Non-Markovian quantum input-output networks

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Quantum input-output response analysis is a useful method for modeling the dynamics of complex quantum networks, such as those for communication or quantum control via cascade connections. Non-Markovian effects are expected to be important in networks realized using mesoscopic circuits, but such effects have not yet been studied. Here we extend the Markovian input-output network formalism to non-Markovian networks. The general formalism can be applied to various examples: (i) we show how non-Markovian coherent feedback can reduce the speed of decoherence for an atom in an optical cavity; (ii) we examine the effect of finite-cavity bandwidths in a superconducting circuit.

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I. INTRODUCTION

There has been tremendous progress in the last few years in experimental efforts to realize quantum networks [1] in various mesoscopic systems. These systems include photonic crystals [2], ion traps [3], and superconducting circuits [4], which also advance related fields such as quantum simulation (for recent reviews, see, e.g., Ref. [5]). The input-output formalism of Gardiner and Collet [6,7] is a useful tool for analyzing such quantum networks. In fact, using the input-output response to analyze or even modify the dynamics is a standard method in engineering, called system synthesis. Up until now, system synthesis for quantum networks has only been studied for Markovian systems. Quantum input-output theory itself has also mainly been limited to the Markovian regime, although it was developed for quantum systems about twenty years ago [6]. It was extended to non-Markovian systems only quite recently [8].

In the existing literature [6,7], quantum input-output theory is mainly applied to optical systems, in which the coupling between the system and its environment is weak and the correlation time (the "memory" of the environment) is small compared with the characteristic time scale of the system dynamics. Under the Markovian assumption, the quantum input-output formalism [6] was extended to cascaded systems [7], and has been used to study quantum coherent feedforward and feedback networks [9–14]. Markovian quantum input-output networks can be described using two alternative formulations: the Hudson-Parthasarathy formalism in the Schrödinger picture [15]; and the quantum transfer function formalism in the Heisenberg picture [16,17]. The general algebraic structure of such systems has been well studied in the language of quantum Wiener and Poisson processes and quantum Ito rules [15].

Although the Markovian assumption is reasonable when considering optical network components, environments in mesoscopic solid-state systems can have correlations on much longer time scales [18–21]. Examples of this are the nuclear spin bath that couples to electron spins in quantum dots [22], and the $1/f$ noise that affects Josephson-junction qubits [23]. It has also been suggested that the damping and decoherence of nanomechanical resonators are due to coupling to a small number of two-level systems [24], which can be expected to induce significant non-Markovian dynamics. In addition, any classical noise with a sufficiently narrow band generates non-Markovian evolution. There has been increasing interest in recent years in non-Markovian open quantum systems, and a number of analytical approaches have been devised to describe them. These include the projection-operator partitioning technique [25], the non-Markovian quantum trajectory approach [26], and very recently a non-Markovian input-output formalism [8].

In this paper we extend the non-Markovian input-output theory to cascaded quantum networks, providing a recipe for obtaining non-Markovian input-output equations for the description of any such network. Naturally, this formalism reduces to the standard input-output network formalism in the Markovian limit. In Ref. [8] the non-Markovian input-output relation was derived, but a quantum measurement was imposed on the output field so that the dynamics of the system is described by a stochastic Schrödinger equation. Such a formalism cannot be used to describe coherent cascade connections between systems because the quantum coherence in the output field is deteriorated by the measurement. In our formalism, without introducing measurements, the system dynamics is described by a non-Markovian quantum stochastic differential equation and a perturbative master equation, which can be naturally extended to a non-Markovian network.

Our formalism is obtained by allowing the coupling to the bath to have an arbitrary frequency dependence. This can be used to describe noise with any frequency profile, and should provide a good model for a wide range of non-Markovian

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environments. A nice feature of the resulting description is that it is exact for both weak and strong coupling. However, in order to perform calculations for nonlinear systems, one must transform the Heisenberg equations of the input-output formalism to a non-Markovian master equation, and this requires further approximations. Here we do this at the simplest level of approximation, by deriving the corresponding master equation to second order in perturbation theory, using the standard Born approximation [25]. Nevertheless, more sophisticated techniques exist for obtaining non-Markovian master equations, and it would be an interesting avenue for future work to examine how these can be used to obtain master equations for non-Markovian cascaded networks. We note that for linear systems the Heisenberg equations of the input-output formalism can be used to obtain exact results. This is especially useful in some cases [27], when the second-order perturbative master equation fails to behave correctly [28]. We expect the network formalism we develop to be useful in describing a range of mesoscopic systems, such as coupled-cavity arrays in photonic crystals [29,30], and nonlinear resonator and qubit networks in solid-state circuits [31–34]. The formalism can also be applied to quantum feedback control networks [35–41] in solid-state systems [42–49].

This paper is organized as follows. In Sec. II, we briefly review the Markovian input-output formalism so that this can be easily compared to the non-Markovian case. In Sec. III, we use an alternative method to derive the non-Markovian input-output relations in Ref. [8], and obtain the dynamical equation for the system such that it can be easily used for networks. We derive these here as a natural extension of the original Collett-Gardiner quantum input-output theory. In Sec. IV, we derive the input-output relations for more complex non-Markovian quantum cascade networks. We then apply this formalism to a non-Markovian damped oscillator, a non-Markovian coherent feedback network in which we show how the non-Markovian feedback suppresses the decoherence effects, and finally two superconducting charge qubits interacting via cavities with Lorentz noise profiles. Section V concludes with a brief summary.

II. BRIEF REVIEW OF INPUT-OUTPUT THEORY FOR MARKOVIAN SYSTEMS

Here we summarize the standard Gardiner-Collet input-output formalism [6,50]. The basic model is a quantum system interacting with a bath, where the bath consists of the modes of an electromagnetic field, or equivalently a continuum of harmonic oscillators. The Hamiltonian for the system and bath is

$$H = H_S + H_B + H_{\text{int}}, \tag{1}$$

with

$$\begin{aligned} H_B &= \int_{-\infty}^{+\infty} \omega b^\dagger(\omega) b(\omega) d\omega, \\ H_{\text{int}} &= i \int_{-\infty}^{+\infty} [\kappa(\omega) b^\dagger(\omega) L - \text{H.c.}] d\omega, \end{aligned} \tag{2}$$

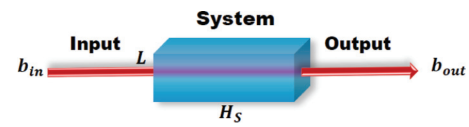

FIG. 1. (Color online) Schematic diagram of the Markovian input-output system.

where $b^\dagger(\omega)$ and $b(\omega)$ are the creation and annihilation operators of the bath mode with frequency ω, which satisfy

$$[b(\omega), b^\dagger(\tilde{\omega})] = \delta(\omega - \tilde{\omega}). \tag{3}$$

