Bell's experiment with intra- and inter-pair entanglement: Single-particle mode entanglement as a case study

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Theoretical considerations of Bell-inequality experiments usually assume identically prepared and independent pairs of particles. Here we consider pairs that exhibit both intrapair and interpair entanglement. The pairs are taken from a large many-body system where all the pairs are generally entangled with each other. Using an explicit example based on single mode entanglement and an ancillary Bose-Einstein condensate, we show that the Bell-inequality violation in such systems can display statistical properties that are remarkably different from those obtained using identically prepared independent pairs. In particular, one can have probabilistic violation of Bell's inequalities in which a finite fraction of all the runs result in violation even though there could be no violation when averaging over all the runs. Whether or not a particular run of results will end up being local realistically explainable is "decided" by a sequence of quantum (random) outcomes.

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I. INTRODUCTION

Entanglement has for decades attracted interest and caused controversy in the physics literature [1]. Studies on this subject usually involve discussions of Bell's inequality (or inequalities) [2], where measurement results cannot be described by a local realistic model. One typically thinks of a Bell-test experiment in terms of a large number of identical pairs, each of which is composed of two particles. One can usually understand such an experiment by analyzing the behavior of a single pair. The single-pair analysis is then cast in statistical terms in order to deduce the outcome of the whole experiment as is usually done in quantum mechanics (By "whole experiment" we mean an experiment that involves repeating the single-pair procedure a large number of times, thus obtaining statistical information about the measurement outcomes.)

Here we are interested in the situation where the pairs that are measured for the Bell test are nonidentical and furthermore entangled with each other. We start by pointing out that although most studies assume identically prepared and independent pairs in the Bell test, no such requirement is used or needed in the derivations of Bell's inequalities. We then analyze the intrapair quantum correlations that can be observed in the presence of interpair quantum correlations. This analysis is somewhat related to but clearly distinct from recent studies that generalize Bell's inequality in order to probe multipartite entanglement [3,4]. Using a specific physical entangled state as an example, we show that rich statistical properties can be obtained in the presence of interpair entanglement, in particular a probabilistic violation of Bell's inequalities (of the bipartite form). We discuss whether this probabilistic violation can be used to exclude explanations based on local-hidden-variable (LHV) models.

II. ASSUMPTION OF IDENTICAL INDEPENDENT PAIRS IN A BELL TEST

We start by noting the point that most studies of entanglement assume the preparation of identical independent pairs, whereas in practice the pairs are typically generated by the same source and can in principle be correlated. The question is therefore whether the preparation of the pairs by a single source constitutes any "loophole" in interpreting the violation of Bell's inequalities as evidence against local realism. Inspection of the derivation of Bell's inequalities [5], however, shows that no particular assumption is made on the possibility of correlations between the different pairs. Interpair correlations or entanglement therefore do not constitute any conceptual hurdle to the interpretation of a Bell-inequality violation as evidence against local realism.

A subtler point, which will be illustrated by an example below, arises in the case when the ensemble average over a large number of experiments does not violate Bell's inequality but a finite fraction of the experimental runs (each one of which involves a large number of pairs) do violate the inequality. As long as the number of pairs in a given run is large enough to render the expected statistical fluctuations around the mean negligible, such a probabilistic violation can be used as evidence to exclude LHV models. The crucial point here is that within the single experimental runs that are accepted, no pairs are excluded (e.g., based on the measurement settings or the outcome of their measurements as in the detection loophole). One must also keep in mind that here we are talking about a finite fraction of all the runs resulting in violation, whereas statistical fluctuations can explain a violation in a small fraction of the runs and the size of this fraction decreases and approaches zero for large numbers of pairs in each run. In short, LHV models would predict that no violation of Bell's inequality can be observed in any experimental run (in the limit that the number of pairs in one
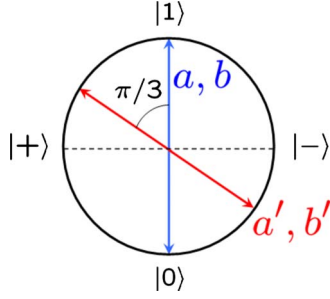
FIG. 1. (Color online) The measurement bases used in the Bell test. The states $|\pm\rangle$ are defined as $|\pm\rangle=(|0\rangle\pm|1\rangle)/\sqrt{2}$. The ideal entangled state is $(|10\rangle+|01\rangle)/\sqrt{2}$. The measurement bases for the first mode (represented by the first symbol inside the ket) are denoted by a and a', and the measurement bases for the second mode (represented by the second symbol inside the ket) are denoted by b and b'. Note that the measurement bases shown above, which we choose in order to simplify our numerical calculations below, differ from the ones that produce maximal violation of Bell's inequality [7].

