Nonlocal macroscopic quantum tunneling and quantum terahertz electrodynamics in layered superconductors: Theory and simulations

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(Received 23 October 2007; published 16 January 2008)

We derive a quantum field theory of Josephson plasma waves (JPWs) in layered superconductors, which describes two types of interacting JPW bosonic quanta (one heavy and one lighter). We propose a mechanism of enhancement of macroscopic quantum tunneling (MQT) in stacks of intrinsic Josephson junctions. Due to the long-range interaction between junctions in layered superconductors, the calculated MQT escape rate Γ has a nonlinear dependence on the number of junctions in the stack. We develop a numerical procedure, based on quantum field tunneling theory, to calculate Γ for the stack of Josephson junctions. We also propose a simple analytical formula to estimate the MQT escape rate. Moreover, we demonstrate that the direct analogy between fluxon tunneling and tunneling of a quantum particle fails even for very thin junction stacks (about 1 μm for $Bi_2Sr_2CaCu_2O_{8+\delta}$) and a field-theoretical approach is necessary. The theory developed here allows to quantitatively describe striking recent experiments in $Bi_2Sr_2CaCu_2O_{8+\delta}$ stacks.

DOI: 10.1103/PhysRevB.77.014509 PACS number(s): 74.72.Hs

I. INTRODUCTION

The recent surge of interest in stacks of intrinsic Josephson junctions is partly motivated by the desire to develop terahertz devices, including emitters,[1,2] filters, detectors, and nonlinear devices.[3] Macroscopic quantum tunneling (MQT) has been, until recently, considered to be negligible in high-T_c superconductors due to the d-wave symmetry of the order parameter. Recent unexpected experimental evidence[4,5] of MQT in layered superconductors could open a new avenue for the potential application of stacks of Josephson junctions in quantum electronics.[6] This requires a quantum theory capable of quantitatively describing this recent stream of remarkable experimental data (e.g., Refs. 4 and 5). In contrast to a *single* Josephson junction, *stacks* of *intrinsic* Josephson junctions are strongly coupled along the direction perpendicular to the layers. This is because the thickness of these layers is of the order of a few nanometers, which is much smaller than the magnetic penetration length. This results in a profoundly *nonlocal* electrodynamics[2] that strongly affects quantum fluctuations in layered superconductors.

The two main results of this work are as follows: first, the quantum electrodynamics of Josephson plasma waves (JPWs), and second, the quantitative description of macroscopic quantum tunneling in stacks of Josephson junctions. Namely, using a general Lagrangian approach, we derive the theory of quantum JPWs, which describes two interacting quantum fields: a heavy JPW and a lighter one. We predict resonances in the amplitudes of quantum processes associated with the creation of pairs of JPW quanta.

Using a general approach, we develop a quantitative theory of MQT in stacks of Josephson junctions. Our study is based on the analysis of coupled sine-Gordon equations and field-theoretical approach to fluxon tunneling, which allows us to adequately describe the *long-range* interactions in $Bi_2Sr_2CaCu_2O_{8+\delta}$ stacks, in contrast to the phenomenological treatment[7,8] of capacitively coupled Josephson junctions. We calculate the MQT escape rate Γ numerically and, following our approach,[9] suggest an approximate analytical formula for Γ. The obtained value of Γ is strongly nonlinear with respect to the number of superconducting layers N and changes to $\Gamma \propto N$ when N exceeds a certain critical value N_c. Our numerical results are in good quantitative agreement with recent exciting experiments[5] and our analytical formulas provide a simple estimate of the MQT escape rate.

II. QUANTUM THEORY FOR LAYERED SUPERCONDUCTORS

A. Lagrangian description for two interacting fields

The electrodynamics of stacks of Josephson junctions can be described by the Lagrangian

$$\mathcal{L} = \sum_n \int dx \left[\frac{1}{2}\dot{\varphi}_n^2 + \frac{1}{2\gamma^2}\dot{p}_n^2 - \frac{1}{2}(\partial_x \varphi_n)^2 - \frac{1}{2}(\partial_y p_n)^2 - \frac{1}{2}p_n^2 + \cos\varphi_n + \frac{1}{2}(\partial_x p_n \partial_y \varphi_n + \partial_y p_n \partial_x \varphi_n) \right], \quad (1)$$

where

$$\varphi_n \equiv \chi_{n+1} - \chi_n - \frac{2\pi s A_y^{(n)}}{\Phi_0}$$

is the gauge-invariant interlayer phase difference and

$$p_n \equiv \frac{s}{\lambda_{ab}}\partial_x \chi_n - \frac{2\pi\gamma s A_x^{(n)}}{\Phi_0}$$

is the normalized superconducting momentum in the nth layer. Here, we introduce the phase χ_n of the order parameter, the interlayer distance s, the in-plane λ_{ab} and out-of-plane λ_c penetration depths, the anisotropy parameter γ

$=\lambda_c/\lambda_{ab}$, flux quantum Φ_0, and vector potential $\vec{A}$. The in-plane coordinate x is normalized by λ_c; the time t is normalized by $1/\omega_J$, where the plasma frequency is ω_J; also, $\partial_x=\partial/\partial x$, $\partial_y f_n=\lambda_{ab}(f_{n+1}-f_n)/s$, $\dot{\varphi}=\partial\varphi/\partial t$. Hereafter, time derivative will be denoted by a dot above the symbol (i.e., $\dot{\varphi}=\partial\varphi/\partial t$). The z axis is pointed along the magnetic field. Here, we ignore dissipation, which was shown[4,10] to be negligible.

Varying the action $\mathcal{S}=\int dt\mathcal{L}$ produces the dynamical equations for the phase difference

$$\ddot{\varphi}_n-\partial_x^2\varphi_n+\sin\varphi_n+\partial_x\partial_y p_n=0,$$

$$\frac{1}{\gamma^2}\ddot{p}_n-\partial_y^2 p_n+p_n+\partial_x\partial_y\varphi_n=0, \tag{2}$$

which reduces to the usual coupled sine-Gordon equations[11] for $\gamma^2\gg 1$. Note that a Lagrangian approach for stacks of Josephson junctions can be formulated only for *two* interacting fields φ and p. This is because the vector potential has two components, A_x and A_y, in stacks of Josephson junctions, in contrast to one-dimensional Josephson junctions where one component of the vector potential is enough.