In the above, H_S is the free Hamiltonian of the system. The bath mode with frequency ω interacts with the system via the system operator L and the coupling strength $\kappa(\omega)$. Hereafter we set $\hbar = 1$. The total Hamiltonian H can be reexpressed in the interaction picture as

$$\begin{aligned} H_{\text{eff}} &= \exp(i H_B t)(H_S + H_{\text{int}}) \exp(-i H_B t) \\ &= H_S + i \int_{-\infty}^{+\infty} [\kappa(\omega) e^{i\omega t} b^\dagger(\omega) L - \text{H.c.}] d\omega. \end{aligned} \tag{4}$$

If the coupling strength is constant for all frequencies, so that

$$\kappa(\omega) = \sqrt{\frac{\gamma}{2\pi}}, \tag{5}$$

then the dynamics of the system will become Markovian. This is the Markovian approximation. The Hamiltonian H_{eff} is now given by

$$H_{\text{eff}} = H_S + i\sqrt{\gamma}[b_{\text{in}}^\dagger(t) L - L^\dagger b_{\text{in}}(t)], \tag{6}$$

where

$$b_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} b(\omega) d\omega \tag{7}$$

is the Fourier transform of the bath modes, and is the time-varying input field that is fed into the system [50,51] (see Fig. 1). In the Heisenberg picture, the system operator $X(t)$ satisfies the following quantum stochastic differential equation (QSDE)

$$\begin{aligned} \dot{X} = &-i[X, H_S] + \frac{\gamma}{2}\{L^\dagger [X, L] + [L^\dagger, X] L\} \\ &+ \sqrt{\gamma}\{b_{\text{in}}[L^\dagger, X] + [X, L] b_{\text{in}}^\dagger\}. \end{aligned} \tag{8}$$

If the input field $b_{\text{in}}(t)$ is in a vacuum state, and we trace it out, we can re-express the system dynamics in the Schrödinger picture as the following master equation

$$\dot{\rho} = -i[H_S, \rho] + \gamma\left(L\rho L^\dagger - \tfrac{1}{2} L^\dagger L \rho - \tfrac{1}{2} \rho L^\dagger L\right). \tag{9}$$

Finally, if $b_{\text{out}}(t)$ is the field just after it has interacted with the system and is about to propagate away, then one has the relation

$$b_{\text{out}}(t) = b_{\text{in}}(t) + \sqrt{\gamma} L(t). \tag{10}$$

This is the Markovian input-output relation.

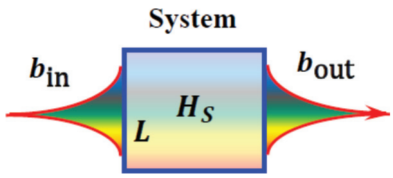

FIG. 2. (Color online) Schematic diagram of the non-Markovian input-output system. The input field is dispersed when it interacts with the system, and the modes with different frequencies in the input field are coupled to the system with different coupling strengths.

III. INPUT-OUTPUT THEORY OF NON-MARKOVIAN SYSTEMS

A. The exact quantum Langevin equations

To derive the input-output relation for a general non-Markovian quantum system (see Fig. 2), we rewrite the Hamiltonian $H_{\rm eff}$ in Eq. (4) as

$$\begin{aligned} H_{\rm eff} &= H_S + i\left\{\left[\int_{-\infty}^{+\infty}\kappa(\tau-t)b_{\rm in}^{\dagger}(\tau)d\tau\right]L - \text{H.c.}\right\} \\ &= H_S + i[\tilde{b}_{\rm in}^{\dagger}(t)L - L^{\dagger}\tilde{b}_{\rm in}(t)], \end{aligned} \tag{11}$$

where

$$\kappa(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-i\omega t)\kappa(\omega)d\omega \tag{12}$$

is the Fourier transform of the coupling strength $\kappa(\omega)$. The input field that interacts directly with the system is now

$$\tilde{b}_{\rm in}(t) = \int_{-\infty}^{+\infty}\kappa(t-\tau)b_{\rm in}(\tau)d\tau \tag{13}$$

and satisfies the new commutation relation

$$[\tilde{b}_{\rm in}(t),\tilde{b}_{\rm in}^{\dagger}(\tilde{t})] = \gamma(t-\tilde{t}), \tag{14}$$

where

$$\gamma(t-\tilde{t}) = \int_{-\infty}^{+\infty}\kappa(t-\tau)\kappa^*(\tilde{t}-\tau)d\tau. \tag{15}$$

We can now proceed to derive the Heisenberg stochastic differential equations for the evolution of the system,

$$\begin{aligned} \dot{X} &= -i[X,H_S] + \{\tilde{b}_{\rm in}(t)[L^{\dagger},X] + [X,L]\tilde{b}_{\rm in}^{\dagger}(t)\} \\ &\quad + \int_0^t\{\gamma^*(t-\tau)L^{\dagger}(\tau)[X(t),L(t)] \\ &\quad + \gamma(t-\tau)[L^{\dagger}(t),X(t)]L(\tau)\}d\tau. \end{aligned} \tag{16}$$

The non-Markovian input-output relation becomes

$$b_{\rm out}(t) = b_{\rm in}(t) + \int_0^t\kappa(t-\tau)L(\tau)d\tau. \tag{17}$$

This relation coincides with Diosi's non-Markovian input-output equation (Eq. (10) in Ref. [8]). We give the details of the derivations of Eq. (16) and output equation (17) in Appendix.

Remark 1. In the Markovian limit, where $\kappa(\omega) = \sqrt{\gamma/2\pi}$, we have $\kappa(t) = \sqrt{\gamma}\delta(t)$, and $\gamma(t-\tilde{t}) = \gamma\delta(t-\tilde{t})$. It can be easily verified that Eq. (16) reduces to the quantum stochastic differential equation (8) and the output equation (17) reduces to Eq. (10).

Note that so far no additional assumptions have been made following the Hamiltonian in Eq. (11). Thus our non-Markovian network formalism is exact for any coupling strength. However, as noted in the introduction, for nonlinear networks we must often resort to the Schrödinger picture to perform calculations.

B. Perturbative master equation

To obtain the second-order perturbative master equation, one averages over the vacuum input field $b_{\rm in}$, which we will take to be in the vacuum state, and uses the Born approximation. The perturbative master equation that corresponds to Eq. (16) is

$$\begin{aligned} \dot{\rho} &= -i H_S,\rho] + \int_0^t\{\gamma^*(t-\tau)[L\rho(\tau),L_{\rm H_S}^{\dagger}(\tau-t)] \\ &\quad + \gamma(t-\tau)[L_{\rm H_S}(\tau-t),\rho(\tau)L^{\dagger}]\}d\tau, \end{aligned} \tag{18}$$

where

$$L_{\rm H_S}(t) = \exp(iH_St)L\exp(-iH_St). \tag{19}$$

The details of the derivation of (18) can also be found in Appendix.

The non-Markovian master equation (18) is in the standard form of the exact Nakajima-Zwanzig (NZ) equation $\dot{\rho}(t) = \int_0^t d\tau\, O(t,\tau)\rho(\tau)$ [52], where the NZ kernel $O(t,\tau)$ is of the time-translationally-invariant form $O(t-\tau)$. Although the properties of the NZ equation have been studied [25,53], it is not yet known what criteria a kernel must satisfy to guarantee complete positivity of the evolution. However, complete positivity has been proved for some simple cases, including noise with a Lorentzian spectrum. We will consider this kind of noise in the examples we analyze below.

