Detecting mode entanglement: The role of coherent states, superselection rules, and particle statistics

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We discuss the possibility of observing quantum nonlocality using the so-called mode entanglement, analyzing the differences between different types of particles in this context. We first discuss the role of coherent states in such experiments, and we comment on the existence of coherent states in nature. The discussion of coherent states naturally raises questions about the role of particle statistics in this problem. Although the Pauli exclusion principle precludes coherent states with a large number of fermionic particles, we find that a large number of fermionic coherent states, each containing at most one particle, can be used to achieve the same effect as a bosonic coherent state for the purposes of this problem. The discussion of superselection rules arises naturally in this context, because their applicability to a given situation prohibits the use of coherent states. This limitation particularly affects the scenario that we propose for detecting the mode entanglement of fermionic particles.

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I. INTRODUCTION

Entanglement is probably the most intriguing aspect of quantum mechanics. It has steadily been the subject of research and controversy ever since it was noted by Schrödinger in 1935 [1–3].

The most studied form of entanglement is the one involving two physical objects with internal degrees of freedom. The quintessential example in the literature is the entanglement in the Bell states of two spin-1/2 particles. This simple form of entanglement, however, is not the only one in nature. In particular, we consider here mode entanglement, which introduces additional intrigue to this subject due to the fact that it can involve the vacuum as a crucial element in the problem, and it can be obtained using a *single* particle. Both of these aspects are commonly seen as foreign to the discussion of entanglement.

In order to capture the essence of mode entanglement, one can consider a single particle in a quantum superposition of being at two different locations. One rarely associates this state with entanglement. However, if the particle is viewed as an excitation of an underlying field, the quantum state takes the form of an entangled state: the first mode of the field containing a particle while the second mode is empty, and vice versa.

The above example shows that the formal expression used to describe a quantum state is not the ideal indicator of the presence of entanglement. Instead, it would be more meaningful to define the presence of entanglement according to the possibility of experimentally observing quantum effects associated with entanglement, e.g., the violation of the Bell inequalities [4]. As we shall discuss in some detail below, the nature of the particles involved in the mode entanglement is a crucial factor in determining whether this entanglement is detectable or not. Analyzing the detectability of mode entanglement for different types of particles is the main subject of this paper.

Several theoretical studies have analyzed the so-called single-photon entanglement in quite some detail [5–10]. In fact, there have been experimental tests of the Bell inequalities using the mode entanglement of single photons [11,12]. Photons, however, represent a single type of particles with specific properties. Here we build on the results of Ref. [13] (see also Refs. [14,15]): we analyze the roles played by particle statistics [16] and superselection rules [17] in the detectability of mode entanglement. We divide our discussion into four cases, depending on the nature of the particles, i.e., bosons or fermions, and whether superselection rules constrain the total particle number to be fixed or not. This division simplifies the task of identifying the roles played by the different physical elements in the problem.

The importance of superselection rules, i.e., the constraint of having a fixed particle number, can be seen by considering a Bell-violation experiment. In such an experiment, it is necessary to perform measurements in a variety of bases. In the case of mode entanglement, the notion of measurements in different bases suggests that one needs to perform measurements in bases of indefinite particle number, e.g., the basis $(|0\rangle \pm |1\rangle)/\sqrt{2}$. If superselection rules apply to the type of particle under consideration, such a measurement is forbidden. Although this difficulty might seem to be a major obstacle to the detectability of mode entanglement under the constraint of superselection rules, we shall present procedures to overcome it by utilizing the indistinguishability between the particle under consideration and other properly prepared ancillary particles. The use of particle indistinguishability in our proposed procedures indicates that particle statistics will also be an important factor in the detectability of mode entanglement, since there are major differences between bosonic and fermionic particles in this regard.

The role of coherent states in a Bell-test experiment can also be seen by considering the need for performing measurements in a variety of bases. Rotations on a quantum state

before the measurement are equivalent to, and sometimes necessary for, changing the measurement basis. Such rotations are commonly induced using coherent states. Thus, in our analysis below we shall deal with questions related to coherent states, superselection rules and particle statistics.