Linearizing Eq. (2) and substituting

$$p,\ \varphi\propto\ \exp(i\omega t+ik_x x+ik_y y),$$

we derive a biquadratic equation,

$$(\omega^2-k_x^2-1)\left(\frac{\omega^2}{\gamma^2}-k_y^2-1\right)-k_x^2k_y^2=0,$$

for the spectrum of the classical JPWs in the continuous limit (i.e., $k_y s\ll 1$) and $\gamma^2\gg 1$. Here, k_x and k_y are the wave vectors (momenta in the quantum description; here, $\hbar=1$) of the JPWs. This equation determines two branches, $\omega=\omega_a(\vec{k})$ and $\omega_b(\vec{k})$, of JPWs,

$$\omega_a(\vec{k})=\left(1+\frac{k_x^2}{1+k_y^2}\right)^{1/2},\quad \omega_b(\vec{k})=\gamma(k_y^2+1)^{1/2}, \tag{3}$$

up to terms of the order of $1/\gamma^2$. The a branch describes Josephson plasmons propagating both along and perpendicular to the layers, while the b branch describes plasmons propagating only perpendicular to the layers.

B. Quantum plasma waves

In order to quantize the JPWs, we use the Hamiltonian

$$\mathcal{H}=\sum_n\int dx(\Pi_{\varphi_n}\dot{\varphi}_n+\Pi_{p_n}\dot{p}_n)-\mathcal{L},$$

with the momenta Π_{φ_n} and Π_{p_n} of the φ_n and p_n fields, and require the standard commutation relations

$$[\varphi_{n'}(x'),\Pi_{\varphi_n}(x)]=i\delta(x-x')\delta_{nn'},$$

$$[p_{n'}(x'),\Pi_{p_n}(x)]=i\delta(x-x')\delta_{nn'}$$

(all other commutators are zero), where δ is either a delta function or Kronecker symbol. Expanding $\cos\varphi_n=1-\varphi_n^2/2+\varphi_n^4/24-\cdots$, we can write

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{\text{an}},$$

where we include terms up to φ_n^2 in $\mathcal{H}_0$ and the anharmonic terms in $\mathcal{H}_{\text{an}}$. Diagonalizing $\mathcal{H}_0$, we obtain the Hamiltonian for the Bosonic free fields a and b,

$$\mathcal{H}_0=\sum_{k_y}\int\frac{dk_x}{2\pi}[\omega_a(\vec{k})a^+a+\omega_b(\vec{k})b^+b].$$

The original fields φ_n, p_n in Eq. (1) are related to the free Bosonic fields a and b by

$$\varphi\approx\frac{a^+ +a}{\sqrt{2\omega_a(\mathbf{k})}}-\mathcal{Z}\frac{b^+ +b}{\gamma\sqrt{2\omega_b(\mathbf{k})}},$$

$$p\approx\mathcal{Z}\frac{a^+ +a}{\sqrt{2\omega_a(\mathbf{k})}}+\gamma\frac{b^+ +b}{\sqrt{2\omega_b(\mathbf{k})}},$$

where $\mathcal{Z}=k_xk_y/(k_y^2+1)$. Equation (3) shows that the "mass" of the a quantum equals 1 and for the heavier b-quasiparticle is γ in our dimensionless units.

C. Analogy with quantum electrodynamics

The interaction between the a and b fields, including the self-interaction, occurs due to the *anharmonic* terms in

$$\mathcal{H}_{\text{an}}\approx-\frac{1}{24}\sum_n\int dx\varphi_n^4+\cdots.$$

In the leading order with respect to the bosonic field interactions, an a particle can create either $a+a$ or $a+b$ pairs. Using Eq. (3), one can conclude that the amplitudes of these processes have energy thresholds

$$\omega_a(\vec{k}_1)=3\ \text{or}\ \gamma+2.$$

Note that it is similar to the $2mc^2$ rest energy threshold for e^-+e^+ *pair creation in usual quantum electrodynamics*. These can result in resonances in the amplitudes of quantum processes (e.g., decay of the a quanta).

III. ENHANCEMENT OF MACROSCOPIC QUANTUM TUNNELING

Now, we apply this field theory to interpret very recent experiments[5] on MQT in $Bi_2Sr_2CaCu_2O_{8+\delta}$. We consider a stack of $N\gg 1$ Josephson junctions having the sizes $D\times L_z$ in the plane of the junctions and L_y across them. To observe MQT, an external current J, close to the critical value J_c, was applied.[5]

A. Effective Lagrangian

In the continuous limit and when $\gamma^2\gg 1$, Eq. (2) can be rewritten as

$$\left(1-\frac{\partial^2}{\partial y^2}\right)\left[\frac{\partial^2\varphi}{\partial t^2}+\sin\varphi\right]-\frac{\partial^2\varphi}{\partial x^2}=0. \tag{4}$$

Here, the coordinate y (transverse to the layers) is normalized by λ_{ab}. This stack bridges two bulk superconducting sheets. The current close to the critical value flows across the stack and the external magnetic field is zero. We neglect the disturbance that the tunneling fluxon produces in the bulk superconductors. The latter assumption is usual for Josephson system and gives a correct result in the limit of a single junction. In this case, the boundary conditions to Eq. (4) are

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=0,d} = \mp \frac{jd}{2}, \qquad \left.\frac{\partial \varphi}{\partial y}\right|_{y=0,L} = 0, \tag{5}$$

where j is the current density flowing through the junctions normalized by the critical current density i_0, $L=Ns/\lambda_{ab}$ is the dimensionless size of the system in the y direction, $d=D/\lambda_c$, and D is the junction width in the x direction.

When tunneling occurs, the phase difference in a junction changes from 0 to 2π, which can be interpreted as the tunneling of a fluxon through the contact. This process can be safely described within a semiclassical approximation and we use the approach developed in Refs. 5, 9, and 12–14 to calculate the escape rate Γ of a fluxon through the potential barrier.

The probability of quantum tunneling in the semiclassical limit is expressed through the classical action of the system in imaginary time. However, the simplified Eq. (4) *has no Lagrangian*. This occurs because we neglect the second order time derivative of the field p in Eq. (2) in the limit $\gamma^2 \to \infty$ and, as a result, obtain a relation between φ and p instead of the second dynamic equation.

