &\qquad \sum_{\lambda}\mathrm{tr}(\sigma_\lambda) \geqslant 1;\\ &\qquad \sigma_\lambda \geqslant 0,\quad \forall\lambda;\end{aligned} \tag{C1}$$

with $p(a|x,\lambda)$ taken as the deterministic strategies, i.e., $\lambda := (\lambda_x)_x$ and $p(a|x,\lambda) := \delta_{a,\lambda_x}$.

In fact, it is sufficient to note that for all $t \geqslant S^{\rm R}_{\rm CSR}$, there exists $\{\sigma'_\lambda\}_\lambda$ such that

$$(1+t)\sum_{\lambda} p(a|x,\lambda)\sigma'_\lambda - \sigma_{a|x} = t\,\mathrm{tr}(\sigma_{a|x})\sigma_B. \tag{C2}$$

Since $\sum_\lambda \mathrm{tr}(\sigma'_\lambda) = 1$, we can absorb the factor $(1+t)$ into the LHS assemblage, i.e., $\sigma_\lambda = (1+t)\sigma'_\lambda$. Note that the first inequality in the definition of the SDP, despite being a weaker condition than the inequality, does not provide a lower value of $S^{\mathrm{R}}_{\mathrm{CSR}}$. To prove this, let us just consider a feasible solution $\{\sigma_\lambda\}$ of the SDP; we then have

$$\sum_\lambda p(a|x,\lambda)\sigma_\lambda - \sigma_{a|x} - \left[\sum_\lambda \mathrm{tr}(\sigma_\lambda) - 1\right]\mathrm{tr}(\sigma_{a|x})\sigma_B =: \eta_{a|x} \geqslant 0. \tag{C3}$$

Then, by summing over a and taking the trace of the left-hand side of (C3), we obtain $\mathrm{tr}(\sum_a \eta_{a|x}) = 0$, for all x, which implies $\eta_{a|x} = 0$ for all a,x, since $\eta_{a|x} \geqslant 0$, by our assumption.

APPENDIX D: LHS SURFACE OF WERNER STATES

Here, we assume that one measurement is the Z measurement and the others are aligned in the XY plane. Since three measurements are orthogonal, the Werner states can be expanded in the Pauli bases, i.e., $\rho = \frac{1}{4}(\mathbb{1} - \sum_{i=1}^3 p\sigma_i \otimes \sigma_i)$. All projective measurements can also be expressed in the Pauli bases, i.e., $E_{\pm|a} = \frac{1}{2}(\mathbb{1} \pm \sum_{i=1}^3 a_i\sigma_i)$, where a can be seen as a vector in the Bloch sphere. Once Alice performs projective measurements on her qubit, Bob's qubit collapses into $\rho^a_\pm = \frac{1}{2}(\mathbb{1} \pm \sum_i p a_i \sigma_i)$.

Now we use spherical coordinates to expand $a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Alice can choose three orthogonal a_i given by

$$\begin{aligned} a_1 &= (\cos\phi, \sin\phi, 0), \\ a_2 &= (\cos(\phi+\pi/2), \sin(\phi+\pi/2), 0), \\ a_3 &= (0,0,1). \end{aligned} \tag{D1}$$

The postmeasurement states which Bob holds are

$$\begin{aligned} \rho^{a_1}_+ &= (p\cos\phi, p\sin\phi, 0), \\ \rho^{a_1}_- &= (-p\cos\phi, -p\sin\phi, 0), \\ \rho^{a_2}_+ &= (-p\sin\phi, p\cos\phi, 0), \\ \rho^{a_2}_- &= (p\sin\phi, -p\cos\phi, 0), \\ \rho^{a_3}_+ &= (0,0,p),\ \rho^{a_3}_- = (0,0,-p). \end{aligned} \tag{D2}$$

Here we already use the Bloch-vector representation of a quantum state. There are eight preexisted quantum states σ_λ, which can be expressed as

$$\begin{aligned} \sigma_{\lambda 1} &= p(\cos\phi - \sin\phi, \sin\phi + \cos\phi, 1), \\ \sigma_{\lambda 2} &= p(\cos\phi + \sin\phi, \sin\phi - \cos\phi, 1), \\ \sigma_{\lambda 3} &= p(-\cos\phi - \sin\phi, -\sin\phi + \cos\phi, 1), \\ \sigma_{\lambda 4} &= p(-\cos\phi + \sin\phi, -\sin\phi - \cos\phi, 1), \\ \sigma_{\lambda 5} &= p(\cos\phi - \sin\phi, \sin\phi + \cos\phi, -1), \\ \sigma_{\lambda 6} &= p(\cos\phi + \sin\phi, \sin\phi - \cos\phi, -1), \\ \sigma_{\lambda 7} &= p(-\cos\phi - \sin\phi, -\sin\phi + \cos\phi, -1), \\ \sigma_{\lambda 8} &= p(-\cos\phi + \sin\phi, -\sin\phi - \cos\phi, -1). \end{aligned}$$

It is obvious that the preexisted quantum states only exist when $p \leqslant 1/\sqrt{3}$ because the radius of a pure-state Bloch vector is equal to one. One can choose four specific preexisted quantum states to mimic the postmeasurement states with equal probability of 1/4. Thus, the LHS states are $\rho^{a_i,\mathrm{US}}_\pm$ and are given by Eqs. (D2), with $p \leqslant 1/\sqrt{3}$ for $i = 1,2$, and 3. As $p > 1/\sqrt{3}$, the LHS states do not exist. We can easily check that the states $\rho^{a_i,\mathrm{US}}_\pm$, given by Eq. (D2) for $p \leqslant 1/\sqrt{3}$, are located at the circle centered at (0,0,0) and with radius p because the Werner states are highly symmetrical. Once we rotate the measurement settings, the new measurements which correspond to the original XY plane are also located on a circle. Thus, the LHS state of the Werner state is a sphere.

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