This paper is organized as follows: In Sec. II we present the basic setup for our analysis. We analyze mode-entanglement-detection (gedanken) experiments using four different types of particles in Secs. III–VI. We conclude by reviewing our main results in Sec. VII.

II. DESCRIPTION OF THE SETUP

Throughout this paper we consider a setup where a particle is prepared in a spatially delocalized state of the form

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle), \tag{1}$$

where the states $|L\rangle$ and $|R\rangle$ are thought of as being localized on opposite sides of the experimental setup. For the case of photons, for example, this state can be obtained by sending a beam into a 50:50 beam splitter. When viewed as a state of the photon field, in the form

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle), \tag{2}$$

one can see that this is an entangled state. The task is now to detect this entanglement, e.g., using a Bell-test experiment [4].

How to proceed in order to probe the entanglement in the state in Eq. (2) depends on the available measurement tools. For example, the experiments on this subject [11,12] probed the entanglement using homodyne detection, mixing the incoming photons with coherent states of known phases. Analyzing the detailed description of such specialized techniques, however, would be a distraction from the aim of this paper. We therefore consider a conceptually simpler scenario: We imagine that the incoming (flying) particle can excite a two-level target particle from its ground state to its excited state. The initial state of the combined system is given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \otimes |gg\rangle, \tag{3}$$

where the first ket describes the state of the flying particle, the second ket describes the state of the two target particles (note that one target particle is placed on each side of the setup), and the symbols g and e are used to denote the ground and excited states of the target particles. Depending on whether the flying particle is absorbed by the target particle during the excitation process or not, one obtains either the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle \otimes (|eg\rangle + |ge\rangle) \tag{4}$$

or the state [18]

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle \otimes |eg\rangle + |01\rangle \otimes |ge\rangle). \tag{5}$$

The state in Eq. (4) is the proper description for an incoming photon that is absorbed by one of two target atoms. However, it cannot be obtained whenever superselection rules apply to the species of flying particles, since the flying particle cannot be annihilated in this case. We shall refer to particles with a conserved total number as massive particles (more as a matter of easily recognizable terminology than fundamental physical arguments [19]).

Note that the number of target particles does not change in the above picture, and they do not move between the two sides of the experimental setup. The discussion of superselection rules is therefore not crucial in regard to the target particles. It is safest, however, to assume that they are different from the flying-particle species, such that we do not need to worry about complications associated with the flying and target particles being indistinguishable and obeying identical-particle symmetry constraints.

III. CASE 1: MASSLESS BOSONS

We start by considering the relatively simple case of a single photon passing through a beam splitter and resulting in an entangled state between the left and right modes of the electromagnetic field. Although the experiments of Refs. [11,12] relied on an auxiliary laser beam as a reference phase standard for the Bell test, it seems conceptually simpler to imagine the incoming photons being absorbed by target atoms and resulting in states of the form given in Eq. (4). The mode entanglement is then transferred to the internal degrees of freedom of the target atoms. One can then conclude that the measurements for the Bell test can be performed straightforwardly on the states of the target atoms.

An important point that was not addressed in the above scenario is the fact that for a Bell test one would need to perform rotations on the states of the target atoms before the measurement (here we are making the realistic assumption that measurements will always be performed in the $\{|g\rangle, |e\rangle\}$ basis). Such rotations are typically performed using classical fields of the same frequency as the incoming photons. For these fields to be classical and useful for our purposes, one must know the relative phase between the fields on the left- and right-hand sides of the beam splitter. In other words, although the entanglement was transferred to the internal states of the target atoms when the incoming photon was absorbed, one still needs to have a common phase reference (typically in the form of photonic coherent states on the two sides of the experimental setup with a known relative phase). The simple-looking scenario of using target atoms therefore does not eliminate the need for a phase reference. It only divides the procedure into two steps, each of which is conceptually simple.

