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An efficient single-step scheme for manipulating quantum information of two trapped ions beyond the Lamb–Dicke limit

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Abstract

Based on the exact conditional quantum dynamics for a two-ion system, we propose an efficient *single-step* scheme for coherently manipulating quantum information of two trapped cold ions by using a pair of synchronous laser pulses. Neither the auxiliary atomic level nor the Lamb–Dicke approximation are needed.

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1. Introduction

The entanglement between different particles has recently become a focus of activity in quantum physics (see, e.g., [1]), because of experiments on non-local features of quantum mechanics and the development of quantum information physics. Einstein–Podolsky–Rosen (EPR) entangled states with two particles have been employed not only to test Bell's inequality [2], but also to realize quantum cryptography and quantum teleportation [1]. Also, entanglement plays a central quantitative role in quantum parallelism [3]. The demonstrations of quantum entanglement to date are usually based on various probabilistic processes, e.g., the generation of photon pairs in parametric down conversion [4]. However, it is very difficult to generate the entanglement of larger numbers of particles, as the probability of randomly generating the appropriate conditions decreases exponentially with the number of particles. Intense activities are now focused on generating an entanglement of particles in a deterministic

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way, i.e., to produce a desired entangled state. For example, the entanglements of two and four trapped ions have been produced experimentally [5,6].

As first suggested by Cirac and Zoller [7], a very promising scenario for implementing a practical quantum information processor is the system of laser-cooled trapped ions, due to its long coherence time [8]. Information in this system is stored in the spin states of an array of trapped cold ions and manipulated by using laser pulses. The ions are held apart from one another by their mutual Coulomb repulsion. Each ion can be individually addressed by focusing laser beams on the selected ion. The collective normal modes of oscillation shared by all of the ions form the information bus, through which all gate operations can be performed. In the past few years, several key features of the proposal in [7], including the production of entangled states and the implementation of quantum controlled operations between a pair of trapped ions, have already been experimentally demonstrated [5,9–11]. Also, several alternative theoretical schemes (see, e.g., [12–19]) have been developed for overcoming various difficulties in realizing a practical ion-trap quantum information processor.

The Lamb–Dicke (LD) approximation is made in *almost all* of previous schemes (see, e.g., [5,10,13,18,19]), wherein the interaction between the internal states $|s\rangle = \{|g\rangle, |e\rangle\}$ and the external motional harmonic oscillator states $\{|n\rangle; n = 0, 1, 2, \ldots\}$ of the ion is usually expanded to the lowest order of the LD parameter η_L. This approximation requires that the coupling between the external and internal degrees of freedom of the ion is very weak, i.e., the spatial dimension of the motion of the ground state of the trapped ion should be much smaller than the effective wavelength of the applied laser field (see, e.g., [7,20]). However, the quantum motion of the trapped ions is *not* limited to the LD regime [21,22]. Inversely, utilizing the laser-ion interaction beyond this limit could be helpful for reducing the noise in the trap and improving the cooling rate (see, e.g., [22]). Therefore, it would be useful to implement the trapped-ion quantum information processing outside the LD regime. Experimentally, the cooling of the motional ground state of two trapped ions has been achieved in a Paul trap in the LD limit [10]. Recently, Morigi et al. has given a proposal for cooling the collective motional states of two trapped ions outside the LD regime [23]. This is a further step towards realizing the ion trap quantum computer operating outside the LD regime. In fact, several schemes [14–16] have been proposed to implement the quantum computation with trapped ions beyond the LD limit by sequently applying a series of pulses.

In this Letter, we propose an alternative scheme for manipulating quantum information, e.g., realizing quantum controlled operations and generating entanglement, of two trapped ions *beyond* the LD limit by using a pair of synchronous laser pulses. The information bus, i.e., the center-of-mass (CM) vibrational quanta of the ions, for communicating different ions, may be either in its ground or an arbitrary excitation state. Neither the auxiliary atomic level nor the Lamb–Dicke approximation are needed in this Letter. The experimental realization of this simple approach is discussed.

2. Conditional quantum dynamics for two trapped ions driven by two synchronous classical laser beams beyond the Lamb–Dicke limit

The ion trap quantum information processor consists of a string of ions stored in a very cold linear radio-frequency trap. The motion of the ions, which are coupled together due to the Coulomb force between them, is quantum mechanical in nature. The ions are sufficiently separated apart (see, e.g., [22,24]) to be easily addressed by different laser beams, i.e., each ion can be illuminated individually by a separate leaser beam. The communication and logic operations between qubits are usually performed by exciting or de-exciting quanta of the collective vibration (i.e., the shared phonon) modes, which act as the information bus (see, e.g., [7,14,17]).