The Born approximation used to derive the perturbative master equation is traditionally introduced when deriving the non-Markovian input-output model from a Markovian input-output system (see Fig. 3). In the Markovian input-output model we consider, the Markovian input and output fields interact with a medium bath, which then interacts with the plant we are interested in. The bath is sometimes a complex and high-dimensional system, which is very hard to describe, and we can only obtain the spectrum of the bath. When we

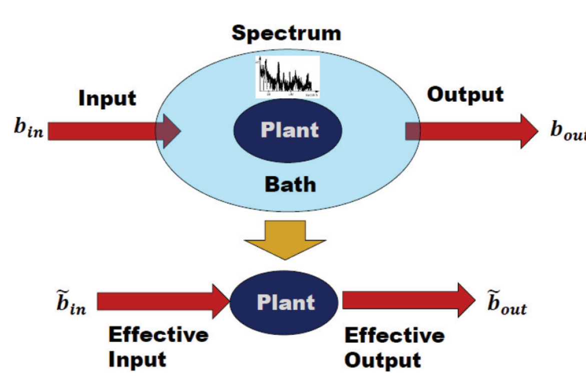

FIG. 3. (Color online) Effective non-Markovian input-output model obtained from a larger Markovian input-output system in which the Markovian input and output fields interact with an intermediate bath, which is then coupled to the plant we consider.

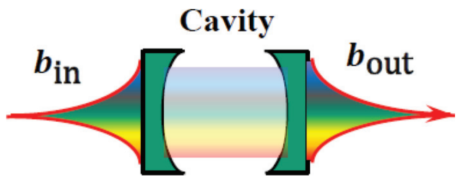

FIG. 4. (Color online) Schematic diagram of the linear non-Markovian cavity.

introduce the Born approximation to omit the backactions on the bath from the plant, the Markovian input-output system can be described by an effective non-Markovian input-output model by averaging out the degrees of freedom of the bath, and the non-Markovian interaction between the effective input field and the plant is determined by the spectrum of the bath. Such a picture can be used, e.g., to describe a model in a complex circuit, in which the input and output fields of the whole circuit interact indirectly with the central element we consider. In general, we do not know the dynamics of the whole circuit, but we know the spectra of the intermediate elements. We can then obtain an effective non-Markovian input-output model for such a complex circuit by using this method.

C. Example: Single-mode cavity

For a single-mode cavity coupled to an external input field (see Fig. 4), the system Hamiltonian H_S and dissipation operator L are given by $H_S = \omega_0 a^\dagger a$ and $L = a$, where ω_0, a ($a^\dagger$) are respectively the frequency and the annihilation (creation) operator of the cavity mode. The mode has an arbitrary non-Markovian coupling to the field modes outside the cavity. The quantum stochastic differential equation for a cavity operator X is then

$$\begin{aligned}\dot{X} &= -i[X,\omega_0 a^\dagger a] + \{\tilde{b}_{\rm in}[a^\dagger,X] + [X,a]\tilde{b}_{\rm in}^\dagger\}\\ &\quad + \int_0^t \{\gamma^*(t-\tau)a^\dagger(\tau)[X(t),a(t)]\\ &\quad + \gamma(t-\tau)[a^\dagger(t),X(t)]a(\tau)\}d\tau. \qquad (20)\end{aligned}$$

From Eq. (20), we can obtain the equation of motion for the cavity mode, which is

$$\dot{a} = -i\omega_0 a - \int_0^t \gamma(t-\tau)a(\tau)d\tau - \tilde{b}_{\rm in}, \qquad (21)$$

and we can solve this linear differential equation using the Laplace transform, defined as

$$O(s) = \int_0^\infty \exp(-st)O(t)dt. \qquad (22)$$

The solution in terms of the Laplace variable s is

$$a(s) = -\frac{\kappa(s)}{s+\gamma(s)+i\omega_0} b_{\rm in}(s), \qquad (23)$$

where $\kappa(s)$, $\gamma(s)$, $a(s)$, $b_{\rm in}(s)$ are the Laplace transforms of $\kappa(t)$, $\gamma(t)$, $a(t)$, $b_{\rm in}(t)$. The resulting input-output relation is

$$b_{\rm out}(s) = \frac{s+\gamma(s)-\kappa^2(s)+i\omega_0}{s+\gamma(s)+i\omega_0}\, b_{\rm in}(s). \qquad (24)$$

Note that this result is exact as far as the frequency dependence of the coupling to the bath is concerned. Because the system is linear we can obtain results without deriving a master equation. This input-output formula shows exactly how the coupling profile applies a low-pass filter to the input field to produce the output field.

Remark 2. If we consider a Markovian cavity with damping rate γ, then we have $\kappa(s) = \sqrt{\gamma}$, $\gamma(s) = \gamma/2$, from which we can obtain the traditional input-output relation for a lossy cavity from Eq. (24) (see Eq. (45) in Ref. [17])

$$b_{\rm out}(s) = \frac{s-\gamma/2+i\omega_0}{s+\gamma/2+i\omega_0}\, b_{\rm in}(s). \qquad (25)$$

D. Example: non-Markovian coherent feedback network

Feedback control for classical systems involves gathering information about the system, and using this to apply controls in real time. For quantum systems this process can be implemented by making explicit measurements, and processing the resulting classical information using a classical system [54–58]. Alternatively a fully quantum version of this process can be realized by coupling the system to a second quantum system without making explicit measurements [9–14,35,62]. It is this second method that we consider here, referred to as coherent feedback control.

Coherent feedback is a superset of measurement-based feedback [59], and it has been shown that coherent feedback can achieve better performance than measurement-based feedback in a number of settings, including noise reduction and cooling in linear systems [60,61], as well as noise reduction under bounded controls [59]. It has also been applied to generating nonlinear effects in mesoscopic systems [13,62]. Previous studies of coherent feedback networks have been restricted to the Markovian setting. Here we show that non-Markovian effects can significantly change the properties of a feedback network.

The coherent feedback control system we consider here is shown in Fig. 5(b). We consider the control of a two-level atom inside an optical cavity by coupling this cavity to a second optical cavity in a feedback loop. We will refer to the cavity containing the atom as the "plant" cavity, and the second cavity as the controller. This configuration can also be realized in a mesoscopic circuit, in which the atom is replaced by a superconducting qubit [4]. Apart from the addition of the atom the configuration of the cavities is the same as that analyzed in Ref. [11]. Each cavity is a four-mirror ring cavity in a folded configuration. We will use our formalism to determine the effect of the finite bandwidth of the cavities on the control loop.

