Supplementary Materials for "$\mathcal{PT}$-Symmetric Phonon Laser"

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This document consists of three parts: (I) amplification factor with varying the optical tunnelling rate J; (II) supermode splitting and linewidth; and (III) stability analysis with the tunable gain-loss ratio δ.

I. Amplification factor with varying J

Figure 2 in the main text shows the steady-state populations of intracavity photons in the passive resonator, by numerically solving Eq. (5). A notable feature of the $\mathcal{PT}$-symmetric COM system is the emerging resonance peaks of the optical amplification factor η around the gain-loss balance $\delta = 1$ (see Fig. 2b), where we choose to change the values of gain-loss ratio δ but with fixed optical tunnelling rate ($J/\gamma = 1$). Here we show in Fig. S1 that for fixed δ, similar features can be observed by changing the optical tunnelling rate J/γ.

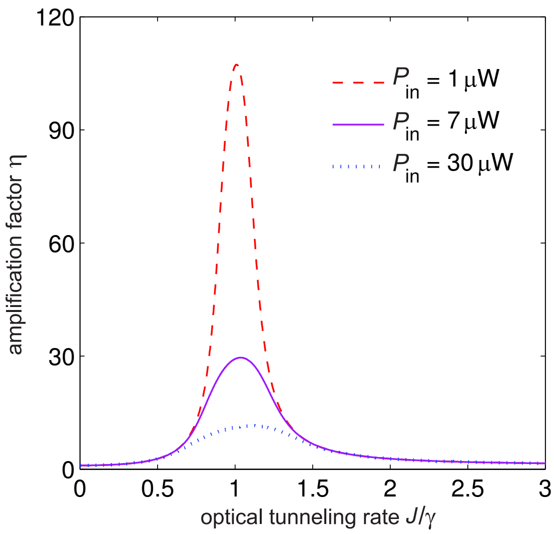

FIG. S1: (Color online) Optomechanical amplification factor η versus the tunable optical tunnelling rate J/γ, for the fixed value of gain-loss ratio $\delta = 1$. Here the amplification factor η is by the Eq. (7) of the main text (having the $\mathcal{PT}$-symmetric result divided by the result for the passive COM case).

Even when comparing with a passive COM system with threshold power $P_{\rm th,p} = 7\,\mu$W, the $\mathcal{PT}$-symmetric COM system performs better for significantly lower input

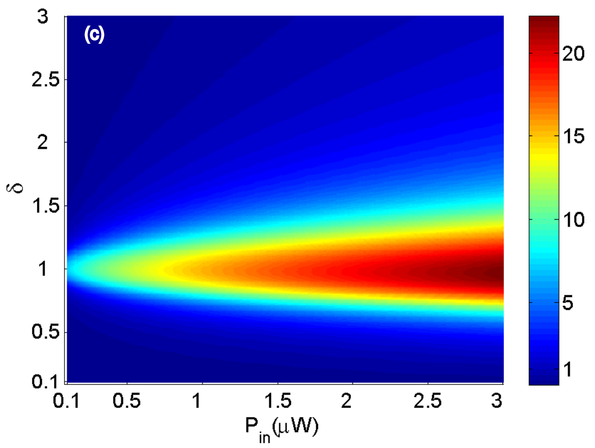

FIG. S2: (Color online) Relative amplification factor ξ versus tunable gain-loss ratio $\delta \equiv \kappa/\gamma$ and the input power $P_{\rm in}$. The vertical color bar refers to the values of ξ.

power, i.e.

$$\xi \equiv \frac{x_s(\delta, P_{\rm in})}{x_{s,{\rm p}}(P_{\rm in} = P_{\rm th,p})} \geq 1,$$

which, for $\delta = 1$, can be realized for $P_{\rm in} \geq 3 \times 10^{-4}\,\mu$W. For instance, $\xi \sim 15.9$ or 29.5 for $P_{\rm in} = 1\,\mu$W or $7\,\mu$W. Figure S2 plots $\xi(\delta, P_{in})$, by comparing the $\mathcal{PT}$-symmetric system working below the threshold $7\,\mu$W and the passive COM system working with $7\,\mu$W. For $\delta \to 1$, $P_{\rm in} > 0.1\,\mu$W, the enhancement effect is significant even in this situation.