We consider an array of N two-level cold ions of mass M trapped in a one-dimensional harmonic potential of frequency ν. The ions are able to perform small oscillations around their equilibrium position z_{i0} $(i = 1, 2, \ldots, N)$, due to the repulsive Coulomb force between them. Each one of the ions is assumed to be individually addressed by a separate laser beam. We consider the case where an arbitrary pair (labeled by $j = 1, 2$) of trapped cold ions from the chain of N trapped cold ions are illuminated independently by two weak classical laser beams. This is different from the scheme proposed in [25,26] of using Raman lasers to drive the ions. The Hamiltonian corresponding to our situation is

$$\hat{H}(t) = \hbar\omega_0 \sum_{j=1}^{2} \frac{\hat{\sigma}_{z,j}}{2} + \hbar\nu\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \sum_{l=1}^{N-1} \hbar\nu_l\left(\hat{b}_l^\dagger\hat{b}_l + \frac{1}{2}\right)$$
$$+ \frac{\hbar}{2}\sum_{j=1}^{2}\left\{\Omega_j\hat{\sigma}_{+,j}\exp\left\{i\left[\eta_j(\hat{a}^\dagger + \hat{a}) + \sum_{l=1}^{N-1}\eta_{j,l}(\hat{b}_l^\dagger + \hat{b}_l) - \omega_j t - \phi_j\right]\right\} + \text{H.c.}\right\}. \tag{1}$$

Here, ν and ν_l $(l = 1, \dots, N-1)$ are the frequencies of the collective center-of-mass (CM) vibrational motion $(l = 0)$ and the higher normal modes $(l \geqslant 1)$ of the trapped ions, respectively; $\hat{a}^\dagger$ and $\hat{a}$ are the ladder operators of CM mode, while $\hat{b}_l^\dagger$ and $\hat{b}_l$ are the ladder operators of the higher normal modes. Ω_j $(j = 1, 2)$ is the carrier Rabi frequency, which describes the coupling strength between the laser and the jth ion and is proportional to the strength of the applied laser. $\hat{\sigma}_{z,j}$ and $\hat{\sigma}_{\pm,j}$ are Pauli operators, $\hbar\omega_0$ is the energy separation of the two internal states $|g\rangle$ and $|e\rangle$ of the ion, and ϕ_j is the initial phase of the applied laser beam. The LD parameters η_j and $\eta_{j,l}$ account for the coupling strength between the internal state of the jth ion and the vibrational states of the CM mode and the higher frequency modes, respectively. Expanding Eq. (1) in terms of creation and annihilation operators of the normal modes, we can rewrite the Hamiltonian of system in the interaction picture as

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1,2}\left\{\Omega_j\hat{\sigma}_{+,j}\hat{G}_j\left[\exp\left(-\frac{\eta_j^2}{2} - i\phi_j\right)\sum_{m,n=0}^{\infty}\frac{(i\eta_j)^{m+n}\hat{a}^{\dagger m}\hat{a}^n}{m!n!}\exp\bigl[i(m-n)\nu t + i\delta_j t\bigr]\right] + \text{H.c.}\right\}, \tag{2}$$

with

$$\hat{G}_j = \prod_{l=1}^{N-1}\exp\left(-\frac{(\eta_{j,l})^2}{2}\right)\sum_{m',n'=0}^{\infty}\frac{(i\eta_{j,l})^{m'+n'}\hat{b}_l^{\dagger m'}\hat{b}_l^{n'}}{m'!n'!}\exp\bigl[i(m'-n')\nu_l t\bigr].$$

We assume that the frequencies of the applied lasers to be tuned resonantly on the same lower red-sidebands of the center-of-mass (CM) vibrational mode, i.e., the frequencies ω_j of the applied lasers are chosen to be $\omega_j = \omega_0 - k_j\nu, k_j = k = 1, 2, \dots$. Then, like the procedure described in [25,27,28], we make the usual rotating wave approximation (RWA) and have the following effective Hamiltonian

$$\hat{H}_{\text{eff}} = \frac{\hbar}{2}\sum_{j=1,2}\left\{\tilde{\Omega}_j\hat{\sigma}_{+,j}\exp\left(-\frac{\eta_j^2}{2} - i\phi_j\right)\sum_{n=0}^{\infty}\frac{(i\eta_j)^{2n+k}\hat{a}^{\dagger n}\hat{a}^{n+k}}{n!(n+k)!} + \text{H.c.}\right\}, \tag{3}$$

for small k values. Here, we are considering the weak excitation regime ($\Omega_j \ll \nu$, i.e., the intensity of the applied laser beams is assumed to be sufficiently weak) [25,27,28]. In this regime, the excitations of the higher-l normal modes $(l \geqslant 1)$ is irrelevant and can be renormalized to the effective Rabi frequencies $\tilde{\Omega}_j$. We stress the following important fact: the effective Hamiltonian (3) reduces to that in previous works (e.g., [5,10,13,19] under the usual LD approximation: $(m+1)\eta_j^2 \ll 1$), to the lowest order of the LD parameter η_j. Here m is the occupation number of the Fock state of the CM vibrational quanta. Also, the Hamiltonian (3) reduces to that in [27] for $k = 1$.