Let ω_q, ω_p, and ω_c be the transition frequency of the atom, the angular frequency of the plant resonator, and the angular frequency of the controller resonator. The coupling constant between the atom and the plant resonator is represented by g, while γ_p and γ_c are the total decay rates of the plant resonator and the controller resonator. The part of these decay rates that are due to transmission through each input/output coupler mirror will be denoted by k. The system parameters are chosen to satisfy the resonance condition $\omega_q = \omega_p = \omega_c$. We further introduce the weak-coupling assumption

$$\gamma_p \gg g^2, \qquad (26)$$

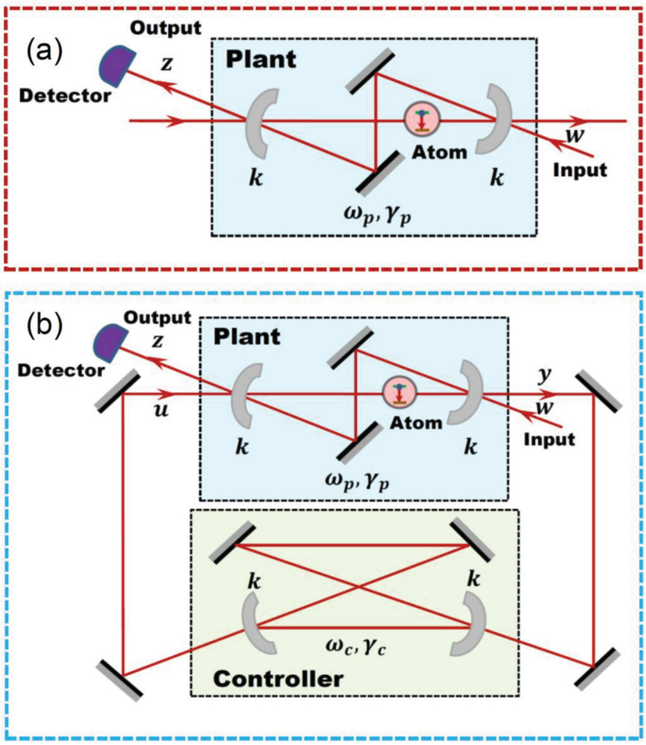

FIG. 5. (Color online) Schematic diagrams of (a) the bare system and (b) the system with coherent feedback modified from the experimental setup in Ref. [11]. The controlled quantum system (plant) and the quantum controller are two four-mirror folded ring resonators with angular frequencies ω_p, ω_c and total decay rates γ_p and γ_c. Also, k is the partial rate associated with transmission through the coupler mirror. A two-level atom with transition frequency ω_q is placed in the plant resonator and the coupling constant between the cavity mode and the atom is g. The control object is to steer the dynamics of the bath of the atom to protect the atom state.

under which we can omit the back-action effects imposed by the atom on the cavity mode [see the derivation presented in Eqs. (52) through (53)]. If the external input field w is in a vacuum state, the cavity mode in the plant resonator can be seen as a vacuum bath interacting with the atom. In this case, the assumption (26) leads to the Born approximation, under which we can obtain the non-Markovian master equation for the degrees of freedom of the atom by Eq. (18) in the interaction picture

$$\dot{\rho} = \int_0^t \{\gamma(t-\tau)[\sigma_-\rho(\tau),\sigma_+] + \text{H.c.}\}d\tau. \tag{27}$$

We can further obtain the following equation for the nondiagonal entries of the density matrix ρ of the atom

$$\langle\dot{\sigma}_\pm\rangle_q = -\int_0^t \gamma(t-\tau)\langle\sigma_\pm\rangle_q(\tau)d\tau, \tag{28}$$

where $\sigma_\pm$ are the raising and lowering operators for the atom and $\langle\sigma_\pm\rangle_q = \text{tr}(\sigma_\pm\rho)$. The function $\gamma(\tau)$ in Eq. (27) is the correlation function of the cavity mode in the plant resonator, which is defined by

$$\gamma(\tau) = g^2\langle a(t)a^\dagger(t+\tau)\rangle_{\text{vac}}, \tag{29}$$

where $a(t)$ ($a^\dagger(t)$) is the annihilation (creation) operator of the cavity mode in the Heisenberg picture and $\langle\cdot\rangle_{\text{vac}}$ is the average taken on the vacuum state. It can be seen from Eq. (28) that the decay rate of the atomic state is determined by $\gamma(\tau)$. The correlation function $\gamma(\tau)$ can be calculated in the frequency domain by

$$\gamma(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\omega\tau)\langle S_a(\omega)\rangle d\omega, \tag{30}$$

where $S_a(\omega) = g^2\langle a(\omega)a^\dagger(\omega)\rangle_{\text{vac}}$ is the power spectrum of the cavity mode in the plant cavity and $a(\omega)$ is the Fourier transform of $a(t)$.

To study the effects induced by the feedback loop, let us compare the dynamics of the atom under open-loop control (without feedback) and feedback control (see Fig. 5). For the open-loop case, we can obtain the equation of the cavity mode

$$\dot{a} = -\gamma_p a - \sqrt{2\kappa}w, \tag{31}$$

which can be solved in the frequency domain as

$$a(\omega) = -\frac{\sqrt{2\kappa}}{-i\omega+\gamma_p}w(\omega), \tag{32}$$

where $w(\omega)$ is the Fourier transform of the external input signal $w(t)$. It can be seen from Eq. (32) that the vacuum input field $w(t)$ is filtered by the cavity to generate a Lorentz-type bath in the plant resonator, which leads to a Lorentz spectrum

$$S_a(\omega) = \frac{2kg^2}{\omega^2+\gamma_p^2},$$

and the correlation function can be calculated by

$$\gamma_{\text{op}}(\tau) = \frac{g^2k}{\gamma_p}\exp(-\gamma_p\tau).$$

From the connections depicted in Fig. 6 we obtain the following relationships between the various quantum signals in frequency space:

$$\begin{aligned} u(\omega) &= K_{uy}(\omega)y(\omega) \\ &= K_{uy}(\omega)[G_{yu}(\omega)u(\omega) + G_{yw}(\omega)w(\omega)], \end{aligned} \tag{33}$$

$$a(\omega) = G_{\text{op}}(\omega)[u(\omega)+w(\omega)]. \tag{34}$$

Here $G_{\text{op}}(\omega) = -\sqrt{2k}/(\gamma_p+i\omega)$ is the open-loop transfer function from $w(t)$ to $a(t)$ obtained from Eq. (31); $G_{yu}(\omega) = -2k/(\gamma_p+i\omega)$ and $G_{yw}(\omega) = 1-2k/(\gamma_p+i\omega)$ are respectively the transfer functions from signals u to y and w to

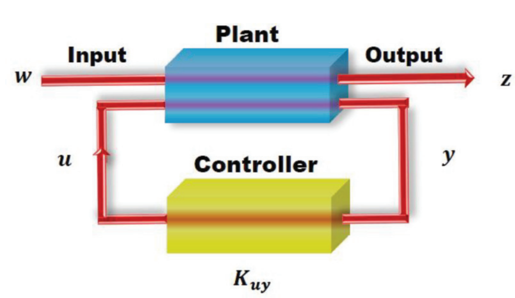

FIG. 6. (Color online) Diagram of the input-output model of the coherent feedback system in Fig. 5(b).