run approaches infinity), and therefore a violation in only a finite fraction of the runs can be used to exclude LHV models. One can also say that typical sub-ensemble-selection loopholes in Bell-inequality tests involve the assumption that the rejected runs would have produced the same correlations as the accepted runs if they had been recorded properly. In contrast, here we are incorporating all the runs (both the accepted and rejected ones) into our analysis, somewhat similarly to what was done in Ref. [6].

III. STATISTICS OF BELL-TEST RESULTS

We now briefly discuss the statistical properties of the Bell-test violation. We use the version of the Bell test where the ideal pair state is the state

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle), \qquad (1)$$

and the measurement bases are taken to be $\{|0\rangle,|1\rangle\}$ (to which we refer as a and b, depending on the subsystem on which the measurement is performed) and $\{\cos\frac{\pi}{3}|0\rangle+\sin\frac{\pi}{3}|1\rangle, \sin\frac{\pi}{3}|0\rangle-\cos\frac{\pi}{3}|1\rangle\}$ (to which we refer as a' and b') [5]. This choice of bases, which is illustrated in Fig. 1, simplifies the analysis below [7]. A large number M of pairs are generated and measured in randomly chosen pairs of bases. Correlation functions given by the statistical averages

$$C_{\alpha,\beta} = \langle \sigma_\alpha \sigma_\beta \rangle \qquad (2)$$

are recorded, where σ_α and σ_β are the Pauli operators along the directions α and β (note that the first and second operators affect the first and second modes, respectively; note also that symbols such as σ_α in our notation are sometimes expressed as $\vec{\alpha}\cdot\vec{\sigma}$ etc. in the literature). One then evaluates the quantity [5]
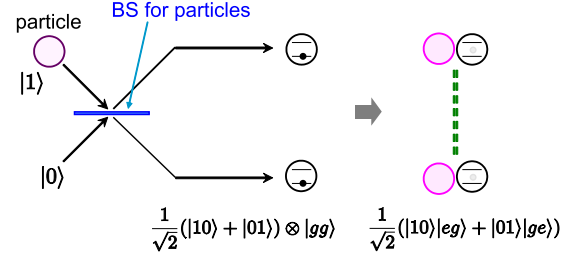

FIG. 2. (Color online) A schematic diagram of the setup under consideration. After "passing through a beam splitter" (BS), a flying particle is in a quantum superposition of being in one of two outgoing paths. At each of the two possible final destinations of the flying particle there is a target particle that is initially in its ground state $|g\rangle$ and is only driven to its excited state $|e\rangle$ if the flying particle arrives at its location.

$$S = |-C_{a,b} + C_{a,b'} + C_{a',b} + C_{a',b'}|. \qquad (3)$$

Bell's inequality (in the Clauser-Horne-Shimony-Holt form) is now expressed as

$$S \leq 2. \qquad (4)$$

The state in Eq. (1) gives the expectation value $S=5/2$, thus violating the inequality [7]. Because of the statistical nature of S, one expects to obtain a different value every time the experiment is performed (with each single experimental run involving a number M of pairs). However, if the number of pairs M in a single run is sufficiently large and the pairs are identical and uncorrelated, the experimentally obtained value of S will, with high probability, be very close to the theoretically calculated single-pair expectation value, with statistical variations in the order of $1/\sqrt{M}$. In this case, S becomes essentially predictable deterministically. In the situation that we consider in this paper, however, all the pairs are entangled with each other. As a result, we are interested in the statistics of the S values that one can expect to obtain in individual experimental runs.