In general, we have to use an action of the form in Eq. (1) with two interacting bosonic fields φ and p, which could produce a rather cumbersome mathematical problem. In order to avoid this difficulty, we follow the approach described in Refs. 9 and 14. First, we seek a solution to Eq. (4) in imaginary time $t=i\tau$ in the form

$$\varphi(\tau,x,y) = \varphi_0(x) + \psi(\tau,x,y),$$

where $\varphi_0(x)$ is a steady-state solution corresponding to an energy minimum of the junction's stack. It does not depend on the y coordinate and satisfies the equation

$$\frac{\partial^2 \varphi_0}{\partial x^2} = \sin \varphi_0. \tag{6}$$

Below, we consider the limit of short junctions $d \ll 1$ ($D \ll \lambda_c$), which corresponds to the usual experimental conditions.[5] In this case, the solution of Eq. (6) with boundary conditions [Eq. (5)] has the form

$$\varphi_0(x) = \arcsin(j) + \frac{j}{2}\left(x - \frac{d}{2}\right)^2 + O(x^4). \tag{7}$$

Substituting the expansion $\varphi=\varphi_0+\psi$ into Eq. (4) and proceeding to the limit $d \ll 0$, we derive the following equation for $\psi(\tau,x,y)$ in imaginary time:

$$\left(1 - \frac{\partial^2}{\partial y^2}\right)\left[-\frac{\partial^2 \psi}{\partial \tau^2} - j(1-\cos\psi) + \sqrt{1-j^2}\sin\psi\right] - \frac{\partial^2 \psi}{\partial x^2} = 0. \tag{8}$$

We assume that the fluxon tunnels mainly through one of the junctions and we can linearize Eq. (8) in all junctions except this one. We denote the number or label for this junction as l. The linearized equation for ψ is

$$\left(1 - \frac{\partial^2}{\partial y^2}\right)\left[-\frac{\partial^2 \psi}{\partial \tau^2} + \bar{\mu}_0 \psi\right] - \frac{\partial^2 \psi}{\partial x^2} = 0, \tag{9}$$

where

$$\bar{\mu}_0 = \sqrt{1-j^2}.$$

This equation is valid in all junctions except for the lth junction, where the flux or tunnels located at the position

$$y_0 = L_1 = \frac{ls}{\lambda_{ab}}.$$

The function $\psi(\tau,x,y)$ satisfies the boundary conditions

$$d\psi/dx = 0 \quad \text{at } x = 0,d$$

and

$$d\psi/dy = 0 \quad \text{at } y = 0,L.$$

Note that the characteristic size of the tunneling fluxon is $\gamma s \ll \lambda_c$ (this will be shown below) and it can be compared to the junction's width D. So, when $D \gtrsim \gamma s$, the x dependence of $\psi(\tau,x,y)$ is essential.

The solution to Eq. (9) with the specified above boundary conditions at $y=0,L$ and continuity condition at $y=L_1$, $\psi(\tau,x,L_1+0)=\psi(\tau,x,L_1-0)$ can be written in the form of the following expansion:

$$\psi(\tau,x,y) = \sum_{n=0}^{\infty} \int_0^{\infty} dp e^{-p\tau} \cos(k_n x) f_n(p,y) \psi_n(p), \tag{10}$$

where $k_n=\pi n/d$ and functions

$$f_n(p,y) = \begin{cases} \dfrac{\cosh[\nu_n(p)y]}{\cosh[\nu_n(p)L_1]}, & y < L_1 \\[2ex] \dfrac{\cosh[\nu_n(p)(L-y)]}{\cosh[\nu_n(p)(L-L_1)]}, & y > L_1, \end{cases} \tag{11}$$

where

$$\nu_n^2 = 1 + \frac{k_n^2}{\bar{\mu}_0 - p^2}.$$

The functions $\psi_n(p)$ in Eq. (10) are derived from the equation for ψ in the lth junction, $\bar{\psi}(\tau,x) \equiv \psi(\tau,x,L_1)$. The latter can be derived from the relation between the phase difference and the magnetic field in the lth layer. Using the standard relation[15]

$$\left.\frac{\partial \psi}{\partial x}\right|_{y=L_1} = \frac{8\pi^2\lambda_{ab}^2\lambda_c}{c\Phi_0}[J_x(y=L+0) - J_x(y=L-0)], \tag{12}$$

from Maxwell's equation, and the formula for the critical current $i_0=c\Phi_0/8\pi^2\lambda_c^2 s$, we obtain

$$\frac{\partial\psi}{\partial x} = \frac{c}{4\pi i_0 \gamma s}\left[\left(\left.\frac{\partial H}{\partial y}\right|_{L_1+0} - \left.\frac{\partial H}{\partial y}\right|_{L_1-0}\right)\right]. \quad (13)$$

Here, J_x is the x component of the current density and we neglect the x dependence of $\varphi_0(x)$ in left-hand side of Eq. (12).

We present the z component of the magnetic field in the form $H=H_0+\bar{H}$, where

$$\frac{\partial H_0}{\partial x} = -\frac{4\pi i_0 \lambda_c}{c}\sin\varphi_0$$

does not depend on y. According to Maxwell's equation, the field $\bar{H}$ linearly depends on ψ at $y\neq L_1$ and can be represented in the form

$$\bar{H}(\tau,x,y) = \sum_{n=1}^{\infty}\int_0^{\infty} dp e^{-p\tau}\sin(k_n x) f_n(p,y) h_n(p), \quad (14)$$

where $h_n(p)$ are functions which are independent of the coordinates x and y. Substituting the expansion [Eq. (14)] into Eq. (13), we obtain the relation between the functions $h_n(p)$ and $\psi_n(p)$,

$$h_n(p) = \left(\frac{2\pi i_0 \gamma s}{c}\right)\frac{k_n \chi_n(p)}{\nu_n(p)}\psi_n(p), \quad (15)$$

where

$$\chi_n = \frac{2\cosh[\nu_n L_1]\cosh[\nu_n(L-L_1)]}{\sinh[\nu_n L]}. \quad (16)$$

The Maxwell equation in the contact at $y=L_1$ is nonlinear with respect to $\bar{\psi}$,

$$-\frac{\partial H}{\partial x} = \frac{4\pi i_0 \lambda_c}{c}\left[\sin(\varphi_0+\bar{\psi}) - \frac{\partial^2\psi}{\partial\tau^2}\right].$$

