Controlled Generation of Squeezed States of Microwave Radiation in a Superconducting Resonant Circuit

A. M. Zagoskin,[1,2,3] E. Il'ichev,[4] M. W. McCutcheon,[5] Jeff F. Young,[2] and Franco Nori[3,6]

[1]*Department of Physics, Loughborough University, Loughborough, Leics LE11 3TU, United Kingdom*
[2]*Department of Physics and Astronomy, The University of British Columbia, Vancouver, BC, V6T 1Z1, Canada*
[3]*Advanced Science Institute, RIKEN, Wako-shi, Saitama 351-0198, Japan*
[4]*Institute of Photonic Technology, P.O. Box 100239, D-07702 Jena, Germany*
[5]*School of Engineering and Applied Sciences, Harvard University, Cambridge, Massachusetts 02138, USA*
[6]*Center for Theoretical Physics, Physics Department, The University of Michigan, Ann Arbor, Michigan 48109, USA*
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Superconducting oscillators have been successfully used for quantum control and readout devices in conjunction with superconducting qubits. Also, squeezed states can improve the accuracy of measurements to subquantum, or at least subthermal, levels. Here, we show theoretically how to produce squeezed states of microwave radiation in a superconducting oscillator with tunable parameters. Its resonance frequency can be changed by controlling an rf SQUID inductively coupled to the oscillator. By repeatedly shifting the resonance frequency between any two values, it is possible to produce squeezed and subthermal states of the electromagnetic field in the (0.1–10) GHz range, even when the relative frequency change is small. We propose experimental protocols for the verification of squeezed state generation, and for their use to improve the readout fidelity when such oscillators serve as quantum transducers.

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The problem of quantum measurements has recently attracted renewed attention. In quantum mechanics, the extraction of information from a quantum system produces an unavoidable disturbance on it. If the object is initially in an eigenstate of the measured observable, a quantum nondemolition (QND) measurement can be realized, where this disturbance is minimal [1]. A well-established type of detector for QND measurements is the parametric transducer (PT) [1] (essentially, an optical or radio-frequency auto-oscillator). A quantum system coupled to a PT changes the phase and/or amplitude of the transducer's oscillations, thereby providing information about the quantum system's dynamics. Since the recent development of superconducting qubits (Refs. [2], for recent reviews see [3]), this approach has been successfully applied to their study. In particular, superconducting resonant tank circuits (high-quality *LC* oscillators made usually of aluminum or niobium) were employed as PTs to measure the quantum state of superconducting flux qubits [4].

The noise of detectors can be decreased below the standard quantum limit (SQL) by employing squeezed states [5]. In this Letter, we show that a superconducting PT can naturally implement this approach since it can be used both *to produce* squeezed states and *to use* them in order to minimize quantum fluctuations. An immediate application of this method would be to suppress the effective noise temperature of the next-stage amplifier, at least to the nominal temperature of the cooling chamber. We emphasize that existing experimental techniques are sufficient for the realization of our proposal.

A system described by a pair of dimensionless conjugate variables, Q and P, is in a squeezed state if, for some times, $\langle \Delta Q^2\rangle \equiv \langle Q^2\rangle - \langle Q\rangle^2 < 1/2$ (i.e., the dispersion of this variable is below the SQL). The uncertainty principle requires that $\langle \Delta P^2\rangle\langle \Delta Q^2\rangle \geq 1/4$, so when one variable is squeezed, the dispersion of the conjugate variable increases. Squeezed states are of great interest due to their usefulness in obtaining SQL resolution in imaging and measurement [5,6]. Their classical analog can be used to obtain subthermal resolution in mechanical measurements [7].

Squeezed states were first introduced in quantum optics, but have since been investigated in other systems, including phonons [8], trapped atoms [9], optical lattices occupied by cold atoms [10], and molecular oscillations [11]. Josephson devices combine tunable nonlinearity and low losses in the microwave range, which makes them a logical choice for various theoretical schemes of producing squeezed states [12–14]; e.g., they have been successfully generated in a Josephson parametric amplifier [15].

If a harmonic oscillator is in a state with equal and minimal uncertainties, $\langle \Delta P^2\rangle = \langle \Delta Q^2\rangle = 1/2$, a *sudden* change of the oscillator frequency would create a squeezed state [16]. The degree of squeezing is given by the ratio $\lambda = \omega/\omega_0$ of the oscillator frequencies before and after the shift, or its inverse, whichever is less than one. This topic was further investigated in Refs. [17–21]. It was shown [18,21] that by repeated and properly timed oscillator frequency shifts, one could reach an *arbitrary* degree of squeezing, even for λ close to unity (neglecting damping and assuming instantaneous frequency shifts).