IV. SPECIFIC SYSTEM: SINGLE-PARTICLE ENTANGLEMENT

From now on we focus on a specific physical system as a demonstrative example of the interesting statistics that can be obtained using a multipartite entangled state. The system that we consider possesses so-called mode entanglement [8–12]. Our starting point is the single-particle state given in Eq. (1) where the first and second quantum numbers now represent the number of particles in two modes that are localized at two spatially separated and ideally distant locations (see Fig. 2). We refer to such delocalized particles as flying particles since we imagine these particles being emitted from a common source somewhere between the two measurement locations. For the purposes of the present analysis, we consider the case where these flying particles cannot be created or annihilated. In other words, there is a conservation law constraining the total number of particles of the flying-particle species to be fixed. As discussed in Refs. [9,10], we imagine that a flying particle can excite a two-level target

particle from its ground state to its excited state. By placing one target particle on each side of the proposed setup, one can prepare the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle \otimes |eg\rangle + |01\rangle \otimes |ge\rangle). \quad (5)$$

One can wonder whether the preparation of the state in Eq. (5) poses any difficulties related to the nonconservation of energy. In principle, energy conservation is not a fundamental difficulty here: one can imagine that the presence of the flying atom modifies the energy levels of the target atom, such that an applied laser field is resonant with the target atom only when the flying atom is on the same side of the apparatus. One can also imagine alternative scenarios where there is no energy difference between the states $|g\rangle$ and $|e\rangle$.

If one performs a measurement on the target particles, the outcome will be consistent with a reduced density matrix where the flying-particle degrees of freedom are traced out. In the basis $\{|gg\rangle, |ge\rangle, |eg\rangle, |ee\rangle\}$, Eq. (5) gives the density matrix

$$\rho_{\mathrm{TP}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (6)$$

This density matrix describes a statistical mixture of the states $|ge\rangle$ and $|eg\rangle$ with no phase coherence between them, i.e., with no entanglement. The reason for the lack of phase coherence is the fact that the which-path information about the location of the excited target particle is also carried by the flying particle.

A. Bose-Einstein condensate as an ancillary phase reference

We now consider an additional resource in the form of a number N of particles of the same species as the flying particle. These ancillary particles are prepared in the state

$$|\Psi_{\mathrm{BEC}}\rangle = \sum_{j=0}^{N} \sqrt{P_j} e^{\varphi_j} |j, N-j\rangle_{\mathrm{anc}}, \quad (7)$$

where the two quantum numbers represent the number of ancillary particles in two modes, each of which is localized on one side of the setup [the subscript Bose-Einstein condensate (BEC) indicates that we are generally assuming N to be a large number, thus forming a Bose-Einstein condensate]. For the case where the two condensates form a single BEC state with equal weights on the two sides and zero relative phase between them [13], the distribution function in Eq. (7) is given by

$$P_j = \frac{1}{2^N} \frac{N!}{j!(N-j)!} \quad (8)$$

and the phases $\varphi_j = 0$.
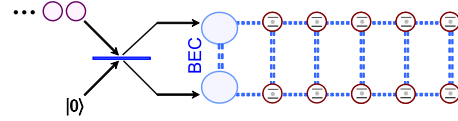
The state of the entire system is now given by

FIG. 3. (Color online) A schematic diagram of a single ancillary BEC that is reused to transfer the mode entanglement of a stream of flying particles to the internal states of the target particles.

$$|\Psi\rangle = \sum_{j=0}^{N} \sqrt{P_j} |j, N-j\rangle_{\mathrm{anc}} \otimes \frac{1}{\sqrt{2}}(|10\rangle \otimes |eg\rangle + |01\rangle \otimes |ge\rangle). \quad (9)$$

Following the procedure proposed in Ref. [9], one can manipulate the state in Eq. (9), controllably and coherently injecting the flying particle into the condensate of particles that are indistinguishable from the flying particle (the application of the injection operation is conditioned on the state of the target atom, ensuring that unitarity is not violated) and obtain the state

$$|\Psi\rangle = \sum_{j=0}^{N} \sqrt{\frac{P_j}{2}}(|j+1, N-j\rangle_{\mathrm{anc}} \otimes |eg\rangle + |j, N-j+1\rangle_{\mathrm{anc}}$$
$$\otimes |ge\rangle). \quad (10)$$