The continuity condition of the y component of the current requires the continuity of the derivative $\partial H/\partial x$ at $y=L_1$. Then, substituting the expansion [Eq. (14)] into the latter equation, we obtain the following system of equations for $\psi_n(p)$:

$$-\frac{\partial^2\bar{\psi}}{\partial\tau^2} + \bar{\mu}_0\sin\bar{\psi} - j(1-\cos\bar{\psi}) = -\frac{s}{2\lambda_{ab}}\sum_{n=1}^{\infty}\int_0^{\infty} dp e^{-p\tau}\cos(k_n x)\frac{k_n^2\chi_n(p)}{\nu_n(p)}\psi_n(p), \quad (17)$$

where

$$\bar{\psi}(\tau,x) = \sum_{n=0}^{\infty}\int_0^{\infty} dp e^{-p\tau}\cos(k_n x)\psi_n(p). \quad (18)$$

It is well known that the wave with the frequency above plasma frequency,

$$\omega_p(j) = \omega_J(1-j^2)^{1/4},$$

can propagate in the Josephson junctions. Thus, the characteristic time of MQT is certainly lower than $1/\omega_p$. On the other hand, the only time scale of the considered problem [Eqs. (4) and (5)] is $1/\omega_p$ and we can consider the MQT as a quasistatic process putting $p=0$ in $\nu_n(p)$ and $\chi_n(p)$ in Eqs. (17). In the case under consideration, $d \ll 1$, we have

$$\nu_n = \frac{k_n}{(\bar{\mu}_0)^{1/2}}$$

for $n>0$. As a result, we reduce Eqs. (17) and (18) to

$$-\frac{\partial^2\bar{\psi}}{\partial\tau^2} + \bar{\mu}_0\sin\bar{\psi} - j(1-\cos\bar{\psi}) = \int_0^d dx' K(x;x')\frac{\partial^2\bar{\psi}(\tau,x')}{\partial x'^2}, \quad (19)$$

where the kernel $K(x;x')$ is

$$K(x;x') = \frac{\gamma s\sqrt{\bar{\mu}_0}}{D}\sum_{n=1}^{\infty}\cos(k_n x)\cos(k_n x')\frac{\chi_N^l(an)}{k_n} \quad (20)$$

and

$$\chi_N^l(a) = \frac{2\cosh(al)\cosh[a(N-l)]}{\sinh(aN)}, \quad a = \frac{\pi\gamma s}{D\sqrt{\bar{\mu}_0}}. \quad (21)$$

If $l,N \gg 1$ and $a\sim 1$, then $\chi_N^l(a)\cong 1$, and the kernel can be calculated explicitly,

$$K(x;x') = -\frac{\gamma s\sqrt{\bar{\mu}_0}}{2\pi\lambda_c}\ln\left|4\sin\left[\frac{\pi(x-x')}{2d}\right]\sin\left[\frac{\pi(x+x')}{2d}\right]\right|. \quad (22)$$

In contrast to Eq. (4), Eq. (19) has a Lagrangian, which can be written in imaginary time $t=i\tau$ as

$$\mathcal{L}_{\text{eff}}(\tau) = \varepsilon_0\int_0^d dx\left[-\frac{1}{2}\left(\frac{\partial\bar{\psi}}{\partial\tau}\right)^2 - \bar{\mu}_0(1-\cos\bar{\psi}) + j(\bar{\psi}-\sin\bar{\psi}) + \frac{1}{2}\bar{\psi}\int_0^d dx' K(x;x')\frac{\partial^2\bar{\psi}}{\partial x'^2}\right], \quad (23)$$

where

$$\varepsilon_0 = i_0 L_z \lambda_c/(2e\omega_J)$$

and L_z is the size of the junctions in the z direction. Indeed, it is easy to check that $\partial^2 K(x;x')/\partial x'^2 = \partial^2 K(x';x)/\partial x^2$, and variation of Eq. (23) with respect to $\bar{\psi}$ gives Eq. (19). Note that, in general, the effective Lagrangian depends on l and N by means of the functions χ_N^l.

B. Field tunneling: Numerical approach

The tunneling escape rate Γ, that is, the tunneling probability per unit time, can be calculated in the semiclassical approach for a system with a Lagrangian in the general form.[12] In the case of tunneling of a fluxon, Γ can be presented as[5,9,16]

$$\Gamma = \omega_p(j)\sum_{l=0}^{N}\sqrt{\frac{30B_N^l}{\pi}}\exp(-B_N^l), \tag{24}$$

where we take into account that the fluxon can tunnel through any junction $0<l<N$ of the stack and the tunneling exponent B_N^l can be expressed via the Lagrangian $\mathcal{L}_{\rm eff}$ in Eq. (23) as

$$B_N^l = -2\int_0^\infty d\tau \mathcal{L}_{\rm eff}(\tau). \tag{25}$$

Since the current I in the stack is assumed to be close to the critical value $I_c(d)$, we calculate B by means of an expansion similar to that developed in our previous paper Ref. 14 for the study of MQT in a single Josephson junction. We expand the Lagrangian [Eq. (23)] and the equation of motion [Eq. (19)] in series of $\bar{\psi}$ and seek function $\bar{\psi}$ of the form

$$\bar{\psi}(\tau,x) = \sum_{n=0}^{\infty} c_n(\tau)\psi_n(x), \tag{26}$$

where ψ_n are orthogonal eigenfunctions of the operator

$$\hat{L} = \bar{\mu}_0 - \int_0^d dx' K(x;x')\frac{\partial^2}{\partial x'^2}. \tag{27}$$

The tunneling exponent can be expanded as[14]

$$B_N^l = \frac{\varepsilon_0}{6}\int_0^\infty d\tau\left[\sum_{nmk} U_{nmk}^{(3)}c_nc_mc_k + \frac{1}{2}\sum_{nmkl} U_{nmkl}^{(4)}c_nc_mc_kc_l + \cdots\right], \tag{28}$$

where

$$U_{n\cdots k}^{(i)} = \int_0^d dx\ \left.\frac{\partial^i(\cos\varphi_0)}{\partial\varphi_0^i}\psi_n\cdots\psi_k\right|_{\varphi_0=\arcsin(j)}. \tag{29}$$

The functions c_n satisfy the system of equations

$$\ddot{c}_n - \mu_n c_n = -\frac{1}{2}\sum_{mk} U_{nmk}^{(3)}c_mc_k - \frac{1}{6}\sum_{mkl} U_{nmkl}^{(4)}c_mc_kc_l - \cdots, \tag{30}$$

with the initial conditions

$$\dot{c}_n(0) = 0, \quad \lim_{\tau\to\infty} c_n(\tau) = 0, \tag{31}$$

where dot means imaginary time derivative and μ_n are eigenvalues of $\hat{L}$.