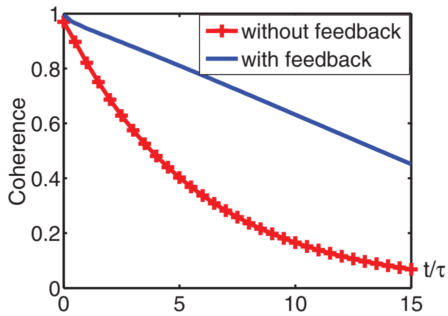

FIG. 7. (Color online) Dynamics of the coherence function $C(t) = \sqrt{\langle\sigma_x\rangle_q^2 + \langle\sigma_y\rangle_q^2}$ of the two-level atom. The red curve with plus sign and the blue solid curve represent the cases (a) without feedback and (b) with coherent feedback. The decoherence of the atom slows down obviously under coherent feedback control. Here $\tau = 1\mu$s.

y; and $K_{uy}(\omega) = 2k/(\gamma_c + i\omega)$ is the transfer function of the controller (the expressions for these transfer functions can be found in, e.g., Ref. [11]).

By substituting Eq. (33) into (34) to eliminate $u(\omega)$, we obtain

$$a(\omega) = G_{\rm op}(\omega)w(\omega)\left\{1 + \frac{K_{uy}(\omega)G_{yw}(\omega)}{1 - K_{uy}(\omega)G_{yu}(\omega)}\right\}, \quad (35)$$

and by substituting this equation into Eq. (30) we obtain the correlation function $\gamma_{fb}(\tau)$ under feedback control.

In Fig. 7 we compare the decay of $\langle\sigma_-\rangle_q$ with and without feedback. Specifically, we plot the coherence function $f(t) = \sqrt{\langle\sigma_x\rangle_q^2 + \langle\sigma_y\rangle_q^2}$ in both cases, with the physical parameters

$$\begin{aligned} \gamma_p/2\pi &= 10 \text{ MHz}, \quad k/2\pi = 3 \text{ MHz}, \\ g^2/2\pi &= 1 \text{ MHz}, \quad \gamma_c/2\pi = 8 \text{ MHz}. \end{aligned} \quad (36)$$

We see that the decoherence of the atom is greatly suppressed by the coherent feedback loop. This is intuitively reasonable because the non-Markovian coherent feedback loop compensates the photon decay before the photon is lost and thus efficiently preserves the coherence dynamics of the cavity mode in the plant resonator.

IV. NON-MARKOVIAN QUANTUM NETWORKS

A. General theory: quantum cascade systems

To derive the input-output relation for complex non-Markovian networks, we should first study the dynamics of a system composed of two cascade-connected subsystems, also known as the "series product" of two subsystems [10]. The Hamiltonian of two cascaded subsystems, depicted in Fig. 8

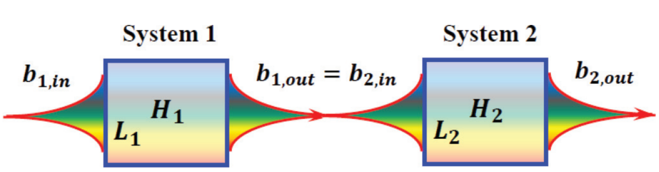

FIG. 8. (Color online) Schematic diagram of the non-Markovian quantum cascade system. The output from the first subsystem is fed into the input of the second subsystem.

can be expressed as

$$\begin{aligned} H_{\rm eff} &= H_1 + i\left\{\left[\int_{-\infty}^{+\infty}\kappa_1(\tau - t)b_{1,\rm in}^\dagger(\tau)d\tau\right]L_1 - \text{H.c.}\right\} \\ &\quad + H_2 + i\left\{\left[\int_{-\infty}^{+\infty}\kappa_2(\tau - t)b_{2,\rm in}^\dagger(\tau)d\tau\right]L_2 - \text{H.c.}\right\}, \end{aligned} \quad (37)$$

where $H_{i=1,2}$ and $L_{i=1,2}$ are the free Hamiltonian and dissipation operator of the ith subsystem; and $\kappa_i(t)$ is the corresponding coupling strength between the ith subsystem and the ith input field. If we omit the time delay for the quantum field transmitting between the two input-output components, then we have

$$b_{2,\rm in}(t) = b_{1,\rm out}(t) = b_{1,\rm in}(t) + \int_0^t \kappa_1(t - \tau)L_1(\tau)d\tau. \quad (38)$$

Substituting Eq. (38) into Eq. (37), we have

$$H_{\rm eff} = H_1 + H_2 + H_{12} + i\sum_{j=1,2}[\tilde{b}_{j,\rm in}^\dagger L_j - L_j^\dagger \tilde{b}_{j,\rm in}]$$

where

$$H_{12} = -i\int_0^t \left[\gamma_{12}^\theta(\tau - t)L_2L_1^\dagger(\tau) - \text{H.c.}\right]d\tau \quad (39)$$

is the interaction Hamiltonian between the two subsystems introduced by the transmitting field; the parameter $\gamma_{12}^\theta(\tau - t)$ is defined by

$$\gamma_{12}^\theta(t - \tilde{t}) = \int_{-\infty}^{+\infty}\kappa_1^*(\tau - t)\kappa_2(\tau - \tilde{t})\theta(t - \tau)d\tau;$$

and $\theta(t)$ is the step function

$$\theta(t) = \begin{cases} 1, & t \geqslant 0; \\ 0, & t < 0. \end{cases} \quad (40)$$

The two equivalent non-Markovian input fields that interact directly with the two subsystems via the dissipation operators L_1 and L_2 are defined as

$$\tilde{b}_{l,\rm in}(t) = \int_{-\infty}^{+\infty}\kappa_l(t - \tau)b_{\rm in}(\tau)d\tau,$$

and these satisfy the following commutation relation

$$[\tilde{b}_{r,\rm in}(t), \tilde{b}_{l,\rm in}^\dagger] = \gamma_{lr}(t - \tilde{t}), \quad (41)$$

where

$$\gamma_{lr}(t - \tilde{t}) = \int_{-\infty}^{+\infty}\kappa_r(t - \tau)\kappa_l^*(\tilde{t} - \tau)d\tau. \quad (42)$$

The dynamics of the total system can then be expressed as the following quantum stochastic differential equation

$$\begin{aligned} \dot{X} &= -i[X, H_S] + \sum_{l=1}^n\{\tilde{b}_{l,\rm in}[L_l^\dagger, X] + [X, L_l]\tilde{b}_{l,\rm in}^\dagger\}, \\ &\quad + \sum_{l,r=1}^2\int_0^t\{\gamma_{lr}^*(t - \tau)L_l^\dagger(\tau)[X(t), L_r(t)] \\ &\quad + \gamma_{lr}(t - \tau)[L_r^\dagger(t), X(t)]L_l(\tau)\}d\tau, \end{aligned} \quad (43)$$

where $H_S = H_1 + H_2 + H_{12}$. The input-output equation of the cascade system can be expressed as

$$b_{\rm out}(t) = b_{\rm in}(t) + \int_0^t \kappa_1(t-\tau)L_1(\tau)d\tau + \int_0^t \kappa_2(t-\tau)L_2(\tau)d\tau. \quad (44)$$