Having established the need for a common phase reference, we are now led to ask whether the two sides of the setup must be entangled in order to have such a common

phase reference. If the answer is yes, one would be led to question whether any observed phenomena probed the mode entanglement of the incoming photons or a combination of the mode entanglement and the preexisting entanglement in the setup. We address the above question next.

Can two coherent states with a known relative phase be prepared independently of each other?

Although the answer to the above question, in the affirmative, is accepted by the majority of physicists, it has generated some controversy in recent years [20,21]. We therefore address it explicitly here for clarity.

The simplest approach to take here is probably to consider the classical problem of, say, radio-frequency antennas. Taking two distant antennas with known relative orientations, and assuming the antennas are controlled by experimentalists with synchronized clocks, the two experimentalists can produce classical waves with a known relative phase. If the setup includes a screen, i.e., a set of detectors, one can predict exactly where the interference maxima and minima will appear on the screen. All that is needed to make this prediction is knowledge of the relative orientation of the antennas and synchronization of the clocks. Although this argument treats relatively low-frequency waves, there is conceptually nothing different when dealing with the optical frequencies. Finally, when this situation is described in quantum-mechanical terms, the predictability of the interference patterns implies that the photon states generated by the two sources must be coherent states.

One can therefore conclude that as long as the two sources share reference frames and synchronized clocks, they can in principle generate coherent states with a known relative phase. The fact that present-day experiments cannot produce two independent optical-frequency lasers with a known relative phase should not be seen as a fundamental obstacle to the existence of coherent states (as was in fact noted in Ref. [20]). The most crucial point here is probably the fact that the two sources generating the mutually coherent waves do not need to share any entanglement.

Turning back to the problem of performing rotations on an atomic state, one can also envision replacing the common phase reference by the application of intense static electric fields (the strength of the field being compared with the frequency of the relevant atomic transition) in order to perform the atomic-state rotation. The phase-standard aspect of the shared reference frame disappears completely in this case. One must keep in mind, of course, that real atoms cannot be approximated by two-level systems under such intense fields. However, this argument demonstrates that sharing a common phase reference is nothing more than sharing a space and time reference frame.

As for the need to share reference frames, this is by no means unique to the case of quantum-optical coherent states. It also applies, e.g., to a Bell-test experiment using spin states. More specifically, take two observers that share maximally entangled pairs of spin-1/2 particles (e.g., in the singlet state). Until the observers establish the proper reference frames for their measurements, they cannot detect the entanglement. Of course they can sacrifice a few pairs in order to establish those proper reference frames, and then they can proceed with the experiment and observe the violation of the Bell inequality. Alternatively, the two observers can scan the entire range of possible measurement directions, thus simultaneously establishing the common reference frame and observing the Bell-inequality violation. The main point here, however, is to note that in many (classical and quantum) physical problems a common reference frame must be established before correct predictions can be made.

IV. CASE 2: MASSIVE BOSONS

This case was analyzed in Ref. [13], and we shall not repeat the analysis here. The main result is that if one takes N ancillary particles of the same species as the flying particles and forms two entangled Bose-Einstein condensates (in a properly prepared state), one can follow the procedure explained in Ref. [13] and detect the mode entanglement in the state of the flying particles. The observable concurrence for each incoming flying particle is given by $1 - 1/(2N)$ for large N.

An important result in this case is that the condensate of N particles can be reused for an arbitrary number of flying particles. The unlimited reusability of the condensate suggests that the condensate can be naturally thought of as playing an auxiliary role in the experiment. This result is also rather counterintuitive, and it stands in contrast with the notion that quantum reference frames are generally degraded as a result of repeated use [22]. A possible explanation of this result is that in the procedure of Ref. [13] no measurements are performed directly on the condensate. In fact, if one performs measurements on the condensate, one can (at least probabilistically) increase the entanglement in the first created pairs of target particles, but the entanglement of subsequent pairs will be degraded. It would be interesting to see if similar ideas can be applied to quantum reference frames in general.