In order to manipulate a pair of trapped ions outside the LD regime, we now wish to solve the quantum dynamical problem associated with the above Hamiltonian (3) *without* using the LD approximation. All operations presented below are based on this solution and do not involve quantum transitions to auxiliary atomic levels. Without loss of generality, the information bus (i.e., the CM vibrational mode of the ions) is assumed to be prepared beforehand in a pure quantum state, e.g., the Fock state $|m\rangle$ with $m < k$. During the time-evolution $\hat{U}(t) = \exp(-it\hat{H}_{\text{eff}}/\hbar)$, the initial state $|m\rangle|g_1\rangle|g_2\rangle$ is unchanged, i.e.,

$$|m\rangle|g_1\rangle|g_2\rangle \xrightarrow{\hat{U}(t)} |m\rangle|g_1\rangle|g_2\rangle, \tag{4}$$

since $\hat{H}_{\rm eff}|m\rangle|g_1\rangle|g_2\rangle = 0$. Using the relations

$$\hat{H}_{\rm eff}|m\rangle|e_1\rangle|g_2\rangle = (-i)^k\hbar e^{i\phi_1}\alpha_1|m+k\rangle|g_1\rangle|g_2\rangle, \qquad \hat{H}_{\rm eff}|m\rangle|g_1\rangle|e_2\rangle = (-i)^k\hbar e^{i\phi_2}\alpha_2|m+k\rangle|g_1\rangle|g_2\rangle,$$

and

$$\hat{H}_{\rm eff}|m+k\rangle|g_1\rangle|g_2\rangle = i^k\hbar\big(e^{-i\phi_1}\alpha_1|m\rangle|e_1\rangle|g_2\rangle + e^{-i\phi_2}\alpha_2|m\rangle|g_1\rangle|e_2\rangle\big),$$

we have the evolutions

$$\begin{cases} |m\rangle|g_1\rangle|e_2\rangle \xrightarrow{\hat{U}(t)} B_1(t)|m+k\rangle|g_1\rangle|g_2\rangle + B_2(t)|m\rangle|g_1\rangle|e_2\rangle + B_3(t)|m\rangle|e_1\rangle|g_2\rangle, \\ |m\rangle|e_1\rangle|g_2\rangle \xrightarrow{\hat{U}(t)} C_1(t)|m+k\rangle|g_1\rangle|g_2\rangle + C_2(t)|m\rangle|g_1\rangle|e_2\rangle + C_3(t)|m\rangle|e_1\rangle|g_2\rangle, \end{cases} \tag{5}$$

with

$$B_1(t) = (-i)^{k+1}e^{i\phi_2}\frac{\alpha_2\sin(\chi t)}{\chi}, \qquad B_2(t) = \frac{\alpha_1^2+\alpha_2^2\cos(\chi t)}{\chi^2}, \qquad B_3(t) = e^{-i(\phi_1-\phi_2)}\frac{\alpha_1\alpha_2[\cos(\chi t)-1]}{\chi^2};$$

$$C_1(t) = (-i)^{k+1}e^{i\phi_1}\frac{\alpha_1\sin(\chi t)}{\chi}, \qquad C_3(t) = \frac{\alpha_2^2+\alpha_1^2\cos(\chi t)}{\chi^2}, \qquad C_2(t) = e^{i(\phi_1-\phi_2)}\frac{\alpha_1\alpha_2[\cos(\chi t)-1]}{\chi^2}.$$

Here, $j = 1, 2$, and

$$\chi = \sqrt{\alpha_1^2+\alpha_2^2}, \quad \alpha_j = \tilde{\Omega}^j_{m,k}, \qquad \beta_j = \tilde{\Omega}^j_{m+k,k}, \quad \tilde{\Omega}^j_{m,k} = \frac{\tilde{\Omega}_j e^{-\eta_j^2/2}}{2}\sqrt{\frac{(m+k)!}{m!}}\sum_{n=0}^{m}\frac{(-i\eta_j)^{2n+k}}{(n+k)!}\binom{n}{m}.$$