Remark 3. In the Markovian limit, we have $\kappa_l(t) = \sqrt{\gamma_l}\delta(t)$, $\gamma_{lr}^{\theta}(t-\tilde{t}) = \gamma_{lr}(t-\tilde{t}) = \sqrt{\gamma_l\gamma_r}\delta(t-\tilde{t})$, and $\tilde{b}_{l,\rm in} = b_{\rm in}$. Thus, the dynamical equation (47) can be re-expressed as

$$\dot{X} = -i[X,H_S] + \tfrac{1}{2}\{L^\dagger[X,L] + [L^\dagger,X]L\}b_{\rm in}[L^\dagger,X] + [X,L]b_{\rm in}^\dagger, \quad (45)$$

where

$$H_S = H_1 + H_2 + \frac{i\sqrt{\gamma_1\gamma_2}}{2}(L_1^\dagger L_2 - L_1 L_2^\dagger)$$
$$L = \sqrt{\gamma_1}L_1 + \sqrt{\gamma_2}L_2,$$

and output equation (50) can be rewritten as

$$b_{\rm out}(t) = b_{\rm in}(t) + \sqrt{\gamma_1}L_1(t) + \sqrt{\gamma_2}L_2(t). \quad (46)$$

These equations coincide with those obtained for Markovian systems using the series product (see, e.g., Ref. [10]).

The dynamical equation (43) can be extended readily to n cascade-connected subsystems to obtain the following quantum stochastic differential equation

$$\dot{X} = -i[X,H_S] + \sum_{l=1}^n \{\tilde{b}_{l,\rm in}[L_l^\dagger,X] + [X,L_l]\tilde{b}_{l,\rm in}^\dagger\} + \sum_{l,r=1}^n \int_0^t \{\gamma_{lr}^*(t-\tau)L_l^\dagger(\tau)[X(t),L_r(t)] + \gamma_{lr}(t-\tau)[L_r^\dagger(t),X(t)]L_l(\tau)\}d\tau, \quad (47)$$

where $\gamma_{lr}(t-\tilde{t})$ is defined by Eq. (42);

$$H_S = \sum_{l=1}^n H_l + \sum_{l<r} H_{lr}; \quad (48)$$

$H_{l=1,\ldots,n}$ is the free Hamiltonian of the lth subsystem; and H_{lr} is the field-mediated interaction Hamiltonian

$$H_{lr} = i\int_0^t \left[\gamma_{lr}^{\theta}(\tau-t)L_r L_l^\dagger(\tau) - {\rm H.c.}\right]d\tau. \quad (49)$$

The function $\gamma_{lr}^{\theta}(t-\tilde{t})$ is defined by

$$\gamma_{lr}^{\theta}(t-\tilde{t}) = \int_{-\infty}^{+\infty} \kappa_l^*(\tau-t)\kappa_r(\tau-\tilde{t})\theta(t-\tau)d\tau,$$

where $\theta(t)$ is the step function defined by Eq. (40). The output equation can be written as

$$b_{\rm out}(t) = b_{\rm in}(t) + \sum_{l=1}^n \int_0^t \kappa_l(t-\tau)L_l(\tau)d\tau. \quad (50)$$

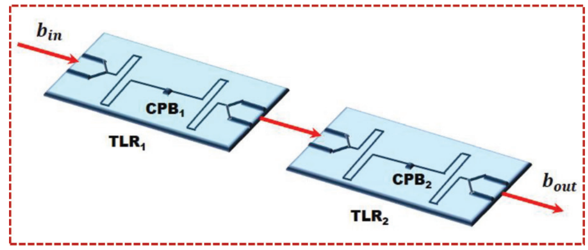

FIG. 9. (Color online) Schematic diagrams of two cascade-connected Cooper-pair box and transmission-line resonator (CPB-TLR) input-output systems.

Transforming this into the Schrödinger picture, we can obtain the following second-order master equation

$$\dot{\rho} = -i[H_S,\rho] + \sum_{l,r=1}^n \int_0^t \{\gamma_{lr}^*(t-\tau)\left[L_r\rho(\tau), L_{{\rm Hs},l}^\dagger(\tau-t)\right] + \gamma_{lr}(t-\tau)\left[L_{{\rm Hs},l}(\tau-t),\rho(\tau)L_r^\dagger\right]\}d\tau,$$

where

$$L_{{\rm Hs},l}(t) = \exp(iH_St)L_l\exp(-iH_St).$$

B. Example: non-Markovian qubit networks in superconducting circuits

As a relatively simple, but experimentally relevant example, we consider how to couple two distant qubits—single Cooper pair boxes (CPBs)—by a microwave field. As shown in Fig. 9 we embed the two CPBs into two superconducting transmission-line resonators (TLRs), and couple the TLRs via a transmission line. When we average over the degrees of freedom of the TLRs, the interactions between the CPBs and the input fields become non-Markovian. This can be understood by noting that the TLRs work as microwave cavities, and these act as low-pass filters. The white-noise input fields are filtered by the TLRs and changed into non-Markovian Lorentz noises, and these in turn interact with the CPBs. Thus, the qubit network considered here is a typical non-Markovian quantum network.

Near the charge-degenerate point, with $n_{gj} = 0.5$, the two lowest-energy levels of the jth CPB are close to each other and far separated from higher-energy levels. Because of this we can treat a single CPB as a two-level system. If we additionally place an ac voltage $V_{gj} = V_{0j}\cos(\omega_{gj}t)$ on the gate of the jth CPB, where V_{0j} and ω_{gj} are the amplitude and frequency of the voltage, the effective Hamiltonian for the jth CPB in the rotating frame becomes $H_{qj} = (\Delta_{qj}/2)\sigma_z^{(j)}$, where $\Delta_{qj} = E_j - \omega_{gj}$ and E_J is the Josephson energy of the CPB. The cavity modes in the TLRs are connected by a transmission line. The total Hamiltonian of the jth SCB-TLR system, and its input field $b_{\rm in,j}$ is

$$H_{\rm tot,j} = \frac{\Delta_{qj}}{2}\sigma_z^{(j)} + \omega_{cj}a_j^\dagger a_j + g_j(a_j^\dagger\sigma_-^{(j)} + a_j\sigma_+^{(j)}) + i\sqrt{\gamma_j}(a_j^\dagger b_{\rm in,j} - b_{\rm in,j}^\dagger a_j), \quad (51)$$

where γ_j is determined by the coupling between the cavity mode and the input field. We now eliminate the degrees of freedom of the cavity mode to obtain a description of the qubits alone. First, it can be shown from Eq. (51) that

$$\dot{a}_j = -\left(i\omega_{cj} + \frac{\gamma_j}{2}\right)a_j + \sqrt{\gamma_j}\, b_{\mathrm{in,j}} - i g_j \sigma_-^{(j)}. \tag{52}$$

We now introduce the following weak-coupling assumption

$$\omega_{cj}, \gamma_j \gg g_j^2,$$

so we can omit the last term in Eq. (52) when we consider the dynamics of the cavity mode, and the resulting solution is

$$a_j(t) = \sqrt{\gamma_j}\int_0^t \exp[-(i\omega_{cj} + \gamma_j/2)(t-\tau)]b_{\mathrm{j,in}}(\tau)d\tau. \tag{53}$$