It should be noted here that as the flying particles come into the proposed setup and are used to excite the target particles then properly discarded into the condensate, some amount of entanglement between the condensate and the target particles is generated. The state of the condensate therefore changes after each measurement on a given pair of target particles. Alternatively, if several entangled pairs are generated before any measurement is performed, the different pairs will be entangled with each other. As such, the different entangled pairs generated in this procedures cannot be considered independent and identically distributed (i.i.d.). Note, however, that whenever the Bell inequalities are violated, the observed correlations cannot be described by local-hidden-variable theories. In other words, using an i.i.d. source is not a requirement of the Bell test.

In principle, it is possible to write down the full (pure) quantum state of the entire system and analyze the entanglement present in different sets of subsystems. However, since our main focus in this paper is the detection of mode entanglement, we only consider the correlations that are present within the individual pairs of target particles, even in the case where a stream of flying particles is used to generate a large number of entangled pairs of target particles. Other correlations in the system give rise to interesting phenomena that are not directly related to the aim of this paper and will be discussed in more detail elsewhere.

Another interesting result in the case of massive bosons is that a single ancillary particle is sufficient to allow the observation of the Bell-inequality violation (ensemble averaging over many setups is needed in order to guarantee the violation, as will be discussed in detail elsewhere). This result can be verified by using the following criterion presented in Ref. [23]. First, following Ref. [13] with N ancillary particles, we calculate the reduced density matrix describing the state of the target particles in the basis $\{|gg\rangle, |ge\rangle, |eg\rangle, |ee\rangle\}$, and we find it to be given by

$$\rho_{\mathrm{TP}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma & 0 \\ 0 & \gamma & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

where $\gamma \approx 1 - 1/(2N)$. (In the following we only need to use the fact that γ is nonzero.) Using this density matrix, we now follow Ref. [23] and define a 3×3 matrix T with entries $T_{ij} \equiv \mathrm{Tr}[\rho_{\mathrm{TP}}(\sigma_i^L \otimes \sigma_j^R)]$ with the standard Pauli matrices σ_1, σ_2, and σ_3 for the left and right particles. Then we compute the three eigenvalues of the matrix $T^\dagger T$ and define a new function $M(\rho)$ as the sum of the two greatest eigenvalues. The necessary and sufficient condition for the violation of the Bell inequality (in the Clauser-Horne-Shimony-Holt version [24]) can be expressed as $M(\rho) > 1$. For the density matrix ρ_{TP} above, we find that $M(\rho) = 1 + |\gamma|^2 \approx 1 + [1 - 1/(2N)]^2$, which is always greater than 1 regardless of N, hence the violation of the Bell inequality.

V. CASE 3: MASSLESS FERMIONS

We now turn to the case of fermionic flying particles. We start by considering the case of massless fermions because it gives conceptually interesting results and serves as an introduction to Sec. VI, regardless of whether it corresponds to any realistic physical situation. The discussion would also be relevant if superselection rules do not have to be obeyed for fermionic particles, a situation predicted by some high-energy theories [14].

We consider a (possibly hypothetical) fermionic analog of photons: we imagine a fermionic species of particles that can be created at will, and any given mode can contain at most one particle. We therefore cannot create coherent states of a form similar to coherent states of bosonic particles, i.e.,

$$|\psi\rangle_{\mathrm{coherent},B} = \exp\left(-\frac{|\eta|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\eta^n}{\sqrt{n!}} |n\rangle. \tag{7}$$

We shall show, however, that the fermionic analog of coherent states can be used to achieve the same result obtained using bosonic coherent states in the context of the present discussion. As mentioned above, we assume that states of the form

$$|\psi\rangle_{\mathrm{coherent},F} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{8}$$

are physical and can be created at will. The above state will be the main building block for the coherent-state-like manipulations below.