It is clearly seen from the expansion [Eq. (20)] for $K(x;x')$ that the orthogonal eigenfunctions $\psi_n(x)$ of the operator $\hat{L}$ are the following:

$$\psi_0(x) = \sqrt{\frac{1}{d}}, \quad \psi_n(x) = \sqrt{\frac{2}{d}}\cos k_nx, \quad n>0, \tag{32}$$

and the corresponding eigenvalues

$$\mu_n = \bar{\mu}_0\left[1 + \frac{an}{2}\chi_N^l(an)\right], \quad a = \frac{\pi\gamma s}{D\sqrt{\bar{\mu}_0}}. \tag{33}$$

In the above equation we just reminded the definition of the parameter a, which was first used in Eq. (21). From Eq. (29), we derive

$$U_{0mk}^{(3)} = j\delta_{mk}/\sqrt{d},$$

$$U_{nmk}^{(3)} = j\frac{\delta_{n,m+k}+\delta_{m,n+k}+\delta_{k,n+m}}{\sqrt{2d}}, \quad n,m,k>0. \tag{34}$$

Note that Eqs. (32) and (33) for ψ_n and μ_n are derived in the limit $D/\lambda_c \ll 1$ when the stationary solution $\varphi_0(x)\approx$ const. The correction to this result can be found in perturbations on D/λ_c.

The lowest eigenvalue

$$\mu_0 = \sqrt{1-j^2}$$

is small when j is close to 1. Therefore, the functions $c_n(\tau)$ should be small and we can neglect all terms in the right-hand side of Eq. (30) except the first one. Let us introduce new variables

$$\alpha_n(\eta) = \frac{jc_n(\tau)}{3\bar{\mu}_0\sqrt{d}}, \quad \eta = \sqrt{\bar{\mu}_0}\tau. \tag{35}$$

Substituting Eqs. (34) and (35) in the system [Eq. (30)], we derive

$$\frac{d^2\alpha_0}{d\eta^2} - \alpha_0 = -\frac{3}{2}\sum_{m=0}^{\infty}\alpha_n^2,$$

$$\frac{d^2\alpha_n}{d\eta^2} - \lambda_n\alpha_n = -3\left(\alpha_n\alpha_0 + \frac{1}{\sqrt{2}}\sum_{m=1}^{\infty}\alpha_m\alpha_{n+m} + \frac{1}{2\sqrt{2}}\sum_{m=1}^{n-1}\alpha_m\alpha_{n-m}\right), \quad n>0, \tag{36}$$

where

$$\lambda_n = \frac{\mu_n}{\bar{\mu}_0}.$$

Equation (28) for the tunneling exponent B_N^l can be rewritten as

$$B_N^l = \frac{J_c}{2e\omega_J}\frac{24(1-j^2)^{5/4}}{5j^2}F(\{\lambda_n\}), \tag{37}$$

where

$$F(\{\lambda_n\}) = \frac{15}{16}\int_0^\infty d\eta\left(\alpha_0^3 + 3\alpha_0\sum_{n=1}^{\infty}\alpha_n^2 + \frac{3}{\sqrt{2}}\sum_{n,m=1}^{\infty}\alpha_n\alpha_m\alpha_{n+m}\right). \tag{38}$$

Substituting Eq. (37) into Eq. (24), we can calculate the escape rate of the fluxon through a set of the junctions.

The tunneling exponent B_N^l depends on l and N via the functions [Eq. (21)]. Below, we assume that $N\gg 1$ and a

~ 1. In this case, the functions $\chi_N^l(an) \cong 1$ and $B_N^l \cong B$ for all l with the exception of $l=0$ and $l=N$. Neglecting these two contributions to tunneling, we derive

$$\Gamma \cong N\omega_J \sqrt{\frac{30\bar{\mu}_0 B}{\pi}} \exp(-B). \tag{39}$$

Under the conditions considered here, we get from Eq. (33),

$$\lambda_n = 1 + \frac{an}{2}.$$

Therefore, the function $F(\{\lambda_n\})$ depends only on the single parameter a, that is, $F(\{\lambda_n\})=F(a)$.

Note that Eq. (39) for Γ is derived here in the limit $N \gg 1$. To find the dependence $\Gamma(N)$, we should solve numerically the problem for each contact $0<l<N$ and perform the summation according to Eq. (24). However, the result of such a procedure is qualitatively the same, as shown in Fig. 1 in Ref. 9.

The analysis of the equation system [Eq. (36)] shows that for any η, we have $\alpha_0(\eta) > \alpha_1(\eta) > \alpha_2(\eta) > \cdots$. Therefore, for a given accuracy, we can consider only the first n_0 equations of the system [Eq. (36)], taking $\alpha_n=0$ for $n \geq n_0$. This closed system of equations is solved numerically. The number of equations that we should take into account depends on of a: the smaller the a, the larger the n_0. The analysis also shows that there exists a critical value $a_c=2.5$ or a critical value of the junction's width,

$$D_c = \frac{2\pi\gamma s}{5(1-j^2)^{1/4}}. \tag{40}$$

If $D<D_c$, all the solutions of Eqs. (36), except $\alpha_0(\eta)$, are explicitly equal to zero. In this case, $F\equiv 1$ and the tunneling exponent in Eq. (37) coincides with calculated under the approximation of the fluxon tunneling by the tunneling of a single quantum particle in the effective potential well.[5,16] Using for an estimate $\gamma=300-500$, $s=1.5$ nm (characteristic for $Bi_2Sr_2CaCu_2O_{8+\delta}$), we find that $D_c \approx 1$ μm.

The function $\Gamma(D)$ is shown in Fig. 1 by the red solid line. This dependence is calculated by means of the numerical procedure described above. In our calculations, we used the parameters of the $Bi_2Sr_2CaCu_2O_{8+\delta}$ sample US1 from Ref. 5. The same figure shows the curve $\Gamma(D)$ calculated using the quantum particle approach [Eq. (39) with $F=1$ in Eq. (37) for the tunneling exponent B_N^l]. It follows from Fig. 1 that the difference between the particle and field approaches becomes significant if D exceeds of about 1.3–1.4 μm.