II. Supermode splitting and linewidth

The non-Hermitian Hamiltonian of the system (comprising the optical gain, the optical supermodes, and the phonon mode) can be written at the simplest level as

$$\begin{aligned} H_{\rm tot} &= (\omega_+ - i\gamma_+)\, a_+^\dagger a_+ + (\omega_- - i\gamma_-)\, a_-^\dagger a_- \\ &\quad + \omega_m b^\dagger b + \frac{g x_0}{2}\left(b a_+^\dagger a_- + a_-^\dagger a_+ b^\dagger\right). \end{aligned} \tag{S1}$$

The weak driving terms are not explicitly shown here. The specific expressions of $\omega_\pm$ and $\gamma_\pm$ are different in two distinct regimes (see the main text): (i) the

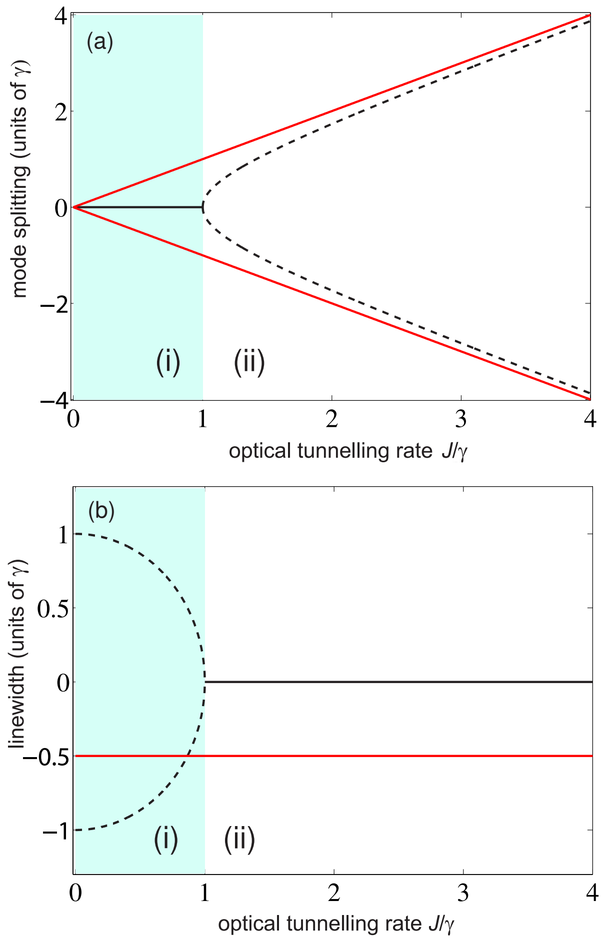

FIG. S3: (Color online) Mode-splitting (a) and linewidth (b) of the supermodes in the $\mathcal{PT}$-symmetric COM system (black curves) or the passive COM system (red lines), as a function of J/γ (see also Ref. [2]). The PT-symmetry holds in (ii) and not in (i).

regime which was identified as the broken-$\mathcal{PT}$-symmetry phase for a purely optical structure [1], characterized by $(\kappa+\gamma)/2 > J$; (ii) the regime with strong inter-cavity coupling $(\kappa+\gamma)/2 \leq J$, which for a purely optical system, was identified experimentally as the unbroken-$\mathcal{PT}$-symmetry phase [1]. We note that only the supermodes with unbroken $\mathcal{PT}$-symmetry can be distributed evenly across the coupled resonators, hence enabling the compensation of loss with gain. In contrast to this, the $\mathcal{PT}$-broken supermodes become spontaneously localized in either the amplifying or lossy resonator, hence experiencing either net gain or loss (see Ref. [2] for more details).

As Fig. S3 shows, the phonon lasing action can exist only in the $\mathcal{PT}$-symmetric regime where the optical supermodes are non-degenerate and thus can exchange energy through the phonon mode. Figure S3 is similar to that as was shown in all $\mathcal{PT}$-symmetric systems, e.g. in Ref. [1]; nevertheless, here we show for the first time that the ultralow-threshold phonon lasing can exist only in the unbroken-symmetry regime, not in the broken-symmetry regime.