We now imagine that the incoming particle is absorbed by one of two target particles as explained in Sec. II. This can be achieved using the effective Hamiltonian

$$\hat{H} = J(i\sigma_+ a - i\sigma_- a^\dagger), \tag{9}$$

where J is the coupling strength, $\sigma_\pm$ are raising and lowering operators of the target-particle state ($\sigma_+|g\rangle = |e\rangle$), and a and $a^\dagger$ are, respectively, annihilation and creation operators of the incoming particle species. After the absorption of the flying particle, the target particles end up in a state of the form given in Eq. (4).

As discussed in Sec. III above, the detection of mode entanglement is now reduced to the ability of performing arbitrary rotations on the states of the target particles. We therefore focus on these rotations for the remainder of this section, and below we give explicit expressions for the representative example of a $\pi/2$ rotation. Note that we do not allow using a bosonic coherent state here; instead we imagine that the target particle can only be manipulated using the same Hamiltonian describing the absorption of the incoming particle [Eq. (9)].

Let us take a target particle in an arbitrary initial state

$$|\psi\rangle_i = \alpha|g\rangle + \beta|e\rangle \tag{10}$$

and try to rotate it to the state

$$|\psi\rangle_{f,\mathrm{ideal}} = \frac{\alpha - \beta}{\sqrt{2}}|g\rangle + \frac{\alpha + \beta}{\sqrt{2}}|e\rangle. \tag{11}$$

The above quantum state can also be described using the density matrix

$$\rho_{f,\mathrm{ideal}} = \frac{1}{2} \begin{pmatrix} |\alpha - \beta|^2 & (\alpha + \beta)^*(\alpha - \beta) \\ (\alpha + \beta)(\alpha - \beta)^* & |\alpha + \beta|^2 \end{pmatrix}. \tag{12}$$

In order to perform the desired rotation, one can try to use an ancillary mode in the state given by Eq. (8) and allow that mode to interact with the target particle using the effective Hamiltonian in Eq. (9) for a duration of $\pi/(4J)$. If we trace out the degrees of freedom of the ancillary mode at the final time, we find that the above operation transforms the initial state of the target particle [Eq. (10)] into a mixed state described by the density matrix

$$\rho_f = \frac{1}{4}\begin{pmatrix} 2|\alpha|^2 + |\alpha - \beta|^2 & \sqrt{2}\alpha(\alpha + \beta)^* + \sqrt{2}(\alpha - \beta)\beta^* \\ \sqrt{2}\alpha^*(\alpha + \beta) + \sqrt{2}(\alpha - \beta)^*\beta & |\alpha + \beta|^2 + 2\beta^2 \end{pmatrix}. \tag{13}$$

The overlap between this state and the ideal state can be calculated using the fidelity

$$F = {}_{f,\text{ideal}}\langle \psi | \rho_f | \psi \rangle_{f,\text{ideal}}. \tag{14}$$

We do not write the long expression for the fidelity in the above example or go further into specific averaging procedures. The main point to note is that the fidelity is clearly smaller than 1 [compare Eqs. (12) and (13)] [25].

Since the fidelity reduction can be attributed to instances where the initial state of the target particle and ancillary mode is given by $|g\rangle \otimes |0\rangle$ or $|e\rangle \otimes |1\rangle$ [26], we now try to reduce the impact of such instances. An obvious approach is to use a large number N of ancillary modes, each in a state of the form given by Eq. (8); the bad states $|g\rangle \otimes |00\cdots 0\rangle$ and $|e\rangle \otimes |11\cdots 1\rangle$ now have very small probability amplitudes. We now perform a numerical simulation: We take the target particle and allow it to interact with each ancillary mode using the Hamiltonian in Eq. (9) for a duration of $\pi/(4JN)$. Without going into the details of the calculation, which parallels the explanation given above for a single ancillary mode, we find that the fidelity, i.e., the overlap between the ideal and actual final states, of the target particle approaches 1, with error proportional to $1/N$.