In this Letter, we show analytically that applying repeated frequency shifts to a superconducting resonant tank circuit can produce GHz squeezed states, which can be then used to improve the sensitivity of such a circuit employed as a PT. The parameters of the circuit can be tuned by controlling the state of an rf SQUID inductively coupled to the superconducting resonant tank circuit [4]. This distinguishes our proposal from Ref. [12], which proposed the generation of squeezed states in an rf SQUID with a tunable junction through a single-step change of the parameters of the junction. The advantage of our method of squeezing state generation compared to the use of a parametric amplifier [15] is that the degree of squeezing and the time of generation are directly controlled through the number of frequency shifts applied.

The assumptions of an instantaneous frequency shift and no damping are convenient for a proof-of-principle analysis, but are not sufficient for the discussion of an experimental realization of the effect. Therefore, we start from the density matrix ρ of the tank circuit coupled to the superconducting oscillator, which satisfies the Liouville equation $i\partial_t\rho = [H(t), \rho]$, where $H(t)$ is the Hamiltonian. Additional Lindblad terms in the r.h.s. can be added to account for dephasing and relaxation [5]. Initially, $H(0) = (\omega_0/2)(a_0^\dagger a_0 + a_0 a_0^\dagger)$. A change of the oscillator frequency, $\omega_0 \to \omega$, transforms [16] the creation operator to $a = [(\omega + \omega_0)a_0 + (\omega - \omega_0)a_0^\dagger](2\sqrt{\omega\omega_0})^{-1}$. This Bogoliubov transformation can be rewritten [16] as $a_0 \to a = U a_0 U^\dagger$, where $U = \exp[(1/4)\ln(\omega/\omega_0)(a_0^{\dagger 2} - a_0^2)]$.

The transformation of the Hamiltonian under the unitary transformation U is given by $H = UH_0U^\dagger - iU\partial_t U^\dagger$; here, $H_0 \equiv H(0)$ is the initial Hamiltonian. The resulting Hamiltonian, expressed in terms of the original operators a_0, $a_0^\dagger$, is

$$H(t) = H_0 + \frac{\omega(t)^2 - \omega_0^2}{4\omega(t)}(a_0^\dagger a_0 + a_0 a_0^\dagger + a_0^{\dagger 2} + a_0^2) - i\frac{\dot{\omega}(t)}{\omega(t)}(a_0^{\dagger 2} - a_0^2). \tag{1}$$

Now we can move to the interaction representation with respect to H_0 by applying to $H(t)$ another unitary transformation, $\tilde{U} = \exp[iH_0 t]$. Then, the H_0-induced evolution of the density matrix ρ is canceled, at the price of the operators acquiring an explicit time dependence: $a_0 \to a_0\exp[-i\omega_0 t]$, $a_0^\dagger \to a_0^\dagger \exp[i\omega_0 t]$. Denoting the resulting Hamiltonian also by $H(t)$, we find

$$H(t) = \frac{\omega(t)^2 - \omega_0^2}{4\omega(t)}(a_0^\dagger a_0 + a_0 a_0^\dagger) + \frac{\omega(t)^2 - \omega_0^2}{4\omega(t)}(a_0^{\dagger 2} e^{2i\omega_0 t} + a_0^2 e^{-2i\omega_0 t}) - i\frac{\dot{\omega}(t)}{\omega(t)}(a_0^{\dagger 2} e^{2i\omega_0 t} - a_0^2 e^{-2i\omega_0 t}). \tag{2}$$