Finally, in order to obtain the time-evolution of the initial state $|m\rangle|e_1\rangle|e_2\rangle$, we solve the Schrödinger equation $i\hbar\partial|\psi(t)\rangle/\partial t = \hat{H}_{\rm eff}|\psi(t)\rangle$ in the invariant subspace $\{|m+2k\rangle|g_1\rangle|g_2\rangle, |m+k\rangle|e_1\rangle|g_2\rangle, |m+k\rangle|g_1\rangle|e_2\rangle, |m\rangle|e_1\rangle \times |e_2\rangle\}$,

$$i\frac{\partial}{\partial t}\begin{pmatrix} D_1(t) \\ D_2(t) \\ D_3(t) \\ D_4(t) \end{pmatrix} = \begin{pmatrix} 0 & (-i)^k\alpha_1 e^{i\phi_1} & (-i)^k\alpha_2 e^{i\phi_2} & 0 \\ i^k\alpha_1 e^{-i\phi_1} & 0 & 0 & (-i)^k\beta_2 e^{i\phi_2} \\ i^k\alpha_2 e^{-i\phi_2} & 0 & 0 & (-i)^k\beta_1 e^{i\phi_1} \\ 0 & i^k\beta_2 e^{-i\phi_2} & i^k\beta_1 e^{-i\phi_1} & 0 \end{pmatrix}\begin{pmatrix} D_1(t) \\ D_2(t) \\ D_3(t) \\ D_4(t) \end{pmatrix},$$

with initial conditions: $D_1(0) = D_2(0) = D_3(0) = 0$, $D_4(0) = 1$. Here, $|\psi(t)\rangle = D_1(t)|m+2k\rangle|g_1\rangle|g_2\rangle + D_2(t)|m+k\rangle|e_1\rangle|g_2\rangle + D_3(t)|m+k\rangle|g_1\rangle|e_2\rangle + D_4(t)|m\rangle|e_1\rangle|e_2\rangle$. After a long but direct algebraic derivation, we obtain the exact time-evolution

$$\begin{aligned} |m\rangle|e_1\rangle|e_2\rangle \xrightarrow{\hat{U}(t)} & D_1(t)|m+2k\rangle|g_1\rangle|g_2\rangle + D_2(t)|m+k\rangle|e_1\rangle|g_2\rangle + D_3(t)|m+k\rangle|g_1\rangle|e_2\rangle \\ & + D_4(t)|m\rangle|e_1\rangle|e_2\rangle, \end{aligned} \tag{6}$$

with

$$D_1(t) = i^{(2k)}e^{i(\phi_1+\phi_2)}\frac{\rho}{\Delta}\big[\cos(\lambda_+t)-\cos(\lambda_-t)\big], \qquad D_4(t) = \frac{1}{\Delta}\big[\zeta_+\cos(\lambda_-t)-\zeta_-\cos(\lambda_+t)\big],$$

$$D_2(t) = (-i)^{k+1}e^{i\phi_2}\frac{\rho}{\Delta}\left[\frac{\alpha_2\rho+\beta_1\zeta_+}{\lambda_+\zeta_+}\sin(\lambda_+t) - \frac{\alpha_2\rho+\beta_1\zeta_-}{\lambda_-\zeta_-}\sin(\lambda_-t)\right],$$

and

$$D_3(t) = (-i)^{k+1}e^{i\phi_1}\frac{\rho}{\Delta}\left[\frac{\alpha_1\rho+\beta_2\zeta_+}{\lambda_+\zeta_+}\sin(\lambda_+t) - \frac{\alpha_1\rho+\beta_2\zeta_-}{\lambda_-\zeta_-}\sin(\lambda_-t)\right].$$

Here,

$$\rho = \alpha_1\beta_2 + \alpha_2\beta_1,$$
$$\zeta_\pm = \lambda_\pm^2 - \sum_{j=1}^{2}\alpha_j^2, \quad \lambda_\pm = \sqrt{\frac{\Lambda \pm \Delta}{2}}, \quad \Lambda = \sum_{j=1}^{2}(\alpha_j^2 + \beta_j^2), \quad \Delta^2 = \Lambda^2 - 4(\alpha_1\beta_1 - \alpha_2\beta_2)^2.$$

Analogously, the effective Hamiltonian and the relevant dynamical evolutions for other driving cases can also be derived exactly. For example, for the case where $k_1 = k_2 = k' < 0$ and $|k'| = k > m$ (i.e., the ions are excited by blue-sideband laser beams with equal frequencies instead of red-sideband ones), the effective Hamiltonian and the relevant dynamics of the system can be easily obtained from Eq. (3) and Eqs. (4)–(6) by making the replacements: $\hat{a} \leftrightarrow \hat{a}^\dagger$ and $|e_j\rangle \leftrightarrow |g_j\rangle$, respectively.