The above procedure can therefore be incorporated into a mode-entanglement experiment, with the conclusion that after the absorption of the incoming particle an arbitrary measurement can be performed on the states of the target particles. This result implies that the mode entanglement would be detectable in a Bell-test experiment.

We should stress here that the coupling between the target particle and the ancillary modes must be done sequentially. If, instead, the target particle is coupled to all ancillary modes simultaneously using the Hamiltonian

$$\hat{H} = J\sum_k (i\sigma_+ a_k - i\sigma_- a_k^\dagger) = J\sqrt{N}\left(i\sigma_+ \sum_k \frac{a_k}{\sqrt{N}} - i\sigma_- \sum_k \frac{a_k^\dagger}{\sqrt{N}}\right), \tag{15}$$

the target particle couples to a single collective mode, defined by the annihilation operator $b \equiv \Sigma_k a_k/\sqrt{N}$. Using this procedure therefore gives the same results as using a single ancillary mode, i.e., a 50% success probability for producing an entangled pair of target particles [13] (here k labels the different ancillary modes).

VI. CASE 4: MASSIVE FERMIONS

Encouraged by the success achieved using fermionic coherent states in Sec. V, we now try to follow a similar procedure for the case of massive fermions.

Since we now want to impose superselection rules (e.g., unlike the scenario of Sec. V, the flying particle is not ab-

sorbed upon exciting the target particle and we cannot create coherent states at will), we must look for alternatives with a fixed particle number for the flying-particle species. We follow a procedure similar to that introduced in Ref. [13] and combine it with the sequential manipulation of Sec. V.

Our starting point is the initial state of the flying particle and two target particles given in Eq. (5). We also assume that we have already created N entangled pairs of ancillary modes (with each pair of modes sharing one particle) of the form

$$|\Psi_{\text{anc}}\rangle = \frac{1}{\sqrt{2}}(|L_{\text{anc}}\rangle + |R_{\text{anc}}\rangle), \tag{16}$$

where the states $|L_{\text{anc}}\rangle$ and $|R_{\text{anc}}\rangle$ describe the ancillary particle being localized on the left- and right-hand side of the beam splitter, respectively. We now want to perform a sequence of local operations, each involving a target particle (on the left- or right-hand side), the corresponding flying-particle mode and an ancillary mode. We shall try to design this sequence of operations such that the flying particle is discarded into one of the ancillary modes by the end of the entire procedure (the key property of this disposal process is that one should no longer be able to deduce the location of the excited target particle from the state of the flying-particle species). The concurrence in the state of the target particles at the end of the sequence of operations can be calculated from the target-particle reduced density matrix, which is obtained by tracing over the degrees of freedom of the flying and ancillary particles at the end of the procedure.

We now focus on a single operation to be performed on one side of the setup; this operation will essentially be the building block from which the entire sequence is constructed. We look for a unitary operation that mixes the states $|e\rangle \otimes |1\rangle_{\text{flying}} \otimes |0\rangle_{\text{anc}}$ and $|e\rangle \otimes |0\rangle_{\text{flying}} \otimes |1\rangle_{\text{anc}}$ with some probability [27]. The desired effect of this operation is that, if the flying particle is on the side of the setup where the operation is performed and the ancillary mode is empty, the flying particle will (with some probability) be discarded into the ancillary mode, thus partially erasing the information in the flying-particle mode. In Ref. [13], a well-merging process was proposed for this purpose. We find the well-merging process unsuitable for generalization to the multistep procedure that we are trying to construct here. It seems that the next closest analog to what was done in Sec. V is to use operations of the form

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{17}$$

in the above basis (i.e., $\{|e\rangle \otimes |1\rangle_{\text{flying}} \otimes |0\rangle_{\text{anc}}, |e\rangle \otimes |0\rangle_{\text{flying}} \otimes |1\rangle_{\text{anc}}\}$), while not affecting any other state [28].

TABLE I. Concurrence between the two target particles (TPs) for one incoming flying particle and maximum number of times the experiment can be repeated (with a given number N of ancillary particles in the case of massive particles) for different types of particles.