In the following, it will be convenient to use, instead of the density matrix ρ, the *Wigner function* W. The latter is related to ρ [5] by a Fourier transform, $W(\alpha, \alpha^*) = \frac{1}{2\pi^2} \times \int d\lambda d\lambda^* e^{-\lambda\alpha^* + \lambda^*\alpha}\mathrm{Tr}[\rho e^{\lambda a_0^\dagger - \lambda^* a_0}]$, and in the classical limit yields the classical distribution function. The Wigner function, as well as ρ, can be expressed both in terms of photon numbers, n, n', and in terms of complex variables α, α^*, which label *coherent states* [5]. A coherent state $|\alpha\rangle$ is the eigenvector of the annihilation operator with the eigenvalue α: $a_0|\alpha\rangle = \alpha|\alpha\rangle$, while $\langle\alpha|a_0^\dagger = \langle\alpha|\alpha^*$. Each coherent state is a superposition of an infinite number of Fock states (states with a definite number of photons, $|n\rangle$) and in the classical limit becomes a *classical state* characterized by a pair of appropriate canonically conjugate classical variables. The complex variables α and α^* can be expressed through their real quadrature components, x, y: $\alpha = x + iy$, $\alpha^* = x - iy$, which can be related to directly observable properties of an oscillator (e.g., current and voltage).

The Liouville equation for the density matrix now yields the equation for W [22]

$$i\partial_t W(\alpha, \alpha^*) = 2\beta(t)[\alpha^*\partial_{\alpha^*} - \alpha\partial_\alpha]W(\alpha, \alpha^*) + 2[\gamma(t)^*\alpha\partial_{\alpha^*} - \gamma(t)\alpha^*\partial_\alpha]W(\alpha, \alpha^*). \tag{3}$$

Here, we have introduced $\beta(t) = \frac{\omega_0}{4}[(\frac{\omega(t)}{\omega_0})^2 - 1] \equiv \frac{\omega_0}{4} \times [\lambda(t)^2 - 1]$; $\gamma(t) = [\beta(t) + i\frac{\dot{\omega}(t)}{\omega(t)}]e^{2i\omega_0 t} \equiv [\beta(t) + i\frac{\dot{\lambda}(t)}{\lambda(t)}]e^{2i\omega_0 t}$.

In the presence of dissipation, Eq. (3) will also contain second-order terms, and would only be tractable numerically; fortunately, in the experimentally relevant (for superconducting PTs) situation, dissipation can be neglected. Then Eq. (3), which is a differential equation of first order, can be solved by the method of characteristics. The characteristics $x(t)$, $y(t)$ satisfy the equations

$$\begin{aligned} \frac{1}{2}\dot{x} &= [\mathrm{Im}\gamma(t)]x(t) + \{\beta(t) - [\mathrm{Re}\gamma(t)]\}y(t); \\ \frac{1}{2}\dot{y} &= -\{\beta(t) + [\mathrm{Re}\gamma(t)]\}x(t) - [\mathrm{Im}\gamma(t)]y(t). \end{aligned} \tag{4}$$

After finding $x(x_0, y_0, t)$; $y(x_0, y_0, t)$ (where x_0, y_0 are the initial conditions) and inverting them to obtain $x_0(x, y, t)$; $y_0(x, y, t)$, we express the Wigner function at an arbitrary time through its value at $t = 0$ via

$$W(x + iy, x - iy, t) = W(x_0(x, y, t) + iy_0(x, y, t), x_0(x, y, t) - iy_0(x, y, t), 0). \tag{5}$$

Equation (5) describes the intuitively clear picture of how the initial value at a given point $W(x_0, y_0, 0)$ is dragged along the characteristic starting in this point. Equation (5) provides the complete solution for the quantum mechanical problem of a harmonic oscillator with variable frequency, as should be expected [6,23].

Equations (4) can only be solved, in general, numerically, but a good analytical approximation can be found under two reasonable assumptions. First, the relative change of the oscillator frequency must be small ($|1 - \lambda| \ll 1$). Second, the frequency must change either very fast ($\dot{\lambda} \gg \omega_0$), or very slowly ($\dot{\lambda} \ll \omega_0$), compared to the oscillator period.

Let us first treat the fast limit. In this case, $\beta(t)$ can be neglected compared to $\gamma(t)$, and Eqs. (4) are reduced to $\dot{\alpha} = -2i\gamma(t)\alpha^*(t)$; $\dot{\alpha}^* = 2i\gamma(t)^*\alpha(t)$. We must keep all terms in $\gamma(t)$ until we transform this system into a second-order equation for $\alpha(t)$. Now, dropping the small terms (assuming $\ddot{\lambda} \ll \dot{\lambda} \equiv v$), we obtain $\ddot{\alpha} + v\dot{\alpha} - 4v^2\alpha = 0$. In the case of a linear frequency change ($v =$ const), this equation is easily solved with the initial conditions $\alpha(0) = \alpha$, $\dot{\alpha}(0) = 2v\alpha^*$, yielding (see Fig. 1)

$$x_0(t, x, y) = x_0(t, x) = \frac{x\sqrt{17}}{(2 - s_-/v)e^{s_+ t} + (s_+/v - 2)e^{s_- t}}; \tag{6}$$

$$y_0(t, x, y) = y_0(t, y) = \frac{y\sqrt{17}}{(s_+/v - 1)e^{s_+ t} + (1 - s_-/v)e^{s_- t}}. \tag{7}$$

Here, $s_+ = v(\sqrt{17} - 1)/2$; $s_- = -v(\sqrt{17} + 1)/2$.

In the slow regime, the terms with $\dot{\lambda}$ can be neglected. The remaining terms are of the same order, but γ-terms oscillate with $2\omega_0$ and average to zero over the period of the oscillator. Therefore, we are left with

$$\frac{1}{2}\dot{x} = \beta(t)y(t); \qquad \frac{1}{2}\dot{y} = -\beta(t)x(t), \tag{8}$$

or $\dot{\alpha} = -2i\beta(t)\alpha(t)$; that is, $\alpha(t) = \alpha(0)\exp\{-i\frac{\omega_0}{2} \times \int_0^t[\lambda^2(s) - 1]ds\}$. Even before solving these equations, it was clear that the slow regime cannot affect squeezing in any way: Eq. (8) describes circles in phase plane; the evolution of the Wigner function given by (5) therefore reduces to its slow rotation as a whole, without changing shape. This conclusion is consistent with [16,21].

The experimental realization of this proposal is nontrivial. There have been several reports of an ultrafast perturbation of optical microcavity modes using the dispersion of injected free carriers [24–26]. However, the small frequency shifts (of the order $|\lambda - 1| \sim 5 \times 10^{-4}$) were on a picosecond time scale, which is slow with respect to the period of one optical cycle of the cavity modes. These processes are therefore in the adiabatic limit, rather than in the sudden-frequency-shift regime [16]. Thus, while applicable to on-chip frequency conversion [27,28], they are not useful for generating nonclassical optical states. Moreover, a repeated application of the perturbation within the subnanosecond lifetime of a microcavity mode is doubtful.

The situation is more promising in the (0.1–10) GHz range, where one can use Josephson junctions as nondissipative nonlinear elements, allowing control of the circuit parameters. If the frequency is low, so that $\hbar\omega \simeq k_BT$, where T is 10–50 mK (dilution refrigerator temperatures), the so-called rf SQUID [29] configuration can be used. The system consists of a superconducting resonant tank circuit, inductively coupled to an rf SQUID (a loop containing the

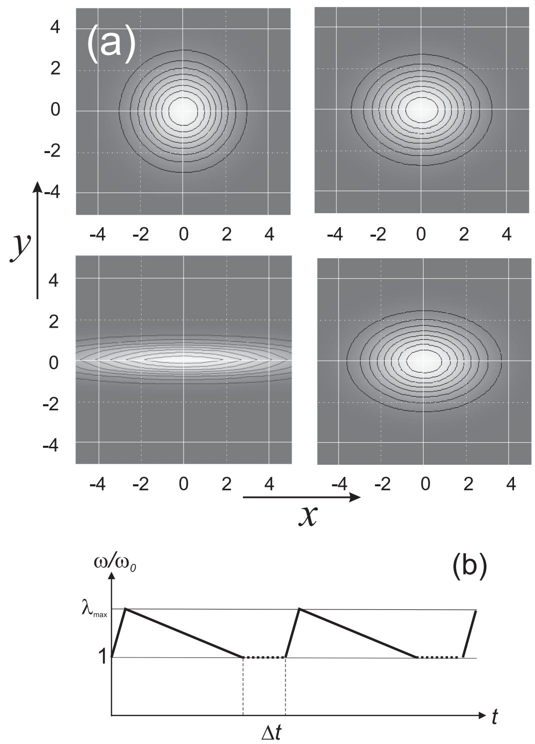

FIG. 1. Squeezing of a thermal state, with $k_BT = 4\omega_0$, by repeated frequency shifts, with $\lambda_{\max} - 1 = 0.05$. (a) Contour plots of the approximate expression of the Wigner function in the interaction representation [see Eqs. (5)–(7)] are shown as a function of the quadrature variables, x and y. Clockwise starting from the upper left: initial thermal state, the state after one, two, and ten cycles. The dark background corresponds to $W = 0$, and white to $W = 0.08$. (b) A schematic frequency-versus-time dependence necessary to produce the results of Fig. 1(a). The idle periods Δt are chosen to ensure that every fast shift occurs in the same phase with respect to the quadrature coordinates. Here, one must account for the fact that in the Schröedinger representation, the whole phase plane rotates around the origin with the base oscillator frequency ω_0. An additional rotation of the Wigner function as a whole during the slow shift can be neglected in comparison, since the additional phase $|\delta\theta| \leq 0.5\omega_0 t|\lambda_{\max}^2 - 1| \ll \omega_0 t$.

Josephson contact). In the dispersive mode of an rf SQUID, the effective inductance of the system becomes [29]

$$L_{\text{eff}} = L_T - \frac{M^2}{L + \mathcal{L}(\varphi)}, \tag{9}$$

where L_T, L are, respectively, the self-inductance of the tank circuit and of the rf SQUID loop, M is their mutual inductance, and $\mathcal{L}(\varphi) = L/(\beta\cos\varphi)$ is the Josephson inductance of the SQUID junction, which depends on the phase bias φ across it. Here, $\beta \equiv 2\pi LI_C/\Phi_0$ and I_C is the critical current of the SQUID Josephson junction. Therefore, by varying φ, one can control the effective inductance and eigenfrequency of the tank:

$$\frac{\omega_{\text{eff}}^2}{\omega_0^2} = \frac{L}{L_{\text{eff}}} \approx 1 + \frac{k^2}{(1 + \beta\cos\varphi)}. \tag{10}$$

Here, $k^2 \equiv M^2/LL_T \ll 1$ (in practice $k \approx 0.3$ can be easily realized) is the coupling coefficient between the tank and the SQUID loop. For a SQUID dispersive mode, the parameter $\beta < 1$, and so the variations $\delta\omega_0$ of the eigenfrequency of the tank satisfy $\delta\omega_0 \simeq (0.1 - 0.01)\omega_0$, which should be enough for our purposes. For higher frequencies $\hbar\omega \gg k_B T$, a tunable superconducting cavity could be used. In such a device, a dc SQUID is incorporated in the strip resonator [30] and $\delta\omega_0 \simeq (0.1 - 0.01)\omega_0$ as well.

For both low and high frequency cases, the phase-changing pulse must be sharp on the scale of ω_0, and this is within the current experimental capabilities. The coherence time of the system is in the range of 1–10 μs. This justifies our neglect of the dissipative terms in Eq. (3) and allows us at least several cycles of frequency change, with the corresponding increase in the squeezing of the final state.

A squeezed state can be measured using the standard homodyne detection, whereby the signal is mixed with the reference signal at the same frequency and with variable phase (local oscillator). The resulting signal is proportional to $[A_x \cos(\omega t) + A_y \sin(\omega t)]\cos(\omega t + \delta) \propto (A_x \cos\delta + A_y \sin\delta + \text{fast oscillating terms})$, which allows a direct determination of either quadrature [31].

In conclusion, we have shown that repeated fast-slow frequency shifts can produce squeezed states in a nonlinear superconducting oscillator, thereby suppressing the fluctuations of the amplitude (phase) of the oscillator along certain directions in phase space, which rotate with the base oscillator frequency ω_0. By making use of this noise suppression, the measurements of the amplitude (phase) of these oscillators can reach a sensitivity below the standard quantum limit, or at least below the thermal level.

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[1] V. B. Braginsky and F. Ya. Khalili, *Quantum Measurements* (Cambridge University Press, Cambridge, 1995).

[2] Y. Nakamura *et al.*, Nature (London) **398**, 786 (1999); D. Vion *et al.*, Science **296**, 886 (2002); I. Chiorescu *et al.*, *ibid.* **299**, 1869 (2003); A. J. Berkley *et al.*, *ibid.* **300**, 1548 (2003); T. Hime *et al.*, *ibid.* **314**, 1427 (2006); S. O. Valenzuela *et al.*, *ibid.* **314**, 1589 (2006); A. O. Niskanen *et al.*, *ibid.* **316**, 723 (2007); A. Wallraff *et al.*, Nature (London) **431**, 162 (2004); M. A. Sillanpaa *et al.*, *ibid.* **449**, 438 (2007); C. M. Wilson *et al.*, Phys. Rev. Lett. **98**, 257003 (2007).

[3] J. Q. You and F. Nori, Phys. Today **58**, No. 11, 42 (2005); G. Wendin and V. S. Shumeiko, in *Handbook of Theoretical and Computational Nanotechnology*, edited by M. Rieth and W. Schommers (American Scientific Publishers, Valencia, CA, 2006), Vol. 3; A. Zagoskin and A. Blais, Physics in Canada **63**, 215 (2007).

[4] E. Il'ichev *et al.*, Phys. Rev. Lett. **91**, 097906 (2003); A. Izmalkov *et al.*, *ibid.* **93**, 037003 (2004); I. Chiorescu *et al.*, Nature (London) **431**, 159 (2004); M. Grajcar *et al.*, Phys. Rev. Lett. **96**, 047006 (2006); S. H. W. van der Ploeg *et al.*, *ibid.* **98**, 057004 (2007); E. Il'ichev *et al.*, Low Temp. Phys. **30**, 620 (2004).

[5] M. Orszag, *Quantum Optics* (Springer, New York, 1999).

[6] V. P. Bykov, Sov. Phys. Usp. **34**, 910 (1991).

[7] D. Rugar and P. Grutter, Phys. Rev. Lett. **67**, 699 (1991).

[8] X. Hu and F. Nori, Phys. Rev. Lett. **76**, 2294 (1996); X. Hu and F. Nori, *ibid.* **79**, 4605 (1997); Phys. Rev. B **53**, 2419 (1996); Physica B (Amsterdam) **263**, 16 (1999).

[9] D. M. Meekhof *et al.*, Phys. Rev. Lett. **76**, 1796 (1996).

[10] P. Rudy, R. Ejnisman, and N. P. Bigelow, Phys. Rev. Lett. **78**, 4906 (1997).

[11] J. Janszky and An. V. Vinogradov, Phys. Rev. Lett. **64**, 2771 (1990).

[12] M. J. Everitt *et al.*, Phys. Rev. A **69**, 043804 (2004); X. Hu and F. Nori (unpublished).

[13] K. Moon and S. Girvin, Phys. Rev. Lett. **95**, 140504 (2005).

[14] T. Ojanen and J. Salo, Phys. Rev. B **75**, 184508 (2007).

[15] B. Yurke *et al.*, Phys. Rev. Lett. **60**, 764 (1988); B. Yurke *et al.*, Phys. Rev. A **39**, 2519 (1989).

[16] R. Graham, J. Mod. Opt. **34**, 873 (1987).

[17] G. S. Agarwal and S. A. Kumar, Phys. Rev. Lett. **67**, 3665 (1991).

[18] J. Janszky and P. Adam, Phys. Rev. A **46**, 6091 (1992).

[19] M. S. Abdalla and P. K. Colegrave, Phys. Rev. A **48**, 1526 (1993).

[20] T. Kiss, J. Janszky, and P. Adam, Phys. Rev. A **49**, 4935 (1994).

[21] I. Averbukh, B. Sherman, and G. Kurizki, Phys. Rev. A **50**, 5301 (1994).

[22] In deriving this equation, we use the correspondence between the action of creation/annihilation operators on ρ and differential operations on $W(\alpha, \alpha^*)$ [5]: $a_0\rho \leftrightarrow [\alpha + (1/2)\partial_{\alpha^*}]W(\alpha, \alpha^*)$; $\rho a_0 \leftrightarrow [\alpha - (1/2)\partial_{\alpha^*}]W(\alpha, \alpha^*)$, etc.

[23] A. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986).

[24] I. Fushman *et al.*, Appl. Phys. Lett. **90**, 091118 (2007).

[25] S. F. Preble, Q. Xu, and M. Lipson, Nat. Photon. **1**, 293 (2007).

[26] M. W. McCutcheon *et al.*, Opt. Express **1**, 293 (2007).

[27] M. Notomi *et al.*, Phys. Rev. Lett. **97**, 023903 (2006).

[28] M. Notomi and S. Mitsugi, Phys. Rev. A **73**, 051803(R) (2006).

[29] A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982).

[30] M. Sandberg *et al.*, arXiv:0801.2479v1.

[31] After this paper was submitted, squeezed states generated by a parametric amplifier were observed [32] using the homodyne detection.

[32] M.-A. Castellanos-Beltran *et al.*, Nature Phys. **4**, 929 (2008); arXiv:0806.0659v1.