We stress that, by increasing the optical coupling rate J, one can realize the transition from the broken to the unbroken $\mathcal{PT}$-symmetric regimes. That is, realizing not only the exchange of two subsystems (micro-resonators), but also changing the gain to loss and vice versa. By tuning the gain-loss ratio, one can realize the transition from linear to nonlinear regimes, i.e. the giant enhancement of the intracavity field intensity and then the mechanical gain. Both of these two conditions (strong J and $\delta = 1$) are required to observe the unidirectional wave propagation in a purely optical system [1] and now the ultralow-threshold phonon laser in a COM system. It is the presence of active gain which makes it possible to realize these two conditions simultaneously.

III. Stability analysis with tunable δ

Finally, we mention that in the vicinity of the gain-loss balance, the stability properties of the COM system can also be significantly modified. To see this we need to study the role of thermal noise on the mechanical response. This is accomplished by linearizing Eqs. (1-3) and then studying the fluctuations of the operators. With the equations of motion as Eqs. (1-3) in the main text, including also the optical detunings $\Delta_i = \omega_{c,i} - \omega_L$ $(i = 1, 2)$ between the two resonators and the input signal laser, the mean values of the optical and mechanical modes then satisfy the following equation

$$
\begin{aligned}
(\kappa - i\Delta_1)\, a_{1,s} - iJa_{2,s} - \sqrt{2\kappa}\Omega_d &= 0, &\text{(S2)}\\
(-\gamma - i\Delta_2)\, a_{2,s} - iga_{2,s}x - iJa_{1,s} &= 0, &\text{(S3)}\\
-m\omega_m^2 x_s - g\,|a_{2,s}|^2 &= 0, &\text{(S4)}
\end{aligned}
$$

with $\Omega_d = \sqrt{P_{\rm in}/\hbar\omega_c}$. From these equations we can obtain the following polynomial for the input power $P_{\rm in}$,

$$
P_{\rm in} = \frac{\omega_{c,1}}{2\kappa J^2}\left[\begin{array}{c} \frac{g^4}{m^2\omega_m^4}\left(\Delta_1^2+\kappa^2\right)N^3 + \frac{g^2}{m\omega_m^2}\left(2\gamma\Delta_1\kappa + 2J^2\Delta_1 - \Delta_1^2\Delta_2 - 2\gamma\kappa\Delta_1 - 2\kappa^2\Delta_2\right)N^2 \\ +\left(\gamma^2\kappa^2 + J^4 - 2\gamma\kappa J^2 + \Delta_1^2\Delta_2^2 + 2\gamma\kappa\Delta_1\Delta_2 - 2J^2\Delta_1\Delta_2 + \gamma^2\Delta_1^2 + \kappa^2\Delta_2^2 - 2\gamma\Delta_1\kappa\Delta_2\right)N \end{array}\right], \quad \text{(S5)}
$$

where N denotes the photon number inside the passive resonator. For a passive COM system, increasing the input power can lead to unstable evolutions of the system.

For our present active system, however, a bistability-free regime is accessible by tuning δ, which is reminiscent of that in a $\mathcal{PT}$-symmetric electronics system [3].