If we now trace out the state of the BEC, we find that the target particles are described by the reduced density matrix

$$\rho_{\mathrm{TP}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma & 0 \\ 0 & \gamma & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (11)$$

where for large N (and, somewhat coincidentally, for $N=1$)

$$\gamma = 1 - \frac{1}{2N}. \quad (12)$$

This state is entangled and allows the violation of Bell's inequality (the violation is obtained even for $N=1$ as was explained in detail in Ref. [10]). It is worth mentioning that for density matrices of the form of Eq. (11) the concurrence [14] is equal to γ.

The above results describe the preparation of a single pair of entangled target particles. We now consider what happens when the same BEC is used (or rather reused) to prepare additional pairs of entangled target particles as illustrated in Fig. 3. Tracing out the target-particle degrees of freedom from Eq. (10) results in a mixed state of the BEC. As was discussed in Refs. [9,10], however, if the measurement on the target particles is made in the $\{|g\rangle, |e\rangle\}$ basis, each one of the two possible final BEC states has the same power as the original BEC in terms of generating entangled pairs. As a result, any time the measurement basis $\{|g\rangle, |e\rangle\}$ is used (even if it is only for one particle in the pair), the BEC is unaffected for the purpose of preparing more entangled pairs. Only when the rotated basis is used for both particles does the BEC undergo nontrivial evolution (averaging over the different outcomes results in the same entangling power as

the original BEC [15]). We analyze this evolution next.

To illustrate the evolution of the ancillary resource following the measurement of the first pair (or pairs) in the system, instead of a BEC we take a single ancillary particle in the state $(|10\rangle+|01\rangle)/\sqrt{2}$. Equation (10) now takes the form

$$|\Psi\rangle = \frac{1}{2}(|20\rangle_{\mathrm{anc}} \otimes |eg\rangle + |11\rangle_{\mathrm{anc}} \otimes |eg\rangle + |11\rangle_{\mathrm{anc}} \otimes |ge\rangle$$
$$+ |02\rangle_{\mathrm{anc}} \otimes |ge\rangle). \qquad (13)$$

The reduced density matrix for the target-particle pair therefore corresponds to a mixed state of the form of Eq. (11) with $\gamma=1/2$, which means that the first pair is entangled with concurrence equal to 1/2. If one or both of the target particles are measured in the $\{|g\rangle,|e\rangle\}$ basis, the ancillary modes are projected onto the state $(|20\rangle+|11\rangle)/\sqrt{2}$ or the state $(|11\rangle+|02\rangle)/\sqrt{2}$, depending on the outcome of the measurement. Obviously, each one of these states has the same number distribution as the original ancillary state apart from an overall shift (thus these states have the same entangling power as the original ancillary state). Let us consider, however, what happens if both target particles are measured in the $(|g\rangle \pm |e\rangle)/\sqrt{2}$ basis. (Note that this basis is different from the one that we use for analyzing the Bell-test experiment but are simpler to analyze in this argument.) If the same measurement outcome is obtained for both target particles of the first pair (a situation that occurs with probability 3/4), the ancillary modes are projected onto the state $(|20\rangle+2|11\rangle+|02\rangle)/\sqrt{6}$. This state can then be used to prepare a second target-particle pair with $\gamma=2/3$, i.e., higher than the value of γ for the first pair. The ancillary resource is therefore enhanced when this outcome is observed. If, on the other hand, opposite measurement outcomes are obtained for the two target particles of the first pair (this situation occurs with probability 1/4), the ancillary modes are projected onto the state $(|20\rangle-|02\rangle)/\sqrt{2}$. This state would give $\gamma=0$ for the second pair, i.e., the second pair would show no sign of entanglement. Thus, although averaging over a large ensemble gives the same value of γ for the second pair (note here that $\frac{3}{4}\times\frac{2}{3}+\frac{1}{4}\times 0=\frac{1}{2}$), knowledge of the first-pair measurement outcome gives additional information about the entanglement in the second pair. This phenomenon is a clear indication of correlation between the different pairs. Naturally, this argument applies to all other pairs that are prepared later in a long sequence as well.