3. Manipulation of quantum information in two trapped cold ions

Based on the conditional quantum dynamics for the two-qubit system derived in the previous section, we now show how to effectively manipulate two-qubit quantum information stored in two ions by applying a pair of simultaneous laser pulses. Generally, the motion state entangles with the spin states during the dynamical evolution. We afterwards focus on how to decouple them and realize *in one step* and beyond the LD limit: either a two-qubit controlled operation or an entanglement between the trapped ions. The state of the information bus (CM mode) remains in its initial state, which is not entangled with the qubits after the operations. This is achieved by properly setting up the controllable experimental parameters, e.g., the Lamb–Dicke parameters η_j, the effective Rabi frequencies $\tilde{\Omega}_j$, the frequencies ω_j $(j = 1, 2)$ and the duration of the applied synchronous pulses.

3.1. Two-qubit controlled operations

As one of the simplest universal two-qubit quantum gates, the C^Z logic operation between the 1st and the 2nd ions

$$\hat{C}_{12}^Z = |g_1\rangle|g_2\rangle\langle g_1|\langle g_2| + |g_1\rangle|e_2\rangle\langle g_1|\langle e_2| + |e_1\rangle|g_2\rangle\langle e_1|\langle g_2| - |e_1\rangle|e_2\rangle\langle e_1|\langle e_2|, \tag{7}$$

means that if the first ion is in the state $|g_1\rangle$, the operation has no effect, whereas if the control qubit (first ion) is in the state $|e_1\rangle$, the state of the second ion is rotated by the Pauli operator $\hat{\sigma}_z$. The first qubit is the control qubit and the second one is the target qubit. The C^Z operation is also known as controlled-rotation (CROT). It is seen from Eqs. (4)–(6) that the $\hat{C}_{12}^Z$ gate can be implemented exactly by a *one-step* operation, if the experimental parameters are set up so that the following conditions are simultaneously satisfied,

$$\cos(\chi\tau_z) = 1, \qquad \cos(\lambda_+\tau_z) = \cos(\lambda_-\tau_z) = -1. \tag{8}$$

Here τ_z is the duration of the two applied synchronous pulses. Notice that the second condition in (8), on $\lambda_\pm$, is equivalent to requiring $|D_4(t)|^2 = 1$, which forces $|D_1(t)|^2 = |D_2(t)|^2 = |D_3(t)|^2 = 0$ due to normalization. The information bus remains in its initial state after the operation. Without loss of generality, we give some solutions of the conditional equations (8) for $m = 0$ in Table 1.

We see from the Table 1 that both the small and large, both the negative and positive, values of the LD parameters may be chosen to satisfy the conditions (8) for realizing the desired two-qubit controlled gate. Our approach does not assume the LD approximation where $\eta_j \ll 1$ for $m = 0$. Thus, the present scheme can operate outside the LD regime and η_j can be large. According to previous works (see, e.g., [7,14]), it is known that an exact two-qubit gate $\hat{C}^Z$ surrounded by two one-qubit rotations on the target qubit can give rise to an exact two-qubit CNOT gate $\hat{C}^X$. The present Letter shows that the CNOT gate between different ions can be realized by using a *three-step* pulse process.

Table 1
A few solutions of Eq. (8) for $m = 0$. These parameters realize a C^Z or CROT gate between two trapped ions. Here, τ_z is the duration of the applied pulses, $\tilde{\Omega}_j$ and η_j $(j = 1, 2)$ are the effective Rabi frequencies and the Lamb–Dicke parameters, respectively. The frequencies of the applied pulses are equal, i.e., $k_1 = k_2 = k = 1, 2, 3$

$\tilde{\Omega}_2/\tilde{\Omega}_1$	k	$\eta_1 = \eta_2 = \eta$	$\tilde{\Omega}_1 \tau_z$	$\tilde{\Omega}_2/\tilde{\Omega}_1$	k	$\eta_1 = \eta_2 = \eta$	$\tilde{\Omega}_1 \tau_z$
2.03951	1	1.93185	18.5069	1.81182	1	2.30578	37.5859
		0.517638	12.2197		2	±0.253727	137.757
	2	±0.915272	14.1979			±2.81702	57.2117
		±2.67624	39.2315		3	0.859544	33.8887
	3	1.12532	17.9115			3.40669	124.419
		2.69702	26.2324	4.02791	1	1.87083	18.6274
		3.34152	96.5506			0.707107	10.9967
0.658331	1	1.76579	56.5182		2	±0.983608	14.3592
		0.939131	34.7414			±2.65189	40.9890
	2	±1.09276	45.1673		3	1.16543	18.4811
		±2.60881	131.088			2.63899	26.2548
	3	1.23348	58.6299			3.11088	62.2365
		2.55336	80.4486			3.33069	102.936