The dependence of the escape rate Γ on the dimensionless current j, calculated using Eq. (39) and the numerical procedure described above, is shown in Fig. 2. The calculations were performed for two $Bi_2Sr_2CaCu_2O_{8+\delta}$ samples described in Ref. 5. The only adjustable parameter is the product γs, which is about 400–800 nm for $Bi_2Sr_2CaCu_2O_{8+\delta}$. We used the value 525 nm (corresponding to $\gamma=350$) for both samples. It is seen from Fig. 2 that the agreement between the calculated and measured value of Γ is quite good. Small

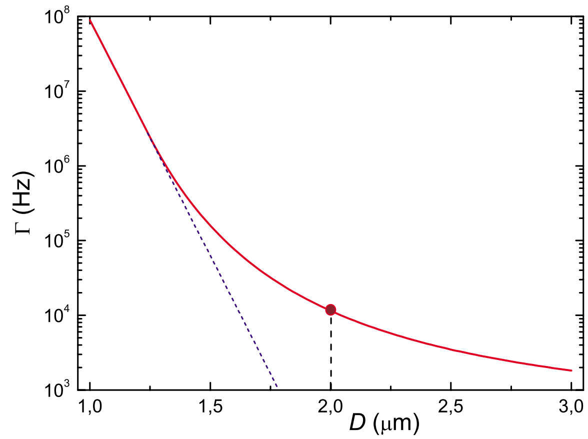

FIG. 1. (Color online) The escape rate Γ versus sample's width D: the red solid curve is numerically calculated using formula (39) for the sample US1 from Ref. 5. The dashed blue line corresponds to the particle tunneling approximation. The parameters ω_J, J_c, and D are taken from the Table I in Ref. 5. The anisotropy coefficient γ and interlayer distance s were chosen as $\gamma=350$ and $s=1.5$ nm. The red point $D\approx 2$ μm indicates the experimental result (Ref. 5).

discrepancies can be attributed to either a violation of the semiclassical approximation or to the two-dimensional nature of the tunneling fluxon.

If $D>D_c$, $\alpha_n(\eta)\neq 0$ for $n>0$ and, for a given accuracy, we should consider n_0 number of equations in the system [Eq. (36)]. The tunneling of the fluxon in this case is similar

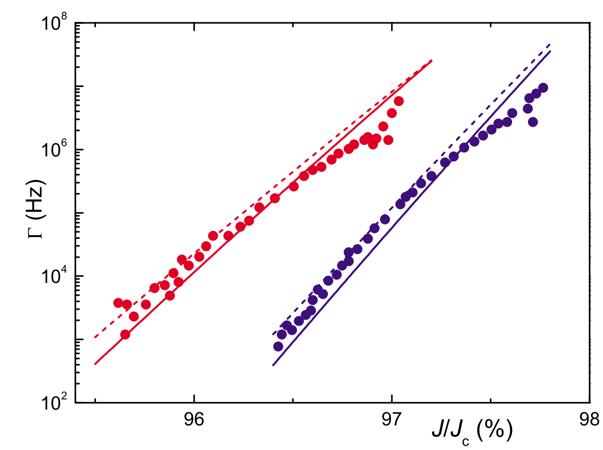

FIG. 2. (Color online) The escape rate Γ versus dimensionless current $j=J/J_c$: red points (on the left) are for the experiment on the $Bi_2Sr_2CaCu_2O_{8+\delta}$ sample US1 and blue points (on the right) are for the sample US4 from Ref. 5; red and blue solid curves are numerically calculated using formula (39) for the samples US1 and US4, respectively. The parameters ω_J, J_c, and D are taken from Table I in Ref. 5; the anisotropy coefficient and interlayer distance were chosen as $\gamma=350$ and $s=1.5$ nm. Dashed red and dashed blue lines are obtained using Eq. (39) and the analytical formula Eq. (49) with $C=0.45$ to calculate B for the same samples. The value of gamma for the blue curve is the same as used for numerical calculations, while $\gamma=455$ for the red curve.

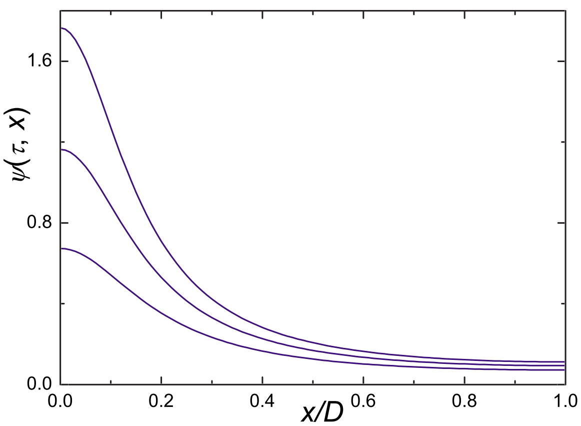

FIG. 3. (Color online) The spatial profile of tunneling fluxon at different values of the imaginary time τ. For the curves from bottom to top, $\eta=\sqrt{\bar{\mu}_0}\tau=-1.6$, $\eta=-1$, and $\eta=0$; $j=0.96$, $\gamma s/D=0.22$.

to the tunneling of a quantum particle in n_0 dimensions, where α_n play the role of the particle coordinates in n_0 dimensional space. The field

$$\bar{\psi}(\tau,x)=\frac{3\bar{\mu}_0\sqrt{d}}{j}\sum_n \alpha_n(\sqrt{\bar{\mu}_0}\tau)\psi_n(x)$$

is strongly inhomogeneous. In Fig. 3, we show the spatial profile $\bar{\psi}(\tau,x)$ of the tunneling fluxon, calculated numerically for different values of the imaginary time τ, which changes from $-\infty$ to zero. The maximum value of $\bar{\psi}$ increases monotonically with τ, while its characteristic size remains practically constant. This analysis shows that the characteristic size of the fluxon is about $aD\sim\gamma s$, as it was mentioned above.

C. Analytical approach

In this section, we obtain a simple analytical formula for calculating the tunneling exponent B_N^l in a stack of Josephson junctions. For this purpose, we reduce the problem of field tunneling to the tunneling of a quantum particle. However, in contrast to the usual approach, we take into account the spatial variation of the gauge-invariant phase difference φ when deriving the effective potential U. In some aspects, the proposed procedure is similar to the method used in Ref. 9. However, here, we use more accurate approximations, which are based on the exact mathematical treatment of the problem presented in the previous sections. We consider here only the case $D\ll\lambda_c$.