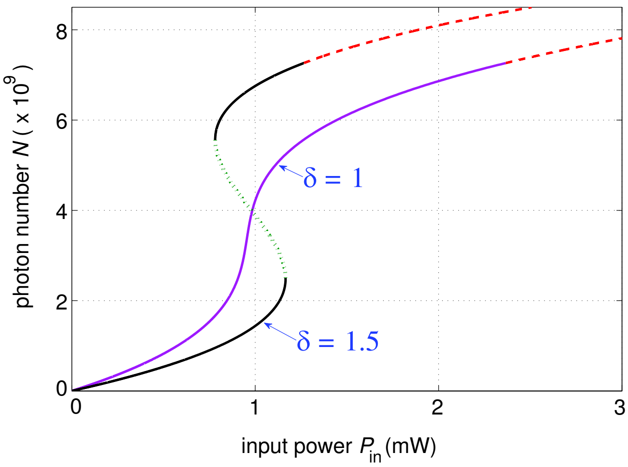

FIG. S4: (Color online) Mean population of photons in the passive resonator containing the mechanical mode. The bistability feature seen for $\delta = 1.5$, which is confirmed to be similar to the passive COM situation, can be removed at the gain-loss balance. Dotted and dashed curves denote different types of instability (see Ref. [5]). All relevant parameter values are given in the main text, including here also the optical detunings $\Delta_1/\omega_m = 0.03$ and $\Delta_2/\omega_m = 0.15$.

Let us now linearize Eqs. (1-3), i.e. expanding every operator as its steady-state value plus a small fluctuation around this value

$$a_1 = a_{1,s} + \delta a_1, \quad a_2 = a_{2,s} + \delta a_2, \quad x = x_s + \delta x,$$

it is straightforward to obtain the linear equations

$$\begin{aligned}\frac{d\delta a_1}{dt} &= \kappa\delta a_1 - iJ\delta a_2 + \sqrt{2\kappa}\delta a^{\text{in}},\\ \frac{d\delta a_2}{dt} &= -\gamma\delta a_2 - iJ\delta a_1 - iga_{2,s}\delta x - igx_s\delta a_2, \qquad \text{(S6)}\\ \frac{d^2\delta x}{dt^2} + \Gamma_m\frac{d\delta x}{dt} + \omega_m^2\delta x &= \frac{g}{m}\left(a_{2,s}^*\delta a_2 + a_{2,s}\delta a_2^*\right) + \frac{\delta\varepsilon^{\text{in}}}{m},\end{aligned}$$

for these fluctuations. The resulting solutions can be compactly written as

$$\delta x[\omega] = \chi[\omega]\,\delta\varepsilon^{\text{in}}, \qquad \text{(S7)}$$

where

$$\begin{aligned}\chi^{-1}[\omega] &= m\left(\omega_m^2 - \omega^2 - i\omega\Gamma_m + \frac{2g^2}{m}|a_{2,s}|^2\mathbf{Re}\mathbb{Y}[\omega]\right),\\ \mathbb{Y}^{-1}[\omega] &= -\omega - i\gamma + gx_s + \frac{J^2}{\omega - i\kappa}, \qquad \text{(S8)}\end{aligned}$$

for a thermally-driven system. Hence, by tuning the gain-loss ratio, COM properties (such as the mechanical susceptibility [4] and the bistability features) can be significantly modified. As a specific example, Fig. S4 shows the stable and unstable parameter regimes, by applying the Routh-Hurwitz criterion [5].

By applying this criterion to the coefficient matrix of these linear equations, we obtain in the following two stability conditions

$$S_1 = \left(\omega_m^2 + \frac{\Gamma_m^2}{4}\right)\left(\triangle_2^2 + \frac{\gamma^2}{4}\right) - 4\omega_m G^2\triangle_2 > 0, \qquad \text{(S9)}$$

$$S_2 = \gamma\Gamma_m\left[\left(\triangle_2^2 - \omega_m^2\right)^2 + \frac{1}{2}\left(\triangle_2^2 + \omega_m^2\right)(\gamma+\Gamma_m)^2 + \frac{1}{16}(\gamma+\Gamma_m)^4\right] + 4G^2\triangle_2\omega_m(\gamma+\Gamma_m)^2 > 0, \qquad \text{(S10)}$$

where $G = gx_0\,|a_{2,s}|$. For $\triangle_2 \geq 0$, the second inequality is always satisfied. With the help of these conditions, the bistability lines can then be plotted by numerically evaluating the polynomial in Eq. (S5). The resulting figure is shown in Fig. S4, with the corresponding stable regimes of parameters. Here, to compare with passive COM systems, we include also optical detunings, ignored previously in order to focus on the role of optical gain. We see that, in general, for stronger input power, bistability appears for higher gain-loss imbalance, as in passive COM systems. In contrast, this effect can now be completely removed at the gain-loss balance (see also Ref. [3]).

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