3.2. Two-qubit entangled states

Recently, the quantum entanglement of two and four trapped ions have been generated experimentally (see, e.g., [5,11]), although the operations are limited to the weak-coupling LD regime. We now show that the entangled states of two trapped ions can also be produced outside the LD limit. Indeed, the dynamical evolutions (4)–(6) clearly reveal that there are many ways to produce various deterministic entangled states of two trapped ions. For example, if the conditions

$$\alpha_j = \beta_j, \qquad \phi_1 = \phi_2, \qquad \alpha_1 \neq \alpha_2, \qquad \cos(\chi \tau_e) = -1, \tag{9}$$

are satisfied, then two equal red-sideband pulses (i.e., $k_1 = k_2 = k > 0$) with frequencies $\omega_1 = \omega_2 = \omega_0 + k\nu$, applied to two ions individually and simultaneously, yield the following dynamical evolutions:

$$\begin{cases} |m\rangle|g_1\rangle|e_2\rangle \longrightarrow |m\rangle \otimes |\psi_{12}^-\rangle, & |\psi_{12}^-\rangle = U|g_1\rangle|e_2\rangle - V|e_1\rangle|g_2\rangle, \\ |m\rangle|e_1\rangle|g_2\rangle \longrightarrow |m\rangle \otimes |\psi_{12}^+\rangle, & |\psi_{12}^+\rangle = -V|g_1\rangle|e_2\rangle - U|e_1\rangle|g_2\rangle, \end{cases} \tag{10}$$

with

$$U = \frac{\alpha_1^2 - \alpha_2^2}{\alpha_1^2 + \alpha_2^2}, \qquad V = \frac{2\alpha_1\alpha_2}{\alpha_1^2 + \alpha_2^2}.$$

Therefore, entangled states $|\psi_{12}^{\pm}\rangle$ can be generated, in a *single-step* process, by the dynamical evolution of the initial non-entangled states $|g_1\rangle|e_2\rangle$ or $|e_1\rangle|g_2\rangle$. We note that the degrees of entanglement for the above entangled states $|\psi_{12}^{\pm}\rangle$ are equivalent. They are

$$E = -U^2 \log_2 U^2 - V^2 \log_2 V^2. \tag{11}$$

Here E is the degree of entanglement defined [29] as $E[\psi] = -\sum_i C_i^2 \log_2 C_i^2$, for an general two-particle entangled pure state $|\psi(A, B)\rangle = \sum_i C_i |\alpha_i\rangle_A \otimes |\beta_i\rangle_B$, $\sum_i |C_i|^2 = 1$. The entangled state of two trapped ions realized experimentally in [11] is not the maximally entangled state. Its degree of entanglement E is 0.94. However, in principle, maximally entangled states with $E = 1$ can be generated deterministically in the present scheme.

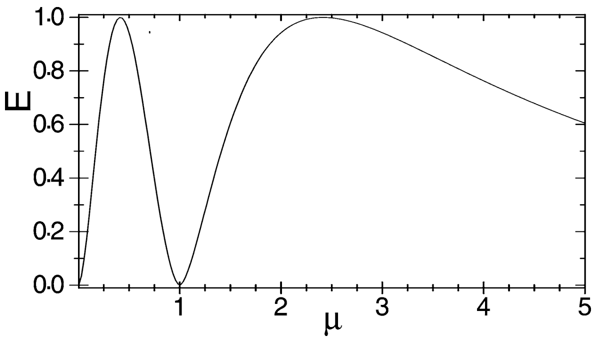

Fig. 1. The degree of entanglement E for the states $|\psi_{12}^{\pm}\rangle$ versus $\mu = \alpha_2/\alpha_1$, which is a function of the LD parameters η_j, effective Rabi frequencies $\tilde{\Omega}_j$, and the laser frequencies ω_j $(j = 1, 2)$. Note that $E = 1$ for $\alpha_2/\alpha_1 = \sqrt{2} \pm 1$, and $E = 0$ for $\alpha_2/\alpha_1 = 1$. For $\alpha_2/\alpha_1 > \sqrt{2} + 1$ the degree of entanglement E decreases when α_2/α_1 increases.