We now change from imaginary τ to real time t in Eq. (19), for the classical field dynamics, and rewrite it in the form

$$\frac{\partial^2\bar{\psi}}{\partial t^2}+\sqrt{1-j^2}\bar{\psi}-j\frac{\bar{\psi}^2}{2}=\int_0^d dx' K(x;x')\frac{\partial^2\bar{\psi}(t,x')}{\partial x'^2}. \quad (41)$$

According to the numerical result shown in Fig. 3, the value $\bar{\psi}(t,x)$ can be approximately presented as a product $\bar{\psi}(t,x)\approx f(x)r(t)$. Following this approach, we seek a real-time solution of Eq. (41) in the form $\bar{\psi}(t,x)=f(x)r(t)$, where $df/dx=0$ at $x=0,d$. We normalize the function $f(x)$,

$$\int_0^d dx f^2(x)=1. \quad (42)$$

We substitute $\bar{\psi}=f(x)r(t)$ in Eq. (41); then, we multiply both sides of this equation by f and integrate along the junction. As a result, we obtain the equation of motion for some hypothetical particle with coordinate $r(t)$, which can be written in the form

$$\frac{d^2r}{dt^2}+\bar{\mu}_0 r-\frac{jr^2}{2}\int_0^d dx f^3(x) = -r\int_0^d dx\int_0^d dx'\frac{df(x)}{dx}P(x;x')\frac{df(x')}{dx'}, \quad (43)$$

where the new kernel $P(x;x')$ is expressed through the kernel $K(x;x')$ [Eq. (20)]. Under the approximation considered here ($D\ll\lambda_c$ and $N\gg1$, $a\sim1$), we derive the explicit analytical formula for $P(x;x')$,

$$P(x;x')=\frac{\gamma s\sqrt{\bar{\mu}_0}}{2\pi\lambda_c}\ln\left|\frac{\sin\left[\frac{\pi(x+x')}{2d}\right]}{\sin\left[\frac{\pi(x-x')}{2d}\right]}\right|. \quad (44)$$

We approximate $f(x)$ by a step function

$$f(x)=\frac{1}{\sqrt{x_0}}\theta(x_0-x). \quad (45)$$

Substituting this function in Eq. (43) and performing integration, we see that the term in the right-hand side of this equation has a logarithmic singularity since $\partial f/\partial x=-\delta(x-x_0)/\sqrt{x_0}$. To cut off this singularity, we take into account that the characteristic scale of change of the phase $\bar{\psi}$ in the stack of Josephson junctions is γs.[17] Thus, performing integration, we put $|x-x'|=C\gamma s$ at $x'\to x$, where C is a constant of the order of unity. Therefore, we obtain the equation of motion for the effective particle in the form $d^2r/dt^2=-\partial U(r)/\partial r$, where the effective potential $U(r)$ can be written as

$$U(r)=\frac{jr^2}{6\sqrt{x_0}}(r_0-r),$$

$$r_0=\frac{3\sqrt{x_0}}{j}\left[\bar{\mu}_0+\frac{\gamma s\sqrt{\bar{\mu}_0}}{2\pi\lambda_c x_0}\ln\left(\frac{2D}{C\pi\gamma s}\right)\right]. \quad (46)$$

Here, we neglect the small term in the kernel $\propto\ln|\sin[\pi(x+x')/2d]|$. Consequently, $P(x;x')\approx P(x-x')$, as was assumed for the corresponding kernel in Ref. 9.

The tunneling exponent for the effective particle in the potential $U(r)$ in a semiclassical approximation reads[12]

$$B = 2\varepsilon_0 \int_0^{r_0} \sqrt{2U(r)}dr. \tag{47}$$

Performing an integration, we obtain

$$B_N^l = \frac{24\varepsilon_0 x_0}{5j^2}\left[\bar{\mu}_0 + \frac{\gamma s\sqrt{\bar{\mu}_0}}{2\pi\lambda_c x_0}\ln\left(\frac{2D}{C\pi\gamma s}\right)\right]^{5/2}. \tag{48}$$

The optimal value for the tunneling of the fluxon is found from the condition of minimum of B: $dB/dx_0=0$, that optimizes the shape or x-dependence of the tunneling flexor (see also animations at http://dml.riken.go.jpMQT/mqt.swf. Thus, finally, we derive

$$B_N^l = \sqrt{\frac{5}{3}}\frac{5J_c}{e\omega_J}\left(\frac{1-j^2}{j^2}\right)\frac{\gamma s}{\pi D}\ln\left(\frac{2D}{C\pi\gamma s}\right). \tag{49}$$

The result obtained is qualitatively similar to the formula for B presented in Ref. 9. We should emphasize, however, that this analytical expression is obtained in the limit $N \gg 1$, while the approximation for the escape probability derived in Ref. 9 allows to describe the dependence of Γ on the number of the layers N. In the limit $N \gg 1$, these approximations practically coincide for usual parameters of layered superconductors. The dependence of Γ on j, calculated by means of Eqs. (39) and (49), is shown by dashed lines in Fig. 2. Calculating these curves, we use γ and C as adjustable parameters. Note that the analytical approach is appropriate to estimate the value of $\ln\Gamma$. However, the numerical results are much more accurate.

IV. CONCLUSIONS

We developed a quantum field Lagrangian approach for two different quantum effects in the layered superconductors. First, we derive the theory of quantum JPWs, which predicts the existence of two types of bosonic elementary excitations, a heavy JPW and a lighter one. Second, we develop a quantitative theory of MQT in stacks of Josephson junctions, which allows to quantitatively describe this effect and agrees well with the experimental observations. We also derive a simple analytical formula for estimation of the MQT escape rate in stacks of Josephson junctions. The proposed numerical approach can also be used to describe quantum tunneling in Josephson junction arrays,[18] as well as in electromechanical[19] and magnetic[20] systems, where the "particle approximation" can be invalid. We stress that the quantum field theory approach predicts an enhancement of the tunneling probability by taking into account the coordinate dependence of the tunneling field. The optimal spatial configuration of the field gives rise to the lowering of the effective potential barrier. In contrast to particle approximation, quantum field theory approach allows an adequate understanding of the observed giant MQT effect in the layered superconductors[4,5] due to the nonlocal electrodynamics of such a media.

A similar approach can also be used to estimate the crossover temperature from the thermal to the quantum regime.[21,22] Finally, computer animations illustrating some of the effects studied here can be found in Ref. 23.