Substituting Eq. (53) into the Hamiltonian $H_{\mathrm{tot,j}}$ in Eq. (51), we can obtain an effective Hamiltonian for the coupling between the jth qubit and the effective input field

$$\tilde{H}_{\mathrm{eff,j}} = \frac{\Delta_{qj}}{2}\sigma_z^{(j)} + i(\tilde{b}_{\mathrm{in,j}}^{\dagger}\sigma_-^{(j)} - \sigma_+^{(j)}\tilde{b}_{\mathrm{in,j}}), \tag{54}$$

where

$$\tilde{b}_{\mathrm{in,j}}(t) = \int_0^t i g_j\sqrt{\gamma_j}e^{-(i\omega_{cj}+\gamma_j/2)(t-\tau)}b_{\mathrm{in,j}}(\tau)d\tau. \tag{55}$$

By comparing Eqs. (55) and (11), we can see that the jth CPB is now directly coupled to the effective non-Markovian field $\tilde{b}_{\mathrm{in,j}}$ with $\kappa_j(t) = i g_j\sqrt{\gamma_j}\exp[-(\omega_{cj} + \gamma_j/2)t]$. Additionally we can see that the total system we consider here is just a cascade-connected two-qubit system mediated by a non-Markovian field. If $b_{\mathrm{in,j}}$ is a white noise, it can be easily verified that the spectrum of $\tilde{b}_{\mathrm{in,j}}(t)$ is of Lorentz type. In fact, it can be calculated in the frequency domain that

$$\begin{aligned}\tilde{b}_{\mathrm{in,j}}(\omega) &= \kappa_j(\omega)b_{\mathrm{in,j}}(\omega)\\ &= \frac{i g_j\sqrt{\gamma_j}}{\gamma_j/2 + i(\omega-\omega_{cj})}b_{\mathrm{in,j}}(\omega),\end{aligned} \tag{56}$$

where $\tilde{b}_{\mathrm{in,j}}(\omega)$, $b_{\mathrm{in,j}}(\omega)$, and $\kappa(\omega)$ are the Fourier transforms of $b_{\mathrm{in,j}}(t)$, $b_{\mathrm{in,j}}(t)$, and $\kappa(t)$. From Eq. (15), it can be shown that

$$\tilde{\gamma}_j(\omega) = |\kappa_j(\omega)|^2 = \frac{g_j^2\gamma_j}{\gamma_j^2/4 + (\omega-\omega_{cj})^2}.$$

Note that

$$\begin{aligned}[\tilde{b}_{\mathrm{in,j}}(t), \tilde{b}_{\mathrm{in,j}}^{\dagger}(\tilde{t})] = \tilde{\gamma}_j(t-\tilde{t}) &= \int_0^{\infty} e^{-i\omega(t-\tilde{t})}\tilde{\gamma}_j(\omega)d\omega\\ &= g_j^2\exp[-\gamma_j|t-\tilde{t}|/2].\end{aligned}$$

We can see that $\tilde{b}_{\mathrm{in,j}}(t)$ is a Lorentz-type noise.

To simplify our discussion, let us assume that the two qubits have the same parameters, so that $\Delta_q = \Delta_{q1} = \Delta_{q2}, g = g_1 = g_2$, and $\gamma = \gamma_1 = \gamma_2$. If we further assume that $\Delta_q = E_J - \omega_g = 0$, we can write down the master equation for this non-Markovian two-qubit system using Eq. (51), and the result is

$$\begin{aligned}\dot{\rho} = &-i[\alpha(t)\sigma_-^{(1)}\sigma_+^{(2)} + \alpha^*(t)\sigma_+^{(1)}\sigma_-^{(2)}, \rho]\\ &+ \int_0^t \{\beta(t-\tau)[J_-\rho(\tau), J_+] + \mathrm{H.c.}\}d\tau,\end{aligned} \tag{57}$$

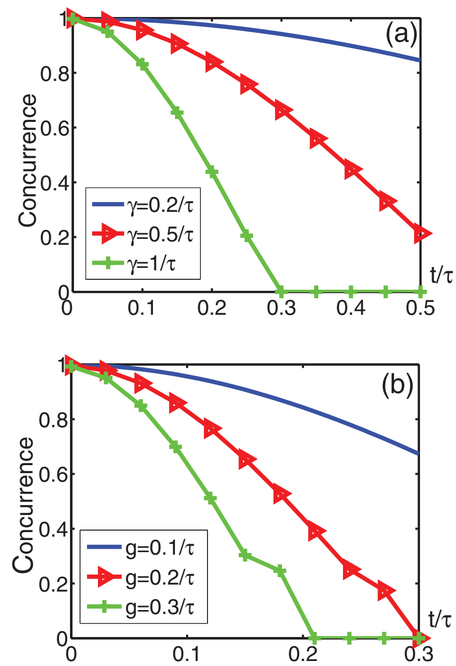

FIG. 10. (Color online) Evolution of the concurrence (a) for various non-Markovian correlation rates γ (the blue curve, red triangle curve, and the green plus-sign curve correspond to $\gamma = 0.2/\tau, 0.5/\tau, 1/\tau$) and (b) for various coupling strengths g (the blue curve, red triangle curve, and the green plus-sign curve correspond to $g = 0.1/\tau, 0.2/\tau, 0.3/\tau$). Here $\tau = 10$ ns. The decay of the concurrence speeds up when increasing the correlation rate γ of the non-Markovian noises and the qubit-environment coupling strength g.

where $J_{\alpha=z,\pm} = \sigma_\alpha^{(1)} + \sigma_\alpha^{(2)}$ is the collective two-qubit operator, and

$$\begin{aligned}\alpha(t) &= i\frac{2g^2}{\gamma}[1-\exp(-\gamma t/2)],\\ \beta(t) &= g^2\exp(-\gamma t/2).\end{aligned} \tag{58}$$

It can be verified that both $\alpha(t)$ and $\beta(t)$ are monotonic functions of γ. This means that both the coherent interaction between the two qubits and the damping induced by the transmitting field decrease as the correlation time of the Lorentz noise increases.

In Fig. 10 we show the evolution of the concurrence of the two qubits. The concurrence is defined by

$$C(\rho) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}.$$

where ρ is the system density matrix given by Eq. (57), the λ_i are the square roots of the eigenvalues of

$$M(\rho) = \rho\left(\sigma_y^{(1)}\sigma_y^{(2)}\right)\rho^*\left(\sigma_y^{(1)}\sigma_y^{(2)}\right);$$

in decreasing order, and ρ^* is the complex conjugate of ρ. From Fig. 10 we see that the damping rate of the concurrence decreases as the bath coupling strength g decreases, but increases when the correlation time of the environment ($\tau_{\mathrm{env}} = 1/\gamma$) increases. This means that in non-Markovian environments the two-qubit entanglement is preserved longer than in a Markovian environment. This is reasonable because the decay rates of the two superconducting microwave cavities decrease in the non-Markovian regime and thus the coherent evolution between the qubits and the cavities protect the quantum entanglement efficiently.