Indeed, it is seen from Fig. 1 that, if the experimental parameters further satisfy the following conditions

$$\frac{\alpha_2}{\alpha_1} = \sqrt{2} \pm 1, \qquad \alpha_1 \tau_e = \frac{(2l-1)\pi}{\sqrt{4 \pm 2\sqrt{2}}}, \quad l = 1, 2, 3, \ldots, \tag{12}$$

the above entangled states $|\psi_{12}^{\pm}\rangle$ become the maximally entangled two-qubit (i.e., EPR) states

$$|\Psi_{12}^{\pm}\rangle = \frac{1}{\sqrt{2}}\big(|g_1\rangle|e_2\rangle \pm |e_1\rangle|g_2\rangle\big). \tag{13}$$

In Fig. 1, we plot the degree of entanglement E for the above entangled states versus the ratio α_2/α_1, which is a function of the LD parameters η_j, effective Rabi frequencies $\tilde{\Omega}_j$, and the laser frequencies ω_j $(j = 1, 2)$. Explicitly,

$$\frac{\alpha_2}{\alpha_1} = \frac{\tilde{\Omega}_2}{\tilde{\Omega}_1}\left(\frac{\eta_2}{\eta_1}\right)^{|k|} \exp\left(-\frac{\eta_2^2 - \eta_1^2}{2}\right) \times \frac{\sum_{n=0}^{m}(-i\eta_2)^{2n} C_m^n/(n+|k|)!}{\sum_{n=0}^{m}(-i\eta_1)^{2n} C_m^n/(n+|k|)!}, \tag{14}$$

for the processes (10). Especially, for the commonly considered case, $m = 0$, the above equation becomes

$$\frac{\alpha_2}{\alpha_1} = \frac{\tilde{\Omega}_2}{\tilde{\Omega}_1}\left(\frac{\eta_2}{\eta_1}\right)^{|k|} \exp\left(-\frac{\eta_2^2 - \eta_1^2}{2}\right). \tag{15}$$

Obviously, the values of E depend on the choice of the experimental parameters $\tilde{\Omega}_j$, η_j and k (thus ω_j) $(j = 1, 2)$. For $\alpha_2/\alpha_2 = 1$ the condition (9) is violated and thus $E = 0$. Similarly,

$$\lim_{\alpha_2/\alpha_1 \to \infty} U = 1, \qquad \lim_{\alpha_2/\alpha_1 \to \infty} V = 0, \quad \text{thus} \quad \lim_{\alpha_2/\alpha_1 \to \infty} E = 0. \tag{16}$$

Inversely, it is seen from Eqs. (12)–(15) that $E = 1$ for $\alpha_2/\alpha_1 = \sqrt{2} \pm 1$. This implies that two-qubit maximally entangled states can be generated deterministically by using a *single-step* operation beyond the LD limit. For example, if the experimental parameters are set up simply as $\eta_1 = \eta_2$, $\tilde{\Omega}_2/\tilde{\Omega}_1 = \sqrt{2} \pm 1$, the EPR state $|\Psi_{12}^{-}\rangle$ can be generated by using a *single-step* synchronous red-sideband π pulses with frequencies $\omega_1 = \omega_2 = \omega_0 + \nu$ and duration $\tau_e = \pi/(\alpha_1\sqrt{4 \pm 2\sqrt{2}})$.

4. Conclusions and discussions

Based on the conditional quantum dynamics for two-qubit system, we have shown that, under certain conditions, the quantum controlled gate or entanglement between a pair of trapped ions can be realized deterministically by *only a single-step operation*, performed by simultaneously applying two laser pulses to two ions. Each of the laser beams interacts with a single ion. Neither auxiliary atomic level nor Lamb–Dicke approximation are required during the operation. The CM mode of the ions always remains in its initial quantum state after the operation.

We now give a brief discussion on the experimental realization of the present scheme. For ion-trap quantum information processing, the information bus, i.e., the usual collective CM vibrational mode, must first be initialized in a pure quantum state, e.g., its ground state. Recently, the collective motion of two and four $^9Be^+$ ions has been successfully cooled to its ground state in the LD regime [5,10]. This is a further step towards realizing the ion trap quantum computer. Once the collective mode of motion of the ions is successfully cooled to the ground state outside the LD regime, the present theoretical scheme may be realized by properly setting up the controllable parameters (e.g., η_j, $\tilde{\Omega}_j$, ω_j) and the durations of the applied laser pulses. Indeed, it is seen from the formulae (see, e.g., [22,28])

$$\eta_j = \cos\theta_j \sqrt{\frac{\hbar\kappa_j^2}{2MN\nu}}, \quad \theta_j = \arccos\left(\frac{\vec{\kappa}_j \cdot \vec{z}_j}{\kappa_j}\right), \tag{17}$$

that the LD parameter η_j can be controlled by adjusting the wave vector $\vec{\kappa}_j$ of the applied laser pulse. Obviously, η_j can be positive or negative, depending on the values of θ_j. Here, MN is the total mass of the ion chain and θ_j is the angle between the laser beam and the z-axis. The effective Rabi frequency $\tilde{\Omega}_j$ of the jth ion can be controlled properly by applying a static electric field [11,30].