B. Bell test without an ancillary condensate: Probabilistic violation

We now turn from the above argument concerning two correlated pairs to analyzing a full Bell-test experiment involving M entangled pairs. We first note that for the three choices (a,b), (a,b'), and (a',b) (which are defined in Fig. 1), the measurement outcomes do not depend on the value of γ and they give
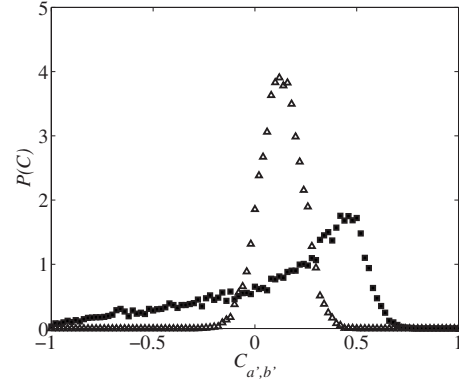

FIG. 4. Probability density $P(C)$ for obtaining a given value of the correlation function $C_{a',b'}$ in a Bell test following our procedure of reusing the ancillary source (squares). Positive values of $C_{a',b'}$ correspond to a violation of Bell's inequality. Here the ancillary source has $N=1$, i.e., a single ancillary particle shared between the two sides of the setup. For comparison we show the probability density if a new ancillary particle was used for each entangled pair (triangles). The results were obtained by constructing a histogram from 10^4 runs, with $M=400$ entangled pairs generated in each run (In the calculation we have assumed that exactly one fourth of these pairs are measured in the a'-b' bases). Doubling the value of M reduces the width of the curve that corresponds to the case of independent pairs (triangles) by a factor of $\sqrt{2}$ but leaves the curve that corresponds to the case of correlated pairs (squares) essentially unchanged.

$$-C_{a,b}+C_{a,b'}+C_{a',b}=2, \qquad (14)$$

up to statistical fluctuations of order $1/\sqrt{M}$ that we ignore here. The condition for the violation of Bell's inequality therefore reduces to

$$C_{a',b'} > 0. \qquad (15)$$

We now consider a single ancillary particle, and we take a stream of flying particles used to produce a large number of entangled target-particle pairs (note that the same ancillary particle, along with the flying particles that are injected into the same modes, are reused to prepare all the entangled pairs). The probability distribution for the values of the correlation function $C_{a',b'}$ that would be observed in experiment is shown in Fig. 4. Unlike the prediction (depicted by the triangles) for identical, independent pairs described by Eq. (11) with $\gamma=1/2$, the distribution is broad and the width reaches a constant value for large M (we have verified this statement by comparing the results for $M=200$, 400, and 800). The average value of $C_{a',b'}$ is the same for both cases, as expected from the fact that, on average, the ancillary resource is neither enhanced nor destroyed after repeated use [9,10].

We next consider the case where no ancillary particles are used ($N=0$; see Fig. 5). This case is perhaps the most relevant one to the probabilistic violation of Bell's inequality, which is the main topic of this paper. In this case one would observe a violation in approximately 40% of the runs and no violation in approximately 60% of the runs, with the average over all the runs being on the nonviolation side. We empha-
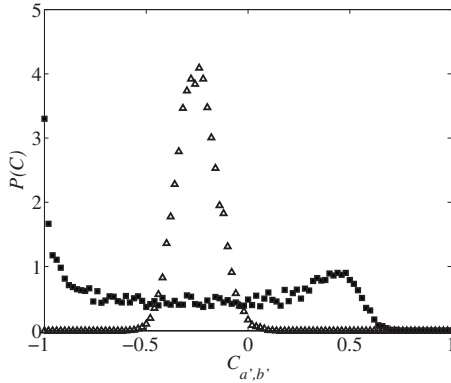
FIG. 5. Same as in Fig. 4, but with $N=0$, i.e., no ancillary particles are used. Note that the finite violation probability in the case of independent pairs (triangles) is a result of the fact that we use a finite number of particles in the numerical calculations. This probability decreases and approaches zero if the number of particles is increased. We have verified, however, that the shape of the curve defined by the squares is essentially unchanged if we change the number of particles (provided that this number is much larger than one).

size here that each run can involve an arbitrarily large number M of pairs. Each run therefore qualifies as a large statistical ensemble for purposes of the Bell test. A violation that is observed with finite probability (for an essentially infinite number of pairs) is therefore sufficient to preclude any LHV explanation of the observed results. Since testing LHV theories is precisely the purpose of performing the Bell test, this probabilistic violation constitutes a successful violation of Bell's inequality.