ACKNOWLEDGMENTS

We gratefully acknowledge partial support from the National Security Agency (NSA), Laboratory Physical Science (LPS), Army Research Office (ARO), National Science Foundation (NSF) Grant No. EIA-0130383, Core-to-Core (CTC) program supported by the Japan Society for the Promotion of Science (JSPS), JSPS-Russian Foundation for Basic Research (RFBR) 06-02-91200, and RFBR 06-02-16691. S.S. acknowledges support from the Ministry of Science, Culture and Sports of Japan via the Grant-in Aid for Young Scientists No. 18740224, the EPSRC via Nos. EP/D072581/1 and EP/F005482/1, and ESF network-programme "Arrays of Quantum Dots and Josephson Junctions." We acknowledge J. Keller for fruitful discussions.

[1] R. Kleiner, F. Steinmeyer, G. Kunkel, and P. Müller, Phys. Rev. Lett. **68**, 2394 (1992); G. Hechtfischer, R. Kleiner, A. V. Ustinov, and P. Müller, *ibid.* **79**, 1365 (1997); E. Goldobin, A. Wallraff, N. Thyssen, and A. V. Ustinov, Phys. Rev. B **57**, 130 (1998); J. Zitzmann, A. V. Ustinov, M. Levitchev, and S. Sakai, *ibid.* **66**, 064527 (2002); M. Tachiki, M. Iizuka, K. Minami, S. Tejima, and H. Nakamura, *ibid.* **71**, 134515 (2005); D. R. Gulevich and F. V. Kusmartsev Phys. Rev. Lett. **97**, 017004 (2006).

[2] S. Savel'ev, V. Yampolskii, A. Rakhmanov, and F. Nori, Phys. Rev. B **72**, 144515 (2005); Physica C **437-438**, 281 (2006); Physica C **445**, 175 (2006).

[3] S. Savel'ev *et al.*, Phys. Rev. Lett. **94**, 157004 (2005); Phys. Rev. B **74**, 184512 (2006); Physica C **445**, 180 (2006); Phys. Rev. Lett. **95**, 187002 (2005); Nat. Phys. **2**, 521 (2006); Phys. Rev. B **75**, 184503 (2007); V. A. Yampol'skii *et al.*, Phys. Rev. B **76**, 224504 (2007); V. A. Yampol'skii *et al.*, Phys. Rev. B **75**, 014527 (2007); H. Susanto, E. Goldobin, D. Koelle, R. Kleiner, and S. A. van Gils, Phys. Rev. B **71**, 174510 (2005).

[4] K. Inomata, S. Sato, Koji Nakajima, A. Tanaka, Y. Takano, H. B. Wang, M. Nagao, H. Hatano, and S. Kawabata, Phys. Rev. Lett. **95**, 107005 (2005); T. Bauch, F. Lombardi, F. Tafuri, A. Barone, G. Rotoli, P. Delsing, and T. Claeson, *ibid.* **94**, 087003 (2005); T. Bauch, T. Lindstrom, F. Tafuri, G. Rotoli, Per Delsing, T. Claeson, and F. Lombardi, Science **311**, 57 (2006).

[5] X. Y. Jin, J. Lisenfeld, Y. Koval, A. Lukashenko, A. V. Ustinov, and P. Müller, Phys. Rev. Lett. **96**, 177003 (2006).

[6] J. M. Martinis, S. Nam, J. Aumentado, and C. Urbina, Phys. Rev. Lett. **89**, 117901 (2002); J. Q. You and F. Nori, Phys. Today **58**(11), 42 (2005); R. McDermott, R. W. Simmonds, M. Steffen, K. B. Cooper, K. Cicak, K. D. Osborn, S. Oh, D. P. Pappas, and J. M. Martinis, Science **307**, 1299 (2005); A. J. Berkley, H. Xu, R. C. Ramos, M. A. Gubrud, F. W. Strauch, P. R. Johnson, J. R. Anderson, A. J. Dragt, C. J. Lobb, and F. C. Wellstood, *ibid.* **300**, 1548 (2003).

[7] M. Machida and T. Koyama, Supercond. Sci. Technol. **20**, S23 (2007).
[8] M. V. Fistul, Phys. Rev. B **75**, 014502 (2007).
[9] S. Savel'ev, A. L. Rakhmanov, and F. Nori, Phys. Rev. Lett. **98**, 077002 (2007); **98**, 269901(E) (2007).
[10] S. Kawabata, S. Kashiwaya, Y. Asano, and Y. Tanaka, Phys. Rev. B **70**, 132505 (2004); **72**, 052506 (2005).
[11] L. N. Bulaevskii, M. Zamora, D. Baeriswyl, H. Beck, and J. R. Clem, Phys. Rev. B **50**, 12831 (1994); T. Koyama and M. Tachiki, *ibid.* **54**, 16183 (1996); S. N. Artemenko and S. V. Remizov, Physica C **362**, 200 (2001).
[12] S. Coleman, Phys. Rev. D **15**, 2929 (1977).
[13] T. Kato and M. Imada, J. Phys. Soc. Jpn. **66**, 1445 (1997).
[14] A. O. Sboychakov, S. Savel'ev, A. L. Rakhmanov, and F. Nori, Europhys. Lett. **80**, 17009 (2007).
[15] A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982).
[16] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981).
[17] A. Gurevich, Phys. Rev. B **46**, 3187 (1992).
[18] E. Goldobin, D. Koelle, and R. Kleiner, Phys. Rev. B **67**, 224515 (2003); E. Goldobin, K. Vogel, O. Crasser, R. Walser, W. P. Schleich, D. Koelle, and R. Kleiner1, *ibid.* **72**, 054527 (2005).
[19] S. Savel'ev, A. L. Rakhmanov, Xuedong Hu, A. Kasumov, and F. Nori, Phys. Rev. B **75**, 165417 (2007).
[20] B. A. Ivanov, D. D. Sheka, V. V. Krivonos, and F. G. Mertens, Phys. Rev. B **75**, 132401 (2007).
[21] P. Hänggi, F. Marchesoni, and P. Riseborough, Europhys. Lett. **13**, 217 (1990); P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
[22] D. Gulevich and F. Kusmartsev, Physica C **435**, 87 (2006).
[23] Computer animations illustrating the tunneling of flux quanta, for different values of the applied current, is available online at http://dml.riken.jp/MQT