Particle type	Concurrence between TPs	Max. number of repetitions
Massless bosons	1	∞
Massive bosons	$1 - 1/(2N)$	∞
Massless fermions	1	∞
Massive fermions	1/2	N

In the case of a single pair of ancillary modes [Eq. (16)], the optimal value of θ is $\pi/2$ (this would be referred to as a π rotation), resulting in a concurrence of 1/2 between the two target particles. In this case one can clearly identify the successful instances as those associated with the subspace $\{|eg\rangle \otimes |L_{\text{flying}}\rangle \otimes |R_{\text{anc}}\rangle, |ge\rangle \otimes |R_{\text{flying}}\rangle \otimes |L_{\text{anc}}\rangle\}$ and the unsuccessful instances as those associated with the subspace $\{|eg\rangle \otimes |L_{\text{flying}}\rangle \otimes |L_{\text{anc}}\rangle, |ge\rangle \otimes |R_{\text{flying}}\rangle \otimes |R_{\text{anc}}\rangle\}$. In particular, if both the flying and ancillary particles end up on the same side of the setup, their indistinguishability cannot be utilized to erase the information about the location of the flying particle.

We now numerically simulate the procedure with two pairs of ancillary modes (i.e., two ancillary particles) and search for the optimal values of θ_1 and θ_2, which represent the two steps in the procedure (we take the same value of θ_j on both sides of the experimental setup in each step). We find that the maximum achievable concurrence is still given by 1/2, and is obtained by taking one of the two angles equal to 0 and the other equal to $\pi/2$. This means that the optimal approach is to use only one of the two available pairs of ancillary modes.

Although the above is one specific example of a procedure attempting to increase the concurrence between the target particles, it seems to be the most natural one combining the results of Sec. V and those of Ref. [13]. We therefore suspect that no other procedure would allow an increase in the concurrence.

Note that the failure to increase the concurrence using a larger number of ancillary particles does not mean that quantum-nonlocal effects cannot be observed in this system. In principle, they are observable [13]. The only concern is that one can raise questions about whether the observed effects should be attributed to the mode entanglement or the combination of the mode entanglement and the entanglement already present within the pair of ancillary modes.

VII. CONCLUSION

In this paper we have analyzed the problem of detecting mode entanglement using various types of particles. The results are summarized in Table I, assuming the existence of N suitably prepared ancillary particles for the case of massive particles. For massless particles, mode entanglement is no different from the Bell-state entanglement in terms of experimental observability, regardless of particle statistics. As we have discussed, coherent states play an important role in this context (note that coherent states can only be used when considering massless particles). For massive particles, i.e., those that must obey particle-number superselection rules, one must make use of additional ancillary particles in order to experimentally detect the mode entanglement. For bosons, an ancillary Bose-Einstein condensate of N particles can be reused arbitrarily many times, which suggests that the condensate should be thought of as a catalyst in the experimental detection of the entanglement. For fermions, we cannot find any procedure that gives better results than using a single ancillary particle. This result suggests that one cannot detect the mode entanglement in this case; one can only detect the entanglement present in the combination of the flying and ancillary particles. If this conclusion is correct, one would have to question whether the mode entanglement of massive fermions can be considered a true (i.e., experimentally observable) form of entanglement.

Finally, we would like to mention that the concept of coherent fermionic states has been used in the literature [29], mainly as a simple calculational tool to analyze the behavior of fermionic many-body systems (This effort was motivated by the fact that coherent states provide invaluable predictive power when studying certain aspects of the behavior of bosonic many-body systems [30].) In this paper, we have analyzed the possibility of using fermionic coherent states to simulate classical fields for the purpose of inducing unitary transformations on the states of the target particles. The sequentiality in our proposed procedure provides some distinguishability between the particles, thus we do not have to deal with anticommutation rules or Grassmann variables. It would be interesting to see if there is a connection between the ability to utilize fermionic coherent states analyzed here and the properties of these states analyzed in previous work.

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