The above argument regarding the interpretation of the probabilistic violation affects mainly the case $N=0$. The reason is that averaging over a large number of runs results in a violation for any finite N, leaving no caveats about the observed violation. The same reason, however, makes the case $N=0$ of special importance for the present discussion of interpreting the probabilistic violation. Even though averaging over many runs shows no violation of Bell's inequality, the probabilistic violation is sufficient to preclude LHV theories and therefore constitutes a successful violation of Bell's inequality, as mentioned above. It is also worth emphasizing here that when $N=0$ no nonlocal ancillary resource is used at all; any correlations between the two observers are carried solely by the flying particles.

One might wonder whether the finite violation probability observed in our numerical calculations for $N=0$ is a result of the fact that the number of pairs M was finite (under 1000 in all of our numerical calculations). It is straightforward to verify, however, that this is not the case. As mentioned above, every time a pair is measured and the results of both measurements are known, the state of the ancillary resource evolves according to these measurement outcomes. If one starts with $N=0$ and takes the experimental runs where the first pair to be measured in the a'-b' bases gives $\sigma_{a'}\sigma_{b'}=+1$, one finds that the ancillary resource evolves into a state equivalent to $(|10\rangle+|01\rangle)/\sqrt{2}$ (the only possible difference from this state is the existence of some additional particles

whose location is known with certainty, which would happen when the first few measurements are performed in bases other than a'-b'). With this new initial state of the ancillary resource, one finds that the ensemble average of all subsequently prepared pairs will have $\gamma=1/2$ (i.e., on the violation side of the inequality). Taking into consideration the fact that the range of $C_{a',b'}$ is finite (from -1 to 1), one can see that the only way for the average over all the runs in this finite subensemble to be on the violation side of the inequality is to have a finite fraction of all these runs being on the violation side. Thus the finite violation probability for an infinite number of runs is proved.

So far we have discussed a sequence of measurements using the same ancillary resource. We now express explicitly how the above analysis can be cast in terms of multipartite-entangled states. If a large number M of target-particle pairs are prepared before performing any measurement, the state of the entire system would be given by

$$|\Psi\rangle = \sum_{n_1=0}^{1} \cdots \sum_{n_M=0}^{1} \sum_{j=0}^{N} \sqrt{\frac{P_j}{2^M}} \left| j + \sum_{k=1}^{M} n_k, N - j + M \right.$$
$$\left. - \sum_{k=1}^{M} n_k \right\rangle_{\text{anc}} \otimes |n_1, 1 - n_1\rangle \otimes \cdots \otimes |n_M, 1 - n_M\rangle,$$

(16)

where, for notational simplicity, the target-particle states $|g\rangle$ and $|e\rangle$ are now expressed as $|0\rangle$ and $|1\rangle$, respectively. For purposes of analyzing the outcomes of measurements performed on the target particles, the flying-particle and ancillary degrees of freedom can be traced out, which still results in a multipartite entangled state for the target particles. As the measurements proceed on target-particle pairs, the state of the remaining pairs evolves according to the initial measurement outcomes, resulting in the observed probabilistic violation. In other words, the entanglement within the unmeasured pairs increases or decreases, depending on the measurement outcomes for the measured pairs.

V. CONCLUSION

In conclusion, we have considered the question of performing Bell-type experiments using pairs that are entangled with each other. We have presented a multipartite entangled physical system where a violation of Bell's inequality would be obtained probabilistically, with the violation or lack thereof being decided by a sequence of quantum (random) outcomes. This probabilistic violation is sufficient to preclude local-realistic models.

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