The two-qubit dynamics given by Eq. (57) can be extended to multiqubit networks [63] to study many-body physical phenomena, such as quantum entanglement and correlations.

V. CONCLUSION

In summary, we have extended quantum input-output theory to arbitrary non-Markovian networks of systems connected via continuous-wave fields. We have derived the Heisenberg picture quantum stochastic differential equation for the systems in the network, the corresponding perturbative master equations, and all the input-output relations. We have applied this general formalism to a model of two superconducting charge qubits interacting via a cascade connection. We showed that this system was non-Markovian because the cavities with which the qubits connect to each other act as filters for the quantum noise. For this system we analyzed the dynamics of the entanglement between the qubits, and showed that it was affected by the non-Markovian nature of the network. We also used our model to analyze the dynamics of a non-Markovian coherent feedback network in which the controller has a fast response time. It is clear from our analysis that non-Markovian effects can change the behavior of mesoscopic quantum networks in significant ways, and the analysis of these effects may be important for future quantum devices.

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APPENDIX: DERIVATIONS OF THE NON-MARKOVIAN DYNAMICAL AND OUTPUT EQUATIONS

From the system Hamiltonian (1), we can obtain the Heisenberg equation of an arbitrary system operator X

$$\dot{X} = -i[X,H_{\rm sys}] + \int d\omega\{\kappa(\omega)b^{\dagger}(\omega,t)[X,L] - \kappa^{*}(\omega)[X,L^{\dagger}]b(\omega,t)\}, \quad (A1)$$

and the equation of the bath operator $b(\omega)$

$$\dot{b}(\omega,t) = -i\omega b(\omega,t) + \kappa(\omega)L. \quad (A2)$$

We can solve Eq. (A2) and obtain

$$b(\omega,t) = e^{-i\omega t}b(\omega) + \kappa(\omega)\int_0^t e^{-i\omega(t-\tau)}L(\tau)d\tau, \quad (A3)$$

where $b(\omega) = b(\omega,0)$ is the initial condition of $b(\omega,t)$. Similarly,

$$b(\omega,t) = e^{-i\omega(t-t_1)}b(\omega,t_1) - \kappa(\omega)\int_t^{t_1} e^{-i\omega(t-\tau)}L(\tau)d\tau, \quad (A4)$$

where $t_1 \geqslant t$. The input and output fields $b_{\rm in}(t)$ and $b_{\rm out}(t)$ are defined as the Fourier transform of $b(\omega)$ and $b(\omega,t_1)$ respectively

$$b_{\rm in}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} b(\omega)e^{-i\omega t}\, d\omega,$$
$$b_{\rm out}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} b(\omega,t_1)e^{-i\omega(t-t_1)}\, d\omega.$$

From Eqs. (A3) and (A4), we have

$$b_{\rm out}(t) = b_{\rm in}(t) + \int_0^{t_1}\kappa(t-\tau)L(\tau)d\tau, \quad (A5)$$

where $\kappa(t)$ is the Fourier transform of $\kappa(\omega)$ defined in Eq. (12). Let $t_1 \to t$, we can obtain the output equation (17).

Furthermore, using the identities

$$\tilde{b}_{\rm in} = \int_{-\infty}^{+\infty}\kappa(t-\tau)b_{\rm in}(\tau)d\tau = \int_{-\infty}^{+\infty}\kappa(\omega)e^{-i\omega t}b(\omega)d\omega,$$

and

$$\gamma(t-\tilde{t}) = \int_{-\infty}^{+\infty}\kappa^{*}(t-\tau)\,\kappa(\tilde{t}-\tau)d\tau = \int_{-\infty}^{+\infty}\kappa(\omega)\,\kappa^{*}(\omega)e^{-i\omega(\tilde{t}-t)}\, d\omega,$$

we can obtain Eq. (16) by substituting Eq. (A3) into Eq. (A1).

To derive the master equation (18), we first change into the interaction picture, in which the effective Hamiltonian $H_{\rm eff}$ can be rewritten as

$$H_{\rm I,eff} = i\left[b_{\rm in}^{\dagger}(t)L_{\rm H_S}(t) - L_{\rm H_S}(t)b_{\rm in}(t)\right], \quad (A6)$$

where $L_{\rm H_S}(t)$ is given in Eq. (19). The density operator $\rho_{\rm I,tot}$ satisfies the following Liouville equation

$$\dot{\rho}_{\rm I,tot} = -i[H_{\rm I,eff}(t),\rho_{\rm I,tot}]. \quad (A7)$$

Integrating the two sides of Eq. (A7), we have

$$\rho_{\rm I,tot}(t) = -i\int_0^t [H_{\rm I,eff}(\tau),\rho_{\rm I,eff}(\tau)]d\tau. \quad (A8)$$

Substituting Eq. (A8) into Eq. (A7), we can obtain

$$\dot{\rho}_{\rm I,tot} = \int_0^t [H_{\rm I,eff}(t),[H_{\rm I,eff}(\tau),\rho_{\rm I,tot}(\tau)]]d\tau. \quad (A9)$$

Tracing over the degrees of freedom of the input field, we can obtain the dynamical equation of the system density operator $\rho_{\rm I} = {\rm tr}_B\rho_{\rm I,tot}$

$$\dot{\rho}_{\rm I} = \int_0^t {\rm tr}_B\{[H_{\rm I,eff}(t),[H_{\rm I,eff}(\tau),\rho_{\rm I,tot}(\tau)]]\}d\tau. \quad (A10)$$

Let us then introduce the Born approximation and assume that the input field stays in the vacuum state, we have

$$\rho_{\mathrm{I,tot}}(t) = \rho_{\mathrm{I}}(t) \otimes |0\rangle_{BB}\langle 0|. \tag{A11}$$

Notice that it can be shown that

$$\langle b_{\mathrm{in}}(t) b_{\mathrm{in}}^{\dagger}(\tilde{t})\rangle = \delta(t-\tilde{t}),$$
$$\langle b_{\mathrm{in}}^{\dagger}(t) b_{\mathrm{in}}(\tilde{t})\rangle = \langle b_{\mathrm{in}}^{\dagger}(t) b_{\mathrm{in}}^{\dagger}(\tilde{t})\rangle = \langle b_{\mathrm{in}}(t) b_{\mathrm{in}}(\tilde{t})\rangle = 0, \tag{A12}$$

where $\langle\cdot\rangle$ is defined by $\langle R\rangle = \langle 0|b|0\rangle_B$. Substituting Eqs. (A6), (A11), and (A12) into Eq. (A10), we can verify that

$$\dot{\rho}_{\mathrm{I}} = \int_0^t \left\{\gamma(t-\tau)\left[L_{H_S}(t)\rho_I(\tau), L_{H_S}^{\dagger}(\tau)\right] + \mathrm{H.c.}\right\}. \tag{A13}$$

We can derive Eq. (18) by transforming Eq. (A13) back into the Schrödinger picture.

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