Finally, we note that the duration of the two applied simultaneous pulses for realizing the above quantum controlled operation is not much longer than that for other schemes (see, e.g., [5,10,13,19]) operating in the LD regime. The shortest duration of the applied synchronous pulses for realizing the above manipulations of two trapped ions is about 10^{-4} seconds, of the same order of the gate speed operating in the LD regime [31], for $\tilde{\Omega}_1/2\pi \approx 225$ kHz [11]. Of course, to excite only the chosen sidebands of the CM mode, the spectral width of the applied laser pulse has to be sufficiently small. It might seem at first, from the above numerical results, that the present scheme for realizing the desired gate operation cannot be easily implemented, as the relevant experimental parameters should be set up accurately. However, this is not the case. Simple numerical analysis shows that the lowest probability of realizing the desired operation is still very high, even if the relevant parameters cannot be set up exactly. For example, the lowest probability of realizing the two-qubit $\hat{C}^Z$ operation is up to 99.97% (99.49%), if the rate of the two effective Rabi frequencies $\tilde{\Omega}_2/\tilde{\Omega}_1$ is roughly set up as 2.03 (2.0), which is 0.5% (1.9%) away from the exact solution of condition (8), see Table 1. Therefore, the proposed scheme may be realizable in the near future.

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References

[1] M.A. Nielsen, I.L. Chuang, Quantum Computation and Quantum Information, Cambridge Univ. Press, Cambridge, 2000.
[2] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. 47 (1935) 777.
[3] P. Shor, in: S. Goldwasser (Ed.), Proceedings of the 35th Annual Symposium on the Foundations of Computer Science, IEEE Computer Society Press, New York, 1994, p. 124.
[4] P.G. Kwiat, et al., Phys. Rev. Lett. 75 (1995) 4337.
[5] C.A. Sackett, et al., Nature (London) 404 (2000) 256.
[6] D. Leibfried, et al., Nature 422 (2003) 412.

[7] J.I. Cirac, P. Zoller, Phys. Rev. Lett. 74 (1995) 4091.
[8] D. Kielpinski, et al., Science 291 (2001) 1013.
[9] Ch. Roos, et al., Phys. Rev. Lett. 83 (1999) 4713.
[10] B.E. King, et al., Phys. Rev. Lett. 81 (1998) 1525.
[11] Q.A. Turchette, et al., Phys. Rev. Lett. 81 (1998) 3631.
[12] L.M. Duan, J.I. Cirac, P. Zoller, Science 292 (2001) 1695.
[13] D. Jonathan, M.B. Plenio, Phys. Rev. Lett. 87 (2001) 127901.
[14] L.F. Wei, S.Y. Liu, X.L. Lei, Phys. Rev. A 65 (2002) 062316.
[15] L.F. Wei, S.Y. Liu, X.L. Lei, Opt. Commun. 208 (2002) 131.
[16] S.S. Sharma, A. Vidiella-Barranco, Phys. Lett. A 309 (2003) 345.
[17] D. Jonathan, M.B. Plenio, P.L. Knight, Phys. Rev. A 62 (2000) 042307.
[18] A.M. Childs, I.L. Chuang, Phys. Rev. A 63 (2000) 012306.
[19] K. Mølmer, A. Sørensen, Phys. Rev. Lett. 82 (1999) 1835.
[20] D.M. Meekhof, et al., Phys. Rev. Lett. 76 (1996) 1796.
[21] D.J. Wineland, W.M. Itano, Phys. Rev. A 20 (1979) 1521.
[22] D. Stevens, J. Brochard, A.M. Steane, Phys. Rev. A 58 (1998) 2750.
[23] G. Morigi, J. Eschner, J.I. Cirac, P. Zoller, Phys. Rev. A 59 (1999) 3797.
[24] D. Leibfried, Phys. Rev. A 60 (1999) R3335.
[25] E. Solano, et al., Phys. Rev. A 62 (2000) 021401.
[26] E. Solano, et al., Phys. Rev. A 59 (1999) R2539.
[27] L.X. Li, G.C. Guo, Phys. Rev. A 60 (1999) 696.
[28] M. Šašura, V. Bužek, J. Mod. Opt. 49 (2002) 1593.
[29] C.H. Bennett, H.J. Bernstein, S. Popescu, B. Schumacher, Phys. Rev. A 53 (1996) 2046.
[30] S.R. Jefferts, C. Monroe, E.W. Bell, D.J. Wineland, Phys. Rev. A 51 (1995) 3112.
[31] C. Monroe, et al., Phys. Rev. Lett. 75 (1995) 4714.