Engineering quantum pure states of a trapped cold ion beyond the Lamb-Dicke limit

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Based on the conditional quantum dynamics of laser-ion interactions, we propose an efficient theoretical scheme to deterministically generate quantum pure states of a single trapped cold ion without performing the Lamb-Dicke approximation. An arbitrary quantum state can be created by sequentially using a series of classical laser pulses with selected frequencies, initial phases and durations. As special examples, we further show how to create or approximate several typical macroscopic quantum states, such as the phase state and the even/odd coherent states. Unlike previous schemes operating in the Lamb-Dicke regime, the present one does well for an arbitrary-strength coupling between the internal and external degrees of freedom of the ion. The experimental realizability of this approach is also discussed.

I. INTRODUCTION

The engineering of quantum states has attracted considerable attention in recent years. This is in order to test fundamental quantum concepts, e.g., nonlocality, and for implementing various potential applications, including sensitive detection and quantum information processing. Recent advances in quantum optics (e.g., micromasers, cavity QED) (see, e.g., [1]) and atomic physics (ion traps) (see, for instance [2]) have allowed a better control of quantum states.

Laser-cooled ions confined in an electromagnetic trap are good candidates for various quantum-state engineering processes. First, the trapped ion system possesses relatively long decoherence times. Second, various interactions including the usual one-quantum transition Jaynes-Cummings (JC) model and also higher order nonlinear models can be implemented in this system by simply choosing the applied laser tunings (see, e.g., [3,4]). Therefore, a trapped ion driven by a classical laser field provides the possibility of conveniently generating various quantum pure states. Indeed, various engineered quantum states of trapped cold ions have been studied. The thermal, Fock, coherent, squeezed, and arbitrary quantum superposition states of motion of a harmonically bound ion have been investigated [5,6]. The manipulation of the entanglement between the external and internal degrees of freedom of the ion and the realization of a fundamental quantum logic gate between them has also been demonstrated experimentally (see, e.g., [7]).

Most of the previous proposals for engineering the quantum state of a single trapped cold ion operate in various extreme experimental conditions, such as the strong Raman excitation or the weak-coupling Lamb-Dicke (LD) approximations. The former (see, e.g., [8]) requires that the Rabi frequency Ω characterizing the laser-ion interaction is much larger than the trap frequency. While the latter (see, e.g., [5,6,9]), requires that the coupling between the external and internal degrees of freedom of the ion is very weak, i.e., the

spatial dimension of the motion of the ground state of the trapped ion should be much smaller than the effective wavelength of the applied laser field (see, e.g., [6]). These approximations can simplify the laser-ion interaction Hamiltonian to certain solvable models. For example, in the LD limit the interaction between the internal states $|s\rangle = \{|g\rangle, |e\rangle\}$ and the external motional harmonic oscillator states $\{|n\rangle; n = 0,1,2,\ldots\}$ of the ion can be expanded to the lowest order of the LD parameter η_L, then the usual JC or anti-JC-type model can be derived. In addition, the coherent state of the motion of the ion can be easily generated in those limits. Therefore, an arbitrary quantum state may be prepared via an atomic interference method by superimposing a finite number of generated coherent states along a line. Almost all the quantum-state engineering implementations in recent ion trap experiments were operated in these limits. Some meaningful second-order modifications of these approximations to the above experimental conditions have been analyzed theoretically [10]. However, in general, these limits are not rigorously satisfied, and higher-order powers of the LD parameter must be taken into account [4]. Indeed, using the laser-ion interaction outside the LD regime could be helpful to reduce the noise in the trap and improve the cooling rate (see, e.g., [11]). Thus, efficiently engineering the quantum state of the trapped cold ion beyond these limits would be useful. Arbitrary Fock states can, in principle, be prepared as a dark motional state of a trapped ion, if the relevant LD parameters can be set with extreme precision [13]. More realistically, Ref. [14] showed that any pure state, including the Fock state, can be effectively approximated by the nonlinear coherent states of the trapped ion. Since these nonlinear coherent states are one of the motional dark states and are insensitive to some motional kick effects, the generation of highly excited Fock states is possible. Recently, a narrow quadrupole $S_{1/2}$ to $D_{5/2}$ transition at 729 nm of a single trapped $^{40}Ca^+$ ion has been successfully manipulated by ac-

curately resolving its vibrational sidebands [12]. The measured lifetime of the excited level $D_{5/2}$ is long enough (≈ 1 s) to allow for a hundred or more quantum operations. Note that the experiments in [12] do not strictly operate in the LD regime (with $\eta \ll 1$), because the corresponding LD parameter is $\eta \approx 0.25$. Therefore, engineering quantum states of a single trapped cold ion by exciting various vibrational sidebands outside the Lamb-Dicke regime is possible to achieve with current technology.

Based on the exact conditional quantum dynamics for the laser-ion interaction, including all orders of the LD parameter, in this paper we propose an efficient scheme for exactly engineering arbitrary motional and entangled states of a single trapped ion beyond the LD limit. In this case, all of the target quantum states are generated deterministically, as any measurement is not required during the quantum state production or manipulations.

This paper is organized as follows. In Sec. II, we solve exactly the quantum dynamics for a trapped cold ion driven by a traveling classical laser beam beyond the LD limit and then introduce some fundamental unitary operations. By repeatedly using these quantum operations, in Sec. III we show how to deterministically generate an arbitrary motional state of a single trapped cold ion. The preparations of arbitrary entangled states between the external and internal degrees of freedom are given in Sec. IV. Conclusions and discussions are given in Sec. V.

II. DYNAMICS OF A TRAPPED COLD ION BEYOND THE LAMB-DICKE LIMIT

We assume that a single two-level ion is stored in a coaxial resonator RF-ion trap [15], which provides pseudopotential oscillation frequencies satisfying the condition $\omega_x \ll \omega_{y,z}$ along the principal axes of the trap. Only the quan-

tized vibrational motion along the x direction is considered for the cooled ion [15]. The dynamics for such an ion, driven by a classical traveling-wave light-field of frequency ω_L and initial phase ϕ_L, can be described by the following Hamiltonian [3,16]

$$\hat{H}(t) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \frac{\hbar\Omega}{2}\{\hat{\sigma}_+\exp[i\eta_L(\hat{a}+\hat{a}^\dagger)] - i(\omega_L t + \phi_L)] + \text{H.c.}\}. \quad (1)$$

The first two terms describe the free motion of the external and internal degrees of freedom of the ion. Here $\hat{a}^\dagger$ and $\hat{a}$ are bosonic creation and annihilation operators of the atomic vibrational quanta with frequency ω. The Pauli operators $\hat{\sigma}_z$ and $\hat{\sigma}_\pm$ are defined by the internal ground state $|g\rangle$ and excited state $|e\rangle$ of the ion as $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|, \sigma_+ = |e\rangle\langle g|$, and $\sigma_- = |g\rangle\langle e|$. These operate on the internal degrees of freedom of the ion of mass M. The final term of $\hat{H}(t)$ describes the interaction between the ion and the light field with wave vector $\vec{\kappa}_L$, and initial phase ϕ_L. Ω is the carrier Rabi frequency, which describes the coupling between the laser and the ion, and is proportional to the intensity of the applied laser. The frequency ω_L and initial phase ϕ_L of the applied laser beam are experimentally controllable [16]. Usually, the atomic transition frequency ω_0 between two internal energy levels is much larger than the trap frequency ω (e.g., in the experiments in [12], $\omega_0 = 2\pi \times 4.11 \times 10^{11}$ kHz and $\omega = 2\pi \times 135$ kHz). Therefore, for lasers exciting at different vibrational sidebands with small k values, the LD parameters

$$\eta = \sqrt{\frac{\hbar\kappa_L^2}{2M\omega}} = \frac{(\omega_0 \pm k\omega)}{c}\sqrt{\frac{\hbar}{2M\omega}}, \quad k = 0,1,2,\ldots, \quad (2)$$

do not change significantly. Here, the Hamiltonian (1) reduces to different forms [4]

$$\hat{H}_{\text{int}} = \frac{\hbar\Omega}{2} \times \begin{cases} \left\{(i\eta)^k\exp\left[-\frac{(\eta)^2}{2} - i\phi^r\right]\hat{\sigma}_+\left(\sum_{j=0}^{\infty}\frac{(i\eta)^{2j}\hat{a}^{\dagger j}\hat{a}^{j+k}}{j!\,(j+k)!}\right) + \text{H.c.}\right\}, & \omega_L = \omega_0 - k\omega, \\[4mm] \left\{(i\eta)^k\exp\left[-\frac{(\eta)^2}{2} - i\phi^b\right]\hat{\sigma}_+\left(\sum_{j=0}^{\infty}\frac{(i\eta)^{2j}\hat{a}^{\dagger(j+k)}\hat{a}^{j}}{j!\,(j+k)!}\right) + \text{H.c.}\right\}, & \omega_L = \omega_0 + k\omega, \\[4mm] \left\{\exp\left(-\frac{\eta^2}{2} - i\phi^c\right)\hat{\sigma}_+\left(\sum_{j=0}^{\infty}\frac{(i\eta)^{2j}\hat{a}^{\dagger j}\hat{a}^{j}}{j!\,j!}\right) + \text{H.c.}\right\}, & \omega_L = \omega_0, \end{cases} \quad (3)$$

in the interaction picture. The usual rotating-wave approximation has been made and all off-resonant transitions have been neglected by assuming a sufficiently weak applied laser field. The applied laser beam tuned at the frequency $\omega_L = \omega_0 - k\omega$ ($\omega_L = \omega_0 + k\omega$) with nonzero integer k being denoted as the kth red (blue) sideband line, because it is red (or blue) detuned from the atomic frequency ω_0. The line for $k=0$ (i.e. $\omega_L = \omega_0$) is

called the carrier. So, we rewrite the initial phase ϕ_L as ϕ^r (ϕ^b, ϕ^c) for a transition process driven by the red-sideband (blue-sideband, carrier) laser beam, respectively.

Previous discussions (see, e.g., [9]) are usually based on the LD perturbation approximation to first order in the LD parameter by assuming η to be very small. However, outside the LD regime the Hamiltonian (3) may provide various

quantum transitions between the internal and external states of the ion. The dynamics for the trapped cold ion governed by the Hamiltonian (3) is exactly solvable [4,17]. For example, if the external state of the system is initially in $|m\rangle$

and the internal state is initially in $|e\rangle$ or $|g\rangle$, then the dynamical evolution of an ion driven by a red-sideband laser beam with frequency $\omega_L = \omega_0 - k\omega$ can be exactly expressed as

$$
|m\rangle \otimes |g\rangle \to \begin{cases} |m\rangle \otimes |g\rangle, & m < k, \\ \cos(\Omega_{m-k,k} t_l^r)|m\rangle \otimes |g\rangle + i^{k-1} e^{-i\phi_l^r} \sin(\Omega_{m-k,k} t_l^r)|m-k\rangle \otimes |e\rangle, & m \geq k, \end{cases}
$$

$$
|m\rangle \otimes |e\rangle \to \cos(\Omega_{m,k} t_l^r)|m\rangle \otimes |e\rangle - (-i)^{k-1} e^{i\phi_l^r} \sin(\Omega_{m,k} t_l^r)|m+k\rangle \otimes |g\rangle, \tag{4}
$$

with Rabi frequency

$$
\Omega_{m,k} = \frac{\Omega \eta^k}{2} \sqrt{\frac{(m+k)!}{(m)!}} e^{-\eta^2/2} \sum_{j=0}^{m} \frac{(i\eta)^{2j}}{(j+k)!} \binom{j}{m},
$$

where m is the occupation number of the initial Fock state of the external vibrational motion of the ion, and t_l^r and ϕ_l^r are the duration and initial phase of the applied red-sideband laser beam, respectively. The above dynamical evolution can be equivalently defined as the kth red-sideband "exciting" quantum operator

$$
\hat{R}_k^r(t_l^r) = \begin{cases} |m\rangle|g\rangle\langle m|\langle g| + \left[(1-|\widetilde{C}_m^{r_l}|^2)^{1/2}|m\rangle|e\rangle + \widetilde{C}_m^{r_l}|m+k\rangle|g\rangle\right]\langle m|\langle e|, & m < k, \\ \left[(1-|C_{m-k}^{r_l}|^2)^{1/2}|m\rangle|g\rangle + C_{m-k}^{r_l}|m-k\rangle|e\rangle\right]\langle m|\langle g| + \left[(1-|\widetilde{C}_m^{r_l}|^2)^{1/2}|m\rangle|e\rangle + \widetilde{C}_m^{r_l}|m+k\rangle|g\rangle\right]\langle m|\langle e|, & m \geq k, \end{cases} \tag{5}
$$

with

$$
C_m^{r_l} = i^{k-1} e^{-i\phi_l^r} \sin(\Omega_{m,k} t_l^r), \quad \widetilde{C}_m^{r_l} = -(C_m^{r_l})^*.
$$

The use of $\hat{R}_k^r$ is advantageous because it is compact, symmetric, and it is simple to iterate. This operator is quite useful for the generation of quantum states. Analogously, exciting the motional state of the ion to the kth blue-sideband, by applying a laser of frequency $\omega_L = \omega_0 + k\omega$, yields a unitary blue-sideband "exciting" quantum operation,

$$
\hat{R}_k^b(t_l^b) = \begin{cases} \left[(1-|C_m^{b_l}|^2)^{1/2}|m\rangle|g\rangle + C_m^{b_l}|m+k\rangle|e\rangle\right]\langle m|\langle g| + |m\rangle|e\rangle\langle m|\langle e|, & m < k, \\ \left[(1-|C_m^{b_l}|^2)^{1/2}|m\rangle|g\rangle + C_m^{b_l}|m+k\rangle|e\rangle\right]\langle m|\langle g| + \left[(1-|\widetilde{C}_{m-k}^{b_l}|^2)^{1/2}|m\rangle|e\rangle + \widetilde{C}_{m-k}^{b_l}|m-k\rangle|g\rangle\right]\langle m|\langle e|, & m \geq k, \end{cases} \tag{6}
$$

with

$$
C_m^{b_l} = i^{k-1} e^{-i\phi_l^b} \sin(\Omega_{m,k} t_l^b), \quad \widetilde{C}_m^{b_l} = -(C_m^{b_l})^*.
$$

Here, t_l^b and ϕ_l^b are the duration and initial phase of the applied blue-sideband laser beam, respectively.

Finally, applying a carrier laser pulse with frequency $\omega_L = \omega_0$, a conditional rotation

$$
\hat{R}_0^c(t_l^c) = \left[(1-|C_m^{c_l}|^2)^{1/2}\hat{I} + C_m^{c_l}|e\rangle\langle g| + \widetilde{C}_m^{c_l}|g\rangle\langle e|\right] \otimes |m\rangle\langle m| \tag{7}
$$

on the internal states of the ion can be implemented. Here, $\hat{I} = |g\rangle\langle g| + |e\rangle\langle e|$ is the identity operator, t_l^c is the duration of the applied carrier laser beam, and

$$
C_m^{c_l} = -i e^{-\phi_l^c} \sin(\Omega_{m,0} t_l^c), \quad \widetilde{C}_m^{c_l} = -(C_m^{c_l})^*,
$$

with

$$
\Omega_{m,0} = \frac{\Omega}{2} \exp\left(-\frac{\eta^2}{2}\right) \sum_{j=0}^{m} \frac{(i\eta)^{2j}}{j!} \binom{j}{m}.
$$

The quantum dynamics of the laser-ion system beyond the LD limit is conditional. This means that the internal and motional degrees of freedom are always coupled. The ion states of two degrees of freedom cannot be operated separately, even if the ion is driven by the carrier line laser. Of course, for a given carrier Rabi frequency Ω (which depends on the intensity/power of the applied laser beam) and the LD parameter, the Rabi frequencies $\Omega_{m,k}$ are sensitive to values of k. See, e.g., Fig. 1 for the same laser power but different LD parameters: $\eta = 0.202$ [5] 0.25 [12], 0.35, 0.5, and 0.9. As seen in Fig. 1, a smaller η corresponds to a larger reduction of the Rabi frequency for certain k values (e.g., in Fig. 1, $\Omega_{0,20}/\Omega \sim 10^{-6}$ for $\eta = 0.202$). However, for any fixed value of k, larger values of η correspond to larger values of the reduced Rabi frequencies $2\Omega_{0,k}/\Omega$. Therefore, fast quantum operations can be obtained outside the LD regime, where η is not small.
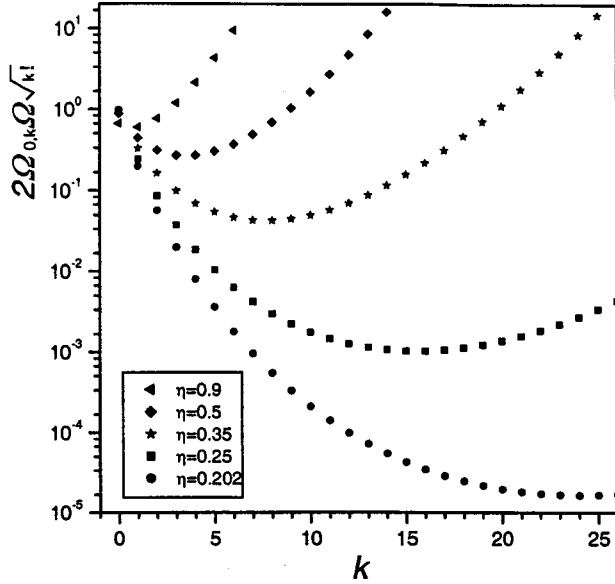
FIG. 1. The k-dependent Rabi frequency $\Omega_{0,k}$ for different LD parameters: $\eta=0.202$ [5], 0.25 [12], 0.35, 0.5, and 0.9.

In general, any quantum process for the laser-ion system can be realized by repeatedly applying the above three kinds of fundamental operations shown in Eqs. (5)–(7). Which operation is applied depends on the laser with the chosen frequency. Below we will use these fundamental unitary operations (5)–(7) to produce or engineer an arbitrary quantum state of a single trapped cold ion beyond the LD limit. The tunable experimental parameters in this process are the frequency ω_L, wave vector $\vec{\kappa}_L$, and the duration of the applied laser pulse. The generation of quantum states described below will start with a common non-entangled initial state $|\psi_0\rangle=|0\rangle\otimes|g\rangle$; that is, the trapped ion has been cooled to its motional ground state and the internal degree of the ion is initially in the low-energy state $|g\rangle$.

III. PREPARATION OF AN ARBITRARY MOTIONAL STATE OF A TRAPPED COLD ION

The first significant quantum state which we want to prepare is the Fock state of the external vibrational quanta of the ion

$$|\psi_1\rangle=|n\rangle, \tag{8}$$

with an arbitrary occupation number $n>0$. The previous schemes (e.g. [5–7,9]) operated in the LD limit only allowed one-quantum transition process (exciting and absorbing a single phonon process, respectively). Thus, at least $(n+1)$ transitions are required between $|j\rangle\otimes|g\rangle\leftrightarrow|j\pm1\rangle\otimes|e\rangle$ to generate the desired state (8). However, for larger values of the LD parameters, the LD approximation is no longer valid and multiquantum transitions must be considered. One can obtain the different phonon transitions, beyond the LD limit, by choosing an appropriate driving laser frequency. For example, the quantum transition: $|0\rangle\otimes|g\rangle\rightarrow|n\rangle\otimes|e\rangle$, can be realized by choosing a blue-sideband driven laser beam with frequency $\omega_L=\omega_0+n\omega$. So the desired Fock state $|n\rangle$ can be

easily obtained by using a single blue-sideband exciting unitary operation $\hat{R}_n^b(t_0^b)$ with the duration t_0^b satisfying the condition: $\sin(\Omega_{0,n}t_0^b)=1$. In other words, $\Omega_{0,n}t_0^b=p\pi/2$ with p an odd integer. Note that the resulting atomic state evolves to its excited state $|e\rangle$ which may transit to the ground state $|g\rangle$ via spontaneous emission. In order to avoid the additional excitation of the desired Fock state due to this emission, an additional operation $\hat{R}_0^c(t_n^c)$ is required to evolve the state $|e\rangle$ into the state $|g\rangle$ keeping the motional state unchanged, with the duration t_n^c satisfying the condition $\sin(\Omega_{n,0}t_n^c)=1$. Therefore, by sequentially performing a $\pi/2$ blue-sideband laser pulse and a $\pi/2$ carrier line laser pulse with initial phases ϕ_0^b and ϕ_n^c, respectively, a relatively steady target Fock state $|\psi_1\rangle$ is generated from the vacuum state $|0\rangle$ as follows:

$$|0\rangle\otimes|g\rangle \xrightarrow{\hat{R}_n^b(t_0^b)} i^{n-1}e^{-i\phi_0^b}|n\rangle\otimes|e\rangle \xrightarrow{\hat{R}_0^c(t_n^c)} -i^n e^{i(\phi_n^c-\phi_0^b)}|n\rangle\otimes|g\rangle. \tag{9}$$

After these unitary operations, the internal electric state returns to its initial ground state $|g\rangle$. The target state $|n\rangle$ can also be prepared by continuously applying a carrier line operation $\hat{R}_0^c(t_0^c)$ and an nth red-sideband exciting operator $\hat{R}_n^r(t_n^r)$, i.e.,

$$|0\rangle\otimes|g\rangle \xrightarrow{\hat{R}_0^c(t_0^c)} -ie^{-i\phi_0^c}|0\rangle\otimes|e\rangle \xrightarrow{\hat{R}_n^r(t_n^r)} -(-i)^n e^{i(\phi_n^r-\phi_0^c)}|n\rangle\otimes|g\rangle, \tag{10}$$

with the durations t_0^c and t_n^r satisfying conditions $\sin(\Omega_{0,0}t_0^c)=1$, $\sin(\Omega_{0,n}t_n^r)=1$. Therefore, two unitary operations are sufficient to deterministically generate an arbitrary Fock state $|n\rangle$ from the initial ground state $|0\rangle$, if the laser-ion interaction is operated outside the LD regime by using the chosen laser beams with desired frequencies.

The more general motional state of the ion which we want to prepare is the following finite superposition of number states

$$|\psi_2\rangle=\sum_{j=0}^{N}c_j|j\rangle, \quad \sum_{n=0}^{N}|c_j|^2=1, \tag{11}$$

with N being a finite integer. For a single-mode light field, this state can be probabilistically generated [18] by physically truncating the photon coherent state. The efficiency of generating a quantum state by the quantum truncation may be quite low due to the necessity of quantum measurements. In the present quantum-state generation the motional vacuum state $|0\rangle$, instead of the motional coherent states $|\alpha\rangle$, is given initially. A quantum-state production scheme, in the LD regime, for generating the desired state (11) has been proposed in [19]. We now extend this scheme to generate the target state $|\psi_2\rangle$ beyond the LD limit. Indeed, sequentially using $N+1$ laser pulses with frequencies $\omega_L=\omega_0, \omega_0-\omega,\ldots,\omega_0-l\omega,\ldots,\omega_0-N\omega$ and durations $t_0^c, t_1^r,\ldots,t_l^r,\ldots,t_N^r$, respectively, the desired state is obtained from the initial state $|\psi_0\rangle$ by a series of dynamical evolutions shown as follows:

$$|\psi_0\rangle \xrightarrow{\hat{R}_0^c(t_0^c)} c_0|0\rangle \otimes |g\rangle - ie^{-i\phi_0^c}(1-c_0^2)^{1/2}|0\rangle \otimes |e\rangle$$

$$\xrightarrow{\hat{R}_1^r(t_1^r)} \left(\sum_{j=0}^{1} c_j|j\rangle\right) \otimes |g\rangle - ie^{-i\phi_0^c}\left(1-\sum_{j=0}^{1}|c_j|^2\right)^{1/2}|0\rangle \otimes |e\rangle \cdots \xrightarrow{\hat{R}_l^r(t_l^r)} \left(\sum_{j=0}^{l} c_j|j\rangle\right) \otimes |g\rangle$$

$$- ie^{-i\phi_0^c}\left(1-\sum_{j=0}^{l}|c_j|^2\right)^{1/2}|0\rangle \otimes |e\rangle \cdots \xrightarrow{\hat{R}_N^r(t_N^r)} \left(\sum_{j=0}^{N} c_j|j\rangle\right) \otimes |g\rangle = |\psi_2\rangle \otimes |g\rangle. \tag{12}$$

The duration of the final unitary operations $\hat{R}_N^r(t_N^r)$ has been set to satisfy the condition: $\sin(\Omega_{0,N}t_N^r)=1$. While, the durations and the initial phases of other applied laser beams can be used to arbitrarily prescribe the weights c_j of the superposed Fock states $\{|j\rangle; j=0,1,\ldots,N\}$, such as

$$c_j = \begin{cases} (1-|C_0^{c_0}|^2)^{1/2} = \cos(\Omega_{0,0}t_0^c), & j=0, \\[2ex] C_0^{c_0}\left[\prod_{l=1}^{j-1}(1-|\tilde{C}_0^{r_l}|^2)\right]^{1/2}\tilde{C}_0^{r_j} = -(-i)^j e^{i(\phi_j^r-\phi_0^c)}\sin(\Omega_{0,0}t_0^c)\prod_{l=1}^{j-1}\cos(\Omega_{0,l}t_l^r)\sin(\Omega_{0,j}t_j^r), & 1\leq j\leq N-1, \\[2ex] C_0^{c_0}\left[\prod_{l=1}^{j-1}(1-|\tilde{C}_0^{r_l}|^2)\right]^{1/2} = -(-i)^j e^{i(\phi_N^r-\phi_0^c)}\sin(\Omega_{0,0}t_0^c)\prod_{l=1}^{N-1}\cos(\Omega_{0,l}t_l^r), & j=N. \end{cases} \tag{13}$$

Similarly, by sequentially applying a series of blue-sideband exciting operators $\hat{R}_j^b(t_j^b)$ with durations $t_j^b, j=1,2,\ldots,N$, after a carrier line operation $\hat{R}_0^c(t_0^c)$, we can implement the following deterministic quantum state generation

$$|\psi_0\rangle \xrightarrow{\hat{R}_N^b(t_N^b)\cdots\hat{R}_1^b(t_1^b)\hat{R}_0^c(t_0^c)} \left(\sum_{j=0}^{N} c_j'|j\rangle\right) \otimes |e\rangle = |\psi_2'\rangle \otimes |e\rangle, \tag{14}$$

with

$$c_j' = \begin{cases} -ie^{-i\phi_0^c}\sin(\Omega_{0,0}t_0^c), & j=0, \\[2ex] i^{j-1}e^{-i\phi_j^b}\cos(\Omega_{0,0}t_0^c)\prod_{l=1}^{j-1}\cos(\Omega_{0,l}t_l^b)\sin(\Omega_{0,j}t_j^b), & 1\leq j\leq N-1, \\[2ex] i^{N-1}e^{-i\phi_N^b}\cos(\Omega_{0,0}t_0^c)\prod_{l=1}^{N-1}\cos(\Omega_{0,l}t_l^b), & j=N. \end{cases}$$

Here, the duration t_N^b of the last operation $\hat{R}_N^b(t_N^b)$ is set as $\sin(\Omega_{0,N}t_N^b)=1$.

So far, we have shown that the desired superposition of a finite set of motional Fock states $\{|j\rangle; j=0,1,\ldots,N\}$ of the ion can be generated deterministically from the ground state $|0\rangle$ by using $N+1$ unitary operations, i.e., a carrier line operation $\hat{R}_0^c$ and N red-sideband $\hat{R}_j^r$ (or blue-sideband $\hat{R}_j^b$) exciting operations performed by using the laser beams with frequencies $\omega_L=\omega_0-j\omega$ (or $\omega_L=\omega_0+j\omega$). It is worth pointing out that the motional state generated above in Eq. (11) or (13) may be reduced to an arbitrary quantum pure state of the external vibration of the trapped cold ion, as any weight c_j in

Eq. (12) for the Fock state $|j\rangle$ can be prescribed arbitrarily. For example, the Pegg-Barnett phase state [20]

$$|\theta\rangle_N = \frac{1}{\sqrt{N+1}}\sum_{j=0}^{N} e^{ij\theta}|j\rangle \tag{15}$$

can be obtained from Eqs. (12) and (13) by setting the initial phases and the durations of the applied laser beams to satisfy the following conditions:

$$\phi_0^c = \frac{\pi}{2}, \quad e^{i\phi_j^r} = i^{j-1}e^{ij\theta}, \tag{16}$$

and

$$c_0 = \cos(\Omega_{0,0}t_0^c) = \cdots = \sin(\Omega_{0,0}t_0^c)\prod_{l=1}^{j-1}\cos(\Omega_{0,l}t_l^r)\sin(\Omega_{0,j}t_j^r)$$

$$= \cdots = \frac{1}{\sqrt{N+1}}, \tag{17}$$

which implies that $t_0^c = 2\exp(\eta^2/2)[2n_0\pi + \arccos(1/\sqrt{N+1})]/\Omega$, and $t_j^r = 2\exp(\eta^2/2)[2n_j\pi + \arcsin(1/\sqrt{N-j+1})]/(\Omega\eta^j)$, $j\neq 0$, with $n_0, n_j = 0, 1, 2, \ldots$. For example, in the typical experimental system [12] where $\omega_0 = 2\pi\times 4.11\times 10^{11}$ kHz, $\omega = 2\pi\times 135$ kHz, $\Omega = 50$ kHz, and $\eta = 0.25$, the simplest phase state $|\theta\rangle_1 = (|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2}$ can be prepared by sequentially applying a resonant laser beam (with frequency $\omega_L = \omega_0$ and initial phase $\phi_0^c = \pi/2$) of the shortest duration $t_0^c \approx 3.24\times 10^{-5}$ s and a red-sideband line (with frequency $\omega_L = \omega_0 - \omega$ and initial phase $\phi_1^r = \theta$) of the shortest duration $t_1^r \approx 2.6\times 10^{-4}$ s.

The superposition state generated above may approach some macroscopic motional quantum states of the ion, if the number N of Fock states $|j\rangle$ is sufficiently large. For example, if the durations of the applied unitary operations are set to satisfy the conditions

$$c_0 = \cos(\Omega_{0,0}t_0^c) = e^{-|\alpha|^2/2}, \quad c_1 = \alpha c_0, \quad c_2 = \alpha^2 c_0/\sqrt{2!},\ldots,$$

$$c_j = \alpha^j c_0/\sqrt{j!},\ldots, \tag{18}$$

the motional superposition state $\Sigma_{j=0}^{N}c_j|j\rangle$ in the limit $N\to\infty$ approaches the usual coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{j=0}^{\infty}\frac{\alpha^j}{\sqrt{j!}}|j\rangle. \tag{19}$$

Similarly, the usual even or odd coherent states may also be approached by the superposition motional states generated by sequentially applying the laser beams with frequencies $\omega_L = \omega - (2l)\omega, l = 0,1,2,\ldots$ or $\omega_L = \omega - (2l+1)\omega$, respectively.

IV. PRODUCING ENTANGLED STATES OF A TRAPPED COLD ION BEYOND THE LAMB-DICKE LIMIT

Before, we discussed how to generate a motional quantum state of the ion. Now, we turn our attention to the problem of how to control the entanglement between these degrees of freedom. It is well known that entanglement is one of the most striking aspects of quantum mechanics and plays an important role in quantum computation [7]. A laser-ion system provides an example for clearly showing how to produce an entanglement between two different quantum degrees of freedom (see, e.g., [21]). Therefore, the third target state which we want to prepare is the general entangled state of the internal and external motion degrees of freedom of the ion

$$|\Psi\rangle = \sum_{j=0}^{N_g}d_j^g|j\rangle\otimes|g\rangle + \sum_{j=0}^{N_e}d_j^e|j\rangle\otimes|e\rangle, \tag{20}$$

where $\Sigma_{j=0}^{N_g}d_j^g|j\rangle$ ($\Sigma_{j=0}^{N_e}d_j^e|j\rangle$) is the external state associated with the internal ground (excited) state. Notice that the coef-

ficients c_j in the state $|\psi_2\rangle$ generated above are prescribed arbitrarily. Thus, applying an additional conditional operation $\hat{R}_0^c(t_{N+1}^c)$, with duration t_{N+1}^c, to the state $|\psi_2\rangle\otimes|g\rangle$, one may prepare an entangled state with $N_g = N_e = N$; i.e.,

$$|\psi_2\rangle\otimes|g\rangle \xrightarrow{\hat{R}_0^c} \sum_{j=0}^{N}(d_j^g|j\rangle\otimes|g\rangle + d_j^e|j\rangle\otimes|e\rangle), \tag{21}$$

with

$$d_j^g = c_j\cos(\Omega_{j,0}t_{N+1}^c), \quad d_j^e = -ic_je^{-i\phi_{N+1}^c}\sin(\Omega_{j,0}t_{N+1}^c).$$

In the sequence of operations shown above, $N+1$ laser-ion interactions (a carrier line and N red-sideband/blue-sideband excitations) are used. We now consider a relatively simple method to generate the entangled quantum states of the ion. That is, by alternatively switching the lasers on the first blue-sideband and the first red-sideband N times, one can generate a typical entangled state [21]

$$|\Psi'\rangle = \sum_{j=0}^{N_g}d_{2j+1}^g|2j+1\rangle\otimes|g\rangle + \sum_{j=0}^{N_e}d_{2j}^e|2j\rangle\otimes|e\rangle, \tag{22}$$

with the odd (even) -number motional states entangled with the ground (excited) internal spin states of the ion. Without any loss of generality, we assume that the ion has been prepared beforehand in a general nonentangled state

$$|\Psi_0\rangle = \hat{R}_0^c(t_0^c)|0\rangle\otimes|g\rangle = d_0^{g0}|0\rangle\otimes|g\rangle + d_0^{e0}|0\rangle\otimes|e\rangle, \tag{23}$$

with $d_0^{g0} = (1-|C_0^{c0}|^2)^{1/2}$, $d_0^{e0} = C_0^{c0}$.

We now first tune the laser beam to the first red-sideband and thus realize the following operation

$$|\Psi_0\rangle \xrightarrow{\hat{R}_1^r(t_1^r)} (d_0^{g1}|0\rangle + d_1^{g1}|1\rangle)\otimes|g\rangle + d_0^{e1}|0\rangle\otimes|e\rangle = |\Psi_1\rangle. \tag{24}$$

Here,

$$d_0^{g1} = d_0^{g0}, \quad d_1^{g1} = d_0^{e0}\tilde{C}_0^{r1}, \quad d_0^{e1} = d_0^{e0}(1-|\tilde{C}_0^{r1}|^2)^{1/2}.$$

Obviously, the state $|\Psi_1\rangle$ is an entangled state. It reduces to the Bell-type state

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}(|0\rangle\otimes|e\rangle + |1\rangle\otimes|g\rangle), \tag{25}$$

if $\phi_1^r = 3\pi/2$ and the durations of the operations $\hat{R}_0^c(t_0^c)$ and $\hat{R}_1^r(t_1^c)$ are set up properly such that $d_0^{e0} = 1$ and $\tilde{C}_{01}^r = 1/\sqrt{2}$, corresponding to the shortest durations $t_0^c \approx 6.48\times 10^{-5}$ s and $t_1^r \approx 2.6\times 10^{-4}$ s.

We then tune the laser beam to the first blue-sideband and implement the evolution

$$|\Psi_1\rangle \xrightarrow{\hat{R}_1^b(t_2^b)} \sum_{j=0}^{1}d_j^{g2}|j\rangle\otimes|g\rangle + \sum_{j=0}^{2}d_j^{e2}|j\rangle\otimes|e\rangle = |\Psi_2\rangle, \tag{26}$$

with

$$d_0^{g2} = d_0^{g1}(1 - |C_0^{b2}|^2)^{1/2}, \quad d_1^{g2} = d_1^{g1}(1 - |C_1^{b2}|^2)^{1/2},$$

and

$$d_0^{e2} = d_0^{e1}, \quad d_1^{e2} = d_0^{g1}C_0^{b2}, \quad d_2^{e2} = d_1^{g1}C_1^{b2}.$$

Repeating the above-mentioned procedure, we realize the following series of evolutions:

$$|\Psi_2\rangle \xrightarrow{\hat{R}_1^r(t_3^r)} |\Psi_3\rangle \cdots \xrightarrow{\hat{R}_1^r(t_{2l}^b)} |\Psi_{2l}\rangle \xrightarrow{\hat{R}_1^r(t_{2l+1}^r)} |\Psi_{2l+1}\rangle \cdots \xrightarrow{\hat{R}_1^r(t_N^b)} |\psi_N\rangle,$$

$$\text{(27)}$$

with

$$|\Psi_{2l}\rangle = \sum_{j=0}^{2l-1} d_j^{g2l}|j\rangle \otimes |g\rangle + \sum_{j=0}^{2l} d_j^{e2l}|j\rangle \otimes |e\rangle,$$

$$|\Psi_{2l+1}\rangle = \sum_{j=0}^{2l+1} d_j^{g2l+1}|j\rangle \otimes |g\rangle + \sum_{j=0}^{2l} d_j^{e2l+1}|j\rangle \otimes |e\rangle.$$

Here,

$$d_j^{g2l} = \begin{cases} d_j^{g2l-1}(1 - |C_j^{b2l}|^2)^{1/2} + d_j^{e2l-1}\widetilde{C}_{j+1}^{b2l}, & 0 \leqslant j \leqslant 2l-2, \\ d_j^{g2l-1}(1 - |C_j^{b2l}|^2)^{1/2}, & j = 2l-1, 2l, \end{cases}$$

$$d_j^{e2l} = \begin{cases} d_j^{e2l-1}, & j = 0 \\ d_j^{e2l-1}(1 - |\widetilde{C}_{j-1}^{b2l}|^2)^{1/2} + d_{j-1}^{e2l-1}C_{j-1}^{b2l}, & 1 \leqslant j \leqslant 2l-2, \\ d_j^{g2l-1}C_j^{b2l}, & j = 2l-1, 2l, \end{cases}$$

and

$$d_j^{g2l+1} = \begin{cases} d_j^{g2l}, & j = 0, \\ d_j^{g2l}(1 - |C_{j-1}^{r2l+1}|^2)^{1/2} + d_{j-1}^{e2l}\widetilde{C}_{j-11}^{r2l+1}, & 1 \leqslant j \leqslant 2l-1, \\ d_j^{e2l}\widetilde{C}_j^{r2l+1}, & j = 2l, 2l+1, \end{cases}$$

$$d_j^{e2l+1} = \begin{cases} d_j^{e2l}(1 - |\widetilde{C}_j^{r2l+1}|^2)^{1/2} + d_{j-1}^{g2l}C_{j-1}^{r2l+1}, & 0 \leqslant j \leqslant 2l-1, \\ d_j^{e2l}(1 - |\widetilde{C}_j^{r2l+1}|^2)^{1/2}, & j = 2l, 2l+1. \end{cases}$$

Therefore, applying N $(N > 0)$ pairs of the first red-sideband and blue-sideband laser beams may generate the desired entangled state

$$|\psi_N\rangle = \sum_{j=0}^{N-1} d_j^{gN}|j\rangle \otimes |g\rangle + \sum_{j=0}^{N} d_j^{eN}|j\rangle \otimes |e\rangle. \quad \text{(28)}$$

If the initial state of the above process of quantum state manipulation is prepared in $|0\rangle \otimes |e\rangle$ by setting the duration of the applied carrier line laser to satisfy condition $|C_0^{c0}|^2 = 1$, then the desired entangled state (22), rewritten as

$$|\psi_N'\rangle = \sum_{j=0}^{[(N-1/2)]} d_{2j+1}^{gN}|2j+1\rangle \otimes |g\rangle + \sum_{j=0}^{[(N/2)]} d_{2j}^{eN}|2j\rangle \otimes |e\rangle, \quad \text{(29)}$$

is obtained by the above process. Here $[x]$ is the largest integer less than x.

V. DISCUSSIONS AND CONCLUSIONS

Based on the conditional quantum dynamics for laser-ion interactions beyond the Lamb-Dicke limit, we have intro-duced three kinds of unitary operations: the simple rotations of the internal states of the ion, the arbitrary red-sideband, and blue-sideband exciting operations. These unitary operations can be performed separately by applying the chosen laser beams with the relevant tunings. In general, any quantum state of the trapped cold ion can be generated deterministically by making use of these unitary operations selectively. Like some of the other schemes presented previously, several laser beams with different frequencies are also required in the present scheme. Tunable lasers provide the tool to create several types of quantum states of trapped ions.

Compared with previous approaches (see, e.g., [7,9]) operated in the LD regime, the most important advantage of the present scheme is that the operations are relatively simple, since various laser-ion interactions may be easily used by choosing the tunings of the applied laser beams. For example, at least n operations are required in the previous schemes to generate the state $|n\rangle \otimes |g\rangle$ from the initial state $|0\rangle \otimes |g\rangle$, as the dynamical process of the multiquantum motional excitation is negligible in the LD regime. However, we have shown here that only two unitary operations beyond the LD limit are sufficient to engineer the same quantum state. In

addition, the generated superposition motional states and the entangled states of the ion are universal and thus may reduce to the various desired special quantum states. The reason is that the weights of the superposed Fock states can be adjusted independently by controlling the relevant experimental parameters; e.g., the durations, initial phases, and frequencies of the applied laser beams. Several typical macroscopic quantum states of the motion of the ion, e.g., the Pegg-Barnett phase state, the coherent state, and the even and odd coherent states, etc. can be created or well-approximated, if the number of the superposition Fock states is sufficiently large.

We now give a brief discussion for the realizability of our approach. First, the present quantum manipulations need to resolve the vibrational sidebands of the ion trap. In fact, it is not difficult to generate the desired laser pulse with sufficiently narrow linewidth with current experimental technology. For example, the linewidth (1 kHz) of the laser beams (at 729 nm) used in Ref. [12] to drive the trapped ion $^{40}Ca^+$ is very small, corresponding to a resolution of better than $\Delta\nu/\nu = 2.5 \times 10^{-12}$. This linewidth is also much smaller than the frequency of the applied trap (138 kHz). Thus, the vibrational sidebands can be well resolved. The expected initial phases of the applied laser beams can be controlled by switching different signal paths [22]. During the very short durations (e.g., $\lesssim 10^{-4}$ s) for implementing the expected quantum operations, the applied laser beams (generated by the Ti:sapphire laser) are sufficiently stable (e.g., the corresponding initial phase-diffusion and frequency-drift times may reach to, e.g., 10 ms [23] and bandwidth $\lesssim 1$ kHz in 1 s averaging time [24], respectively). In fact, the proposed engineering scheme could also, in principle, be used for Raman excitation, where the phases of the applied laser beams can be well controlled (see, e.g., [2,5,6]).

Second, in the present scheme, an arbitrary Fock state can be prepared by using only two operations [see, e.g., Eqs. (9) and (10)]. The duration of operation depends on the value of k, once the LD parameter and the intensity of the applied laser beam are given. The Rabi frequency does not significantly reduce for large LD parameters (e.g., $\eta \gtrsim 0.5$). However, for small values of η (e.g., $\eta \lesssim 0.25$), the Rabi frequency decreases fast for sufficiently small values of k. Small values of the Rabi frequency correspond to a long duration of quantum operations, and the allowed number of operations will be reduced. For example, if $\eta = 0.25, \Omega = 2\pi \times 50$ kHz, then the duration of the transition $|g\rangle \rightarrow |e\rangle$ is $t_0^c \approx 10$ μs. Adding a $\pi/2$ pulse with duration $t_1^r \approx 40$ μs (or

$t_{10}^r \approx 0.3$ s), the Fock state: $|1\rangle$ (or $|10\rangle$) can be generated. Note that, compared to the t_1^r, t_0^r, the duration of operation $\hat{R}_{10}^r$ is relatively long, as the Rabi frequency is relatively small; $\Omega_{0,10}/\Omega \approx 2 \times 10^{-5}$ for the same laser intensity (see Fig. 1). In principle, this decreased Rabi frequency can be effectively compensated by enhancing the power (i.e., intensity) of the applied laser beam. In fact, the power of the laser applied to drive the trapped cold ion is generally controllable (e.g., the Ti:sapphire laser used in experiment [12] is adjustable in the range from a few μW to a few hundred mW). Therefore, the corresponding durations can be shorter by 2–5 orders of magnitude. For example, for $\eta = 0.25$, if the power of the applied laser beam is adjusted from a few μW to a few mW, the Rabi frequency $\Omega_{0,10}$ of the transition $|0\rangle \otimes |e\rangle \rightarrow |10\rangle|g\rangle$ can be enhanced to the same order of magnitude of the carrier Rabi frequency $\Omega(=2\pi \times 50)$kHz. The duration of the corresponding quantum operation is then shortened to 10^{-5} s. The smaller LD parameters η correspond to lasers with larger adjustable power ranges; e.g., for $\eta = 0.202$, the adjustable power range should be five orders of magnitude larger. Therefore, it is difficult to realize transitions with higher k in the LD regime, where $\eta \ll 1$.

Finally, the generation of the macroscopic superposed Fock states is limited in practice by the existing decays of the vibrational and atomic states. The lifetime of the atomic excited state $|e\rangle$ reaches up to 1 s [12] allows, in principle, to perform $10^3 - 10^4$ manipulations. Also, the recent experiment [24] showed that coherence for the superposition of $|n=0\rangle$ and $|n=1\rangle$ was maintained for up to 1 ms. Usually, the lifetime (i.e., relaxation time) of the state $|1\rangle$ should be longer than this dephasing time. Therefore, roughly say, preparing a superposition (e.g., phase state) from ground state $|n=0\rangle$ to the excited motional Fock states $|n\rangle$ with $n > 10$ is experimentally possible, as the durations of quantum operation are sufficiently short, e.g., $< 10^{-4}$ s. Improvements might be expected by considering the more realistic dynamics [14] that includes the decays of the excited atomic